The derived category \( \mathcal{D}(R) \) is formed from the homotopy category \( \mathcal{K}(R) \) of cofunctor complexes of \( R \)-modules by "inverting quasi-isomorphisms." This is a process analogous to the localization of a ring (for instance, the field of fractions of an integral domain).

**Def.** Let \( S \) be a collection of morphisms in a category \( \mathcal{C} \). A localization of \( \mathcal{C} \) wrt. \( S \) is a category \( \mathcal{S}^{-1}\mathcal{C} \) and a functor \( q : \mathcal{C} \to \mathcal{S}^{-1}\mathcal{C} \) such that:
1. \( \forall s \in S, \ q(s) \) is an isomorphism in \( \mathcal{S}^{-1}\mathcal{C} \)
2. If \( F : \mathcal{C} \to \mathcal{D} \) is a functor such that \( F(s) \) is an isomorphism in \( \mathcal{D} \) for all \( s \in S \), then \( \exists F' : \mathcal{S}^{-1}\mathcal{C} \to \mathcal{D} \) such that \( F = F'q \).

Because this is a universal property, \( \mathcal{S}^{-1}\mathcal{C} \) is unique if it exists.

**Def.** Let \( Q = \{ f : X \to X' \mid f \text{ is quasi-isoto} \} \subset \mathcal{K}(R) \). Then the derived category of \( R \)-modules is
\[
\mathcal{D}(R) = Q^{-1}\mathcal{K}(R)
\]

**Proposition.** \( \mathcal{D}(R) \) exists.

Roughly, morphisms in \( \mathcal{D}(R) \) are equivalence classes of "fractions" \( fs^{-1} : X \xrightarrow{s^{-1}} X' \) where \( s \in Q \).

This creates a set-theoretic issue, since, for a given \( X \), the class of quasi-isomorphisms \( X \xleftarrow{s} X' \) may not be a set.
Nevertheless, it is possible to model the equivalence classes of fractions by sets, and this is how $D(R)$ is constructed. This is why "$D(R)$ exists" is a proposition.

Prop $D(R)$ is a triangulated category, and $q: K(R) \to D(R)$ is an exact functor (commutes with shift, takes exact triangles to exact triangles.)

See Weibel Ch. 10 for proofs.

At the end of the day, morphisms in $D(R)$ are related to something that we knew from homological algebra.

Def Given $A, B \in \text{Ch}(R)$, define the hyperext groups

$$\text{Ext}^n(A, B) := \text{Hom}_{D(R)}(A, B[n])$$

If $A, B \in \text{mod-}R$ are regarded as complexes in degree 0, $\text{Ext}^n(A, B)$ is the usual Ext group.

Note that for $A, B \in \text{mod-}R$,

$$\text{Hom}_{K(R)}(A, B[n]) = \begin{cases} \text{Hom}_{\text{mod-}R}(A, B), & n = 0 \\ 0 & , n \neq 0 \end{cases}$$

showing that $K(R)$ really differs from $D(R)$.

Recall $\text{Ext}: A, B \in \text{mod-}R$. Replace $A$ by a projective resolution (e.g. a free resolution)

$$\ldots \xrightarrow{f_2} P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} A \to 0$$

then compute $\text{Hom}_{\text{mod-}R}(P_i, B)$, take cohomology.
\[ \text{Ext}_2(\mathbb{Z}_2, \mathbb{Z}_2) : \quad \text{Free resolution } 0 \to \mathbb{Z}^2 \to \mathbb{Z} \to \mathbb{Z}_2 \to 0 \]

\[ \text{Hom}(\mathbb{Z}, \mathbb{Z}_2) \leftarrow \text{Hom}(\mathbb{Z}, \mathbb{Z}_2) \]

\[ \mathbb{Z}_2 \]

So \[ \text{Ext}_2^0(\mathbb{Z}_2, \mathbb{Z}_2) = \mathbb{Z}_2 = \text{Hom}_\mathbb{Z}(\mathbb{Z}_2, \mathbb{Z}_2) \]

\[ \text{Ext}_2^1(\mathbb{Z}_2, \mathbb{Z}_2) = \mathbb{Z}_2, \text{ the nonidentity element corresponds to the extension } 0 \to \mathbb{Z}_2 \to \mathbb{Z}_4 \to \mathbb{Z}_2 \to 0. \]

So \[ \text{Hom}_{D(\mathbb{R})}(\mathbb{Z}_2, \mathbb{Z}_2) = \mathbb{Z}_2 \]

\[ \text{Hom}_{D(\mathbb{R})}(\mathbb{Z}_2, \mathbb{Z}_2[1]) = \mathbb{Z}_2 \]

The nontrivial morphism \[ \mathbb{Z}_2 \to \mathbb{Z}_2[1] \] is represented by the fraction \( s^{-1} \)

\[ 0 \to 0 \to \mathbb{Z}_2 \to 0 \]

\[ s \uparrow \quad s \uparrow \quad s \text{ is a quasi isomorphism} \]

\[ 0 \to \mathbb{Z}^2 \to \mathbb{Z} \to 0 \]

\[ f \downarrow \quad f \downarrow \]

\[ 0 \to \mathbb{Z}_2 \to 0 \to 0 \]