Lecture 17  Proving relations using compactified moduli spaces.

\[ \mathcal{D} : \text{CF}^*(L_0, L_1) \rightarrow \text{CF}^*(L_0, L_1) \text{ counts (mod 2) } \]

\[ \begin{array}{c}
\text{strip modulo translation} \\
\gamma \downarrow \quad \downarrow \gamma^+ \\
L_0 \quad L_1 \\
\end{array} \]

\[ M^*_\mathbb{Z}(y_-, y_+) \]

\[ \mathcal{D} \circ \mathcal{D} \text{ would then count configurations such as } \]

\[ \begin{array}{c}
\gamma_0 \downarrow \\
L_0 \quad L_1 \\
\gamma_1 \downarrow \\
\gamma_2 \uparrow \\
\end{array} \]

This configuration is in the boundary of \( \overline{M}^*_\mathbb{Z}(y_0, y_2) \).

If conversely, we know that every boundary point of \( \overline{M}^*_\mathbb{Z}(y_0, y_2) \)

corresponds to such a configuration, then the coefficient of \( y_0 \) in \( \partial(\mathcal{D}(y_2)) \) is zero. This coefficient counts these
configurations (mod 2), and a compact 4-manifold with
boundary always has an even number of boundary points.

In general, \( \overline{M}^*_\mathbb{Z}(y_0, y_2) \) may have other boundary points
such as

\[ \begin{array}{c}
\gamma_0 \downarrow \\
L_0 \quad L_1 \\
\gamma_1 \downarrow \\
y_2 \uparrow \\
\end{array} \]

we will later introduce hypotheses that rule out such configurations,
making the previous argument valid, and guaranteeing \( \partial^2 = 0 \).

We denote \( \text{HF}^*(L_0, L_1) = \ker \partial / \text{Im} \partial \).
Next, $\mathcal{C}_S : \otimes \text{CF}_+^c(L_{S^0}, L_{S^1}) \rightarrow \otimes \text{CF}_-^c(L_{S^0}, L_{S^1})$

checks (mod 2) $M_S(\xi y_S, y_S + 3)^0$

The boundary of $M_S(\xi y_S, y_S + 3)^1$ contains configurations such as

\[
\text{contributes to } \mathcal{C}_S \circ \mathcal{C}_S
\]

Again, once we rule out other degeneracies, the fact that $M_S(\xi y_S)^1$ has an even number of boundary points shows

$\partial \circ \mathcal{C}_S + \mathcal{C}_S \circ \partial = 0 \pmod 2$, or $\partial \circ \mathcal{C}_S = \mathcal{C}_S \circ \partial$

so that $\mathcal{C}_S$ is a chain map, and induces a map

$\Psi_S : \otimes \text{HF}_+^c(L_{S^1}, L_{S^0}) \rightarrow \otimes \text{HF}_-^c(L_{S^1}, L_{S^0})$

which are the TFT maps.