Lecture 13  Almost complex structures and
Pseudo-holomorphic curves.

Let $(M, \omega)$ be a symplectic manifold: $\omega \in \Omega^2(M)$, $\omega = 0$, $\omega^n > 0$.

We can think of $\omega$ as a skew-symmetric 2-form on $TM$. $\omega: TM^{\otimes 2} \to \mathbb{R}$.

An almost complex structure $J$ on $M$ is a section of $\text{End}(TM)$, i.e., a bundle map $J: TM \to TM$ such that $J^2 = -\text{Id}$.

A choice of an almost complex structure makes each tangent space $T_p M$ into a $C$-vector space with $(a + bi) \cdot v = a \cdot v + b J(v)$.

An almost complex structure $J$ is called compatible with $\omega$ if $g(u, v) = \omega(u, J(v))$ is a positive definite symmetric bilinear form, i.e., a Riemannian metric.

Proposition  The space of almost complex structures compatible with a given symplectic form is contractible.

There is a fairly constructive proof of this fact due to B. Sévennec.

Lemma  Let $(\mathbb{C}^n, J_0 = i, g, \omega)$ denote the standard complex space with hermitian metric $\langle u, v \rangle = g(u, v) + i \omega(u, v)$.

$$\langle (z_i), (w_i) \rangle = \sum_i z_i \overline{w_i}$$

Let $J_C = \{ J \in \text{Mat}_{2n \times 2n} \mid J^2 = -\text{Id}, J \text{ compatible with } \omega \}$.

Then, the "Cayley transform" $J \mapsto S := \frac{J - J_0}{J + J_0}$ induces a diffeomorphism

$$J_C \to \{ S \in \text{Mat}_{2n \times 2n} \mid \| S \|_g < 1, J_0 S + S J_0 = 0, S^T = S \}$$

Proof is a (somewhat lengthy) linear algebra exercise.
Now observe that the conditions \( J_0 S + SJ_0 = 0 \) and \( S^T = S \) are linear, while the condition \( \| S \|_g < 1 \) is convex. This shows that \( J_0 \) is contractible.

Now, given \( (M, \omega) \), let \( J_c(\omega) \rightarrow M \) be the fiber bundle whose fiber at \( p \in M \) is the space of linear maps \( J_p : T_p M \rightarrow T_p M, J^2 = 1 \) which are compatible with \( \omega_p : T_p M \rightarrow IR \). Then by the lemma, \( J_c(\omega) \rightarrow M \) has contractible fibers. The space \( J_c(\omega) \) of compatible a.c.s. on \( M \) is the space of sections of \( J_c(\omega) \), so it is also contractible.

Remark: Why "almost"? On an almost complex manifold \( (M, J) \), there need not exist local holomorphic coordinates. If such do exist, the a.c.s. is called integrable. Cf. Newlander-Nirenberg.

Now let \( (\Sigma, j) \) be a Riemann surface, \( \dim IR \Sigma = 2 \), \( j \) an a.c.s. on \( \Sigma \). There is a good theory of pseudo-holomorphic maps \( u : (\Sigma, j) \rightarrow (M, J) \), and a good moduli theory when \( J \) is compatible with \( \omega \).

Consider a smooth map \( u : \Sigma \rightarrow M \). For \( p \in \Sigma \), \( d_p u : T_p \Sigma \rightarrow T_{u(p)} M \) is a linear map. Now \( (T_p \Sigma, j_p) \) and \( (T_{u(p)} M, J_p) \) are complex vector spaces, but \( d_p u \) is only \( IR \)-linear in general. We can take the \( \mathbb{C} \)-linear and \( \mathbb{C} \)-anti-linear components.

\[
\Theta u = (du)^{1,0} = \frac{1}{2} (du - J du \circ j) \quad \Theta u \circ j = J du
\]

\[
\bar{\Theta} u = (du)^{0,1} = \frac{1}{2} (du + J du \circ j) \quad \bar{\Theta} u \circ j = -J du
\]

\( u \) is pseudo-holomorphic if \( (du)^{0,1} = 0. \)
The energy of a map is \( E(u) = \frac{1}{2} \int_{\Sigma} |dw|^2 \, dv_{\Sigma} \)

where \( |dw|^2 \) is calculated using the metric \( g(u,v) = \omega(u, Jv) \)

It may appear to depend on a metric on \( \Sigma \) as well, but it is actually conformally invariant, so it only depends on the complex structure of \( \Sigma \).

Let \( h \) be any metric compatible with the complex structure of \( \Sigma \) means \( h(v, Jv) = 0 \) for all \( v \).

Any other metric with this property is \( h' = e^{2\phi} h \) for some function \( \phi : \Sigma \to \mathbb{R} \).

Then \( |dw|_{g, h'}^2 = e^{-2\phi} |dw|_{g, h}^2 \) and \( dv_{h'} = e^{2\phi} dv_h \)

so \( |dw|_{g, h'}^2 dv_{h'} = |dw|_{g, h}^2 dv_h \)

Prop: If \( u : \Sigma \to M \) is pseudo-holomorphic, and \( J \) is compatible with \( \omega \), then \( E(u) = \int_{\Sigma} u^* \omega \)

Proof: Choose a local holomorphic coordinate \( z = x + iy \) such that \( h(\overline{\partial}_s, \partial_s) = 1 \). Then \( h(\overline{\partial}_t, \partial_t) = 1 \), \( h(\overline{\partial}_s, \partial_t) = 0 \).

\[
g(du(\overline{\partial}_s), du(\partial_s)) = g(du(\overline{\partial}_t), du(\partial_t)) = g(J du(\overline{\partial}_s), J du(\overline{\partial}_s)) = g(du(\overline{\partial}_s), du(\overline{\partial}_s))
\]

and \( g(du(\partial_s), du(\partial_t)) = g(du(\partial_s), J du(\partial_s)) = 0 \)

If \( \xi \) is an \( h \)-unit vector on \( \Sigma \), \( \xi = \cos \theta \overline{\partial}_s + \sin \theta \partial_t \)

Then \( g(du(\xi), du(\xi)) = \cos^2 \theta g(du(\partial_s), du(\partial_s)) + \sin^2 \theta g(du(\partial_t), du(\partial_t)) = g(du(\partial_s), du(\partial_s)) \).

\[
|dw|_{g, h}^2 = \max_{\theta, \xi} g(du(\xi), du(\xi)) = g(du(\partial_s), du(\partial_s)) = \omega(du(\partial_s), J du(\partial_s)) = \omega(du(\partial_s), du(\partial_t)) = \omega(du(\partial_t), du(\overline{\partial}_s)) = \omega(du(\partial_t), du(\partial_t))
\]

so \( |dw|_{g, h}^2 dv_h = \omega(du(\partial_s), du(\partial_t)) dv_h dt = u^* \omega \).