Lecture 36  Ideals

Let \((R, +, \cdot)\) and \((S, +, \cdot)\) be rings.
Let \(\varphi : R \to S\) be a ring homomorphism.

**Def:** The kernel of \(\varphi\) is
\[
\ker \varphi = \varphi^{-1}(0) = \{ r \in R \mid \varphi(r) = 0 \}
\]

**Lemma:** \(\varphi\) is injective if and only if \(\ker \varphi = \{0\}\).
This is true because a ring homomorphism is always a homomorphism of groups \(\varphi : (R, +) \to (S, +)\).

Now for groups, the kernel is always a normal subgroup.
For rings, the kernel is a special kind of subgroup called an ideal:

**Def:** An ideal in a ring \(R\) is a subset \(I \subseteq R\) such that
- \(I\) is a subgroup of \(R\) with respect to addition:
  \(a, b \in I \Rightarrow a + b \in I\) and \(-a \in I\).
- \(I\) is closed under multiplication by elements of \(R\):
  \(a \in I, r \in R \Rightarrow ra, rI \in I\) and \(ar \in I\).

In the case where \(R\) is non-commutative, we say that
- \(I\) is a left ideal if \(a, r \in I \Rightarrow ra \in I\)
  (but not necessarily \(ar \in I\))
- \(I\) is a right ideal if \(a, r \in I \Rightarrow ar \in I\)
  (but not necessarily \(ra \in I\)).

In this context, we say \(I\) is a two-sided ideal (or simply \(I\) ideal) if it is both a left and right ideal.
Proposition (6.2.15) If \( \varphi : R \to S \) is a ring homomorphism, then \( \ker(\varphi) \) is an ideal in \( R \).

**Proof:** Since \( \varphi : (R,+) \to (S,+) \) is a homomorphism of groups, its kernel is a subgroup.

Let \( r \in R \) and \( a \in \ker(\varphi) \) then
\[
\varphi(ra) = \varphi(r) \varphi(a) = \varphi(r) \cdot 0 = 0 \Rightarrow ra \in \ker(\varphi)
\]
\[
\varphi(ar) = \varphi(a) \varphi(r) = 0 \cdot \varphi(r) = 0 \Rightarrow ar \in \ker(\varphi).
\]

**Example (i)** \( \varphi : \mathbb{Z} \to \mathbb{Z}_n \) \( \varphi(k) = [k]_n \)

\( \ker(\varphi) = n\mathbb{Z} = \{ n \cdot k \mid k \in \mathbb{Z} \} \) all multiples of \( n \).

(ii) Let \( K \) be a field, \( a \in K \) define \( \varphi_a : K[x] \to K \) to be the unique homomorphism such that \( \varphi_a(x) = r \) for \( r \in K \) and \( \varphi_a(x) = a \). If \( f(x) \) is a polynomial, we have
\[
\varphi_a(f(x)) = f(a).
\]

So \( \ker(\varphi_a) = \{ f \mid f(a) = 0 \} \). This is the set of polynomials that become 0 under the substitution \( x \to a \). This is the set of polynomials that have \( a \) as a root.

Proposition (a) The intersection of ideals is an ideal:

If \( \{ I_a \}_{a \in A} \) are ideals in \( R \), then \( \bigcap_{a \in A} I_a \) is an ideal in \( R \).

(b) If \( I \) and \( J \) are ideals in \( R \), then
\[
IJ = \{ \sum_{i=1}^s a_i b_i \mid s \geq 1, a_i \in I, b_i \in J \}
\]

is an ideal in \( R \) and \( IJ \subseteq I \cap J \).

(c) If \( I \) and \( J \) are ideals in \( R \) then
\[
I + J = \{ a + b \mid a \in I, b \in J \}
\]
is an ideal in \( R \).