Math 595  Homological Mirror Symmetry

Two geometries

A) Symplectic manifold \((M, \omega)\)
   - \(M\) smooth manifold of dimension \(2n\)
   - \(\omega \in \Omega^2(M)\), \(d\omega = 0\), \(\omega^n > 0\).

B) Algebraic variety \(X\)
   - e.g. compact complex submanifold of \(\mathbb{C} \mathbb{P}^n\)
   - or more abstractly, a scheme.

In homological mirror symmetry (HMS),
Geometries (A) and (B) meet in the form of
triangulated categories.

\[
\begin{align*}
\text{Symplectic manifolds} & \quad \rightarrow \quad \text{Triangulated categories} & \quad \leftarrow \quad \text{Algebraic varieties} \\
(M, \omega) & \quad \rightarrow \quad \mathcal{F}(M, \omega) \quad \text{"Fukaya Category"} & \quad \leftarrow \quad \mathcal{D}^b(X) & \quad \leftarrow \quad X
\end{align*}
\]

Where \(\mathcal{F}(M, \omega)\) and \(\mathcal{D}^b(X)\) are "equivalent," we say
\((M, \omega)\) and \(X\) are "HMS partners" and that
"HMS holds" for this pair.

There is a lot of motivation and history behind this idea,
but for now, let's have a look at a precise theorem that
we can (hope to) prove this semester.

*Note:* Some of the concepts in what follows are likely new
to you. In this course we will provide complete
definitions. (Just not all today.)
Seidel's Quartic Surface Theorem

\textbf{Coefficients:} let \( \Lambda = \mathbb{C}[[q]] \) be ring of formal power series
let \( \Lambda_q = \mathbb{C}[[q]] \) be formal Laurent series
let \( \Lambda_q \) be algebraic closure of \( \Lambda_q \)

\( \Lambda_q \) consists of fractions series
\( f = \sum_{m \in \mathbb{Q}} a_m q^m \)
such that \( a_m = 0 \) for \( m \ll 0 \) and all \( m \) such that \( a_{m+0} \) can be taken to have a common denominator.

The \( q \)-adically continuous Galois group of \( \Lambda_q / \mathbb{C} \)
consists of formal changes of variable
\( \nu^*: q \mapsto \nu(q) \) for \( \nu \in \mathbb{C}[[z]] \)

\textbf{Symplectic side} let \( \mathbb{C}P^3 \) be the complex projective space
let \( M_p \subset \mathbb{C}P^3 \) be a submanifold defined by
the vanishing of a quartic homogeneous polynomial \( \varphi \)
\( \varphi(x_0, x_1, x_2, x_3) = x_0^4 + x_1^4 + x_2^4 + x_3^4 = 0 \)

\( \mathbb{C}P^3 \) carries a symplectic Fubini-Study form \( \omega_{FS} \)
let \( \omega_p = \omega_{FS} |_{M_p} \); this is a symplectic form on \( M_p \)

\textbf{Lemma:} Up to symplectic diffeomorphism \( (M_p, \omega_p) \)
does not depend on the choice of the quartic polynomial \( \varphi \)
(All that matters is that \( 0 \) be a regular value of \( \varphi \)
so that \( M_p \) is a smooth manifold)
To \((M_p, w_p)\) we associate the split-closed derived Fukaya category \(D^{TT}F(M_p, w_p)\) which is a \(\Lambda^Q\)-linear triangulated category (Hom sets are \(\Lambda^Q\)-vector spaces) defined using (rational) Lagrangian submanifolds of \(M_p\) and pseudo-holomorphic curves.

**Algebraic side:** Inside \(\mathbb{P}^3_{\Lambda^Q}\) consider the quartic surface

\[ Y = \{ y_0 y_1 y_2 y_3 + q(y_0^4 + y_1^4 + y_2^4 + y_3^4) = 0 \} = \mathbb{P}^3_{\Lambda^Q} \]

There is a group action by \(\Gamma_{16} \cong \mathbb{Z}_4 \times \mathbb{Z}_4\)

\[ \Gamma_{16} = \left\{ \begin{bmatrix} \alpha_0 & \alpha_1 \\ \alpha_2 & \alpha_3 \end{bmatrix} \right\} \mid \alpha_k^4 = 1, \text{ } \alpha_0 \alpha_1 \alpha_2 \alpha_3 = 1 \right\} \subset \text{PSL}(4, \Lambda^Q) \]

by scaling the coordinates.

Denote by \(\Sigma^*\) the unique minimal surface resolving the singularities of \(Y / \Gamma_{16}\).

(Since \(\Lambda^Q\) is an algebraically closed field of characteristic zero, this is classical.)

To \(\Sigma^*\) we associate the Bounded derived category of coherent sheaves \(D^b\text{Coh}(\Sigma^*)\).

Also a \(\Lambda^Q\)-linear triangulated category.

**Theorem (Seidel)** There is a \(\Psi \in \text{Cl}(\Sigma)\) and an equivalence of categories

\[ D^{TT}F(M_p, w_p) \cong \hat{\Psi}^* D^b\text{Coh}(\Sigma^*) \]
Here $\hat{\Psi}$ is any lift of $\Psi \in \text{Gal}_V(L_2/C)$ to $\text{Gal}_V(L_0/C)$ and $\hat{\Psi}^* \text{D}^b \text{Coh}(Z_{V_0}^*)$ means we twist the $L_0$-linear structure by this automorphism. 
[\Psi$ corresponds to the "mirror map" between moduli spaces.]

What are triangulated categories, and how do they arise from geometry?

Homological Algebra: let $R$ be an associative ring, and let mod-$R$ denote the category of right $R$-modules.

Objects = right $R$-modules
Morphisms = $R$-module homomorphisms.

mod-$R$ has more structure than just a category, because we can do things like add objects (AB3) and morphisms, we can take kernels of morphisms, we have exact sequences, and so on.

We can look at cochain complexes of $R$-modules

$$\ldots \rightarrow M_{i-1} \xrightarrow{d_{i-1}} M_i \xrightarrow{d_i} M_{i+1} \xrightarrow{d_{i+1}} M_{i+2} \rightarrow \ldots$$

each $M_i$ an $R$-module, $d_i$'s module maps

$(\forall i)$ \(d_i \circ d_{i-1} = 0$

Cohomology modules $H^i(M) = \frac{\ker d_i}{\text{Im} d_{i-1}}$

There is a category Ch(mod-$R$) of cochain complexes an cochain maps $f : M \rightarrow N$

\[ M_i \xrightarrow{d_i} M_{i+1} \quad f_i \downarrow \quad \downarrow f_{i+1} \]

\[ N_i \xrightarrow{d_i} N_{i+1} \]
Then we form $K(\text{mod}-R)$ = homotopy category of $\text{Ch}(\text{mod}-R)$
    same objects but morphisms are homotopy classes of cochain maps.

Then we form $D(\text{mod}-R)$ from $K(\text{mod}-R)$
by taking the “Verdier quotient” by the subcategory
of acyclic complexes (ones for which $H^i(M) = 0$ $\forall i$)

This $D(\text{mod}-R)$ is called the Derived category of R-modules
and it is the right setting for the “homotopy theory”
of R-modules.

The notion of “Triangulated category” is the abstraction
of the salient features of $D(\text{mod}-R)$