Start with Example on p. 4 of previous notes.
Finish previous notes, then:

Sketch of proof that \( K(R) \) is a triangulated category:

Axiom TR1 is clear from the construction: every map has a cone.

For axiom TR2, we may assume the exact triangle is of the form

\[
A \to B \to \text{Cone}(u) \to A[1]
\]

We claim

\[
B \to \text{Cone}(u) \to A[1] \to B[1]
\]

which is to say that \( \text{Cone}(B \to \text{Cone}(u)) \) is chain homotopy equivalent to \( A[1] \).

\[
\text{Cone}(u) = \begin{pmatrix} B \oplus A[1], d_{\text{cone}u} = \begin{pmatrix} d_B & u \\ 0 & -d_A \end{pmatrix} \end{pmatrix}
\]

\[
\text{Cone}(v) = \begin{pmatrix} (B \oplus A[1]) \oplus B[1], d = \begin{pmatrix} d_B & u & 1_B \\ d_A & 0 & 0 \\ 0 & 0 & -d_B \end{pmatrix} \end{pmatrix}
\]

So \( \text{Cone}(v) \) contains a subcomplex \( (B \oplus B[1], (d_B, 1_B)) \) which is clearly contractible: indeed \( p = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \) is a map such that

\[
dP + Pd = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} d & 1 \\ 0 & -d \end{pmatrix} + \begin{pmatrix} d & 1 \\ 0 & -d \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1_{B \oplus B[1]}
\]

Using this we can construct a homotopy equivalence

\[
\text{Cone}(v) \cong \text{Cone}(v) / B \oplus B[1] \cong A[1]
\]

Homework: check the other rotation.
Axiom TR^3 expresses the naturality of the mapping cone construction.

**TR^4**: Now the dreaded octahedral axiom: Let \( A \to B \to C \) be maps where \( C' = \text{cone}(u) \), \( A' = \text{cone}(v) \), \( B' = \text{cone}(vu) \).

The axiom asserts existence of maps \( f : C' \to B' \) and \( g : B' \to A' \) with certain properties.

Now \( C' = (B \oplus A[i], (d_{B} u, d_{A}) \) \) \( B' = (C \oplus A[i], (d_{C} vu, d_{A}) \) \)
\( A' = (C \oplus B[i], (d_{C} v, d_{B}) \) \)

Define \( f : C' \to B' \) by \( f(b, a) = (vb, a) \) \)
\( g : B' \to A' \) by \( g(c, a) = (c, u(a)) \)

Check: These are chain maps, there is a natural map \( A' \to \text{Cone}(f) \)

since \( \text{cone}(f) = C \oplus A[i] \oplus B[i] \oplus A[2] \)

It remains to check all of the compatibilities in the octahedral diagram, in particular, that \( \text{cone}(f) \) is homotopy equivalent to \( A' \).

(omitted.)

Given a cochain complex of \( R \)-modules \( \ldots M_{i-1} \to M_{i} \to M_{i+1} \ldots \) we can take the \( i \)-th cohomology \( R \)-module

\[ H^{i}(M) = \frac{\ker(d : M_{i} \to M_{i+1})}{\text{Im}(d : M_{i+1} \to M_{i})} \]

By homotopy invariance of cohomology, \( H^{i} : K(R) \to \text{mod-R} \) is a functor on the homotopy category.
The collection of functors \( \{ \mathcal{H}^i \}_{i \in \mathbb{Z}} \) interact nicely with exact triangles:

Prop. If \( \mathcal{C}^{X \to Y} \) is an exact triangle in \( \mathcal{K}(R) \), then

\[
\begin{array}{ccc}
A & \overset{u}{\rightarrow} & B \\
\downarrow & & \downarrow \\
\mathcal{H}^{-} & \overset{v^*}{\rightarrow} & \mathcal{H}^{+}
\end{array}
\]

there is a long exact sequence in \( \text{mod}-R \):

\[\ldots \rightarrow \mathcal{H}^{-1}(C) \overset{u^*}{\rightarrow} \mathcal{H}^{-1}(A) \overset{v^*}{\rightarrow} \mathcal{H}^{-1}(B) \overset{v^*}{\rightarrow} \mathcal{H}^{-1}(C) \overset{u^*}{\rightarrow} \mathcal{H}^{-1}(A) \rightarrow \ldots\]

Remk: Axiomatizing this proposition as a property of \( \{ \mathcal{H}^i \} \) yields the notion of a "cohomological functor" on a triangulated category.

For some purposes, \( \mathcal{K}(R) \) is not quite the right category for the homotopy theory of chain complexes: There are two related issues:

Def. A cochain complex \( \{ M^d \}_{d \geq 0} \) is called \emph{acyclic} if \( \mathcal{H}^i(M) = 0 \) for all \( i \).

A map \( f : M^d \rightarrow N^d \) is called a \emph{quasi-isomorphism} if the induced map on cohomology, \( \mathcal{H}^i(f) : \mathcal{H}^i(M) \rightarrow \mathcal{H}^i(N) \) is an isomorphism for all \( i \).

For example, a chain homotopy equivalence is always a quasi-isomorphism.

The two (related) problems are: In \( \mathcal{K}(R) \),

(i) There are quasi-isomorphisms that are not homotopy equivalences.

(ii) There are acyclic complexes that are not contractible.