Lecture 28  The orbit-counting theorem
(Burnside’s Lemma)

Last time: orbit-stabilizer theorem. If a group $G$ acts on a set $X$, and $x \in X$, there is a bijection.

$$\mathcal{Y}: G/\text{Stab}(x) \to G \cdot x$$

In particular, if $X$ and $G$ are finite sets, we have an equation.

$$|G \cdot x| = |G| / |\text{Stab}(x)|$$

Here is another question: Assume $X$ and $G$ are finite. How many orbits are there? There is a rather nice formula for this that follows from the orbit-stabilizer theorem together with a bit of clever arithmetic.

First: recall $\text{Stab}(x) = \{ g \in G \mid g \cdot x = x \}$ $\subseteq G$

Also define $\text{Fix}(g) = \{ x \in X \mid g \cdot x = x \}$ $\subseteq X$

$\text{Fix}(g)$ is the set of fixed points of $g$ acting in $X$.

Orbit-counting theorem Assume $G$ and $X$ are finite.

Then

$$\text{# of orbits} = \frac{1}{|G|} \sum_{g \in G} |\text{Fix}(g)|$$

"The number of orbits equals the average number of fixed points of an element of $G"
Proof: consider the set \( \Gamma = \{(g, x) \mid g \cdot x = x \} \subseteq G \times X \) of pairs of a group element and a point fixed by it.

We count \( |\Gamma| \) in two ways

\[
|\Gamma| = \sum_{x \in X} \left( \# g \text{ such that } (g, x) \in \Gamma \right) = \sum_{x \in X} |\text{Stab}(x)|
\]

\[
|\Gamma| = \sum_{g \in G} \left( \# x \text{ such that } (g, x) \in \Gamma \right) = \sum_{g \in G} |\text{Fix}(g)|
\]

So \( \sum_{x \in X} |\text{Stab}(x)| = \sum_{g \in G} |\text{Fix}(g)| \)

Now \( |\text{Stab}(x)| = |G| / |G \cdot x| \) by orbit-stabilizer theorem,

so \( \sum_{x \in X} \frac{|G|}{|G \cdot x|} = \sum_{g \in G} |\text{Fix}(g)| \)

and \( \sum_{x \in X} \frac{1}{|G \cdot x|} = \frac{1}{|G|} \sum_{g \in G} |\text{Fix}(g)| \) (divide both sides by \(|G|\))

So it remains to show that the left-hand-side is the number of orbits

Let \( \Theta_1, \Theta_2, \ldots, \Theta_r \) be a complete list of pairwise distinct orbits

Recall these form a partition of \( X \), and \( x \in \Theta_i \iff G \cdot x = \Theta_i \)

\[
\sum_{x \in X} \frac{1}{|G \cdot x|} = \sum_{i=1}^{r} \sum_{x \in \Theta_i} \frac{1}{|G \cdot x|} = \sum_{i=1}^{r} \left( \sum_{x \in \Theta_i} \frac{1}{|G \cdot x|} \right)
\]

= \( \sum_{i=1}^{r} 1 = r = \# \text{ of orbits} \)

This completes the proof.
How many necklaces can be made from 4 red beads and 3 blue beads?

To make a necklace:

(1) Arrange red and blue beads along a string:

(2) Tie the ends together:

Some different choices in step (1) lead to the same necklace:

More precise setup:

(1) Arrange beads at sites labeled 1-7

(2) Transfer this to the vertices of a regular 7-gon
The dihedral group \( D_7 \) acts on the set of these pictures, and two necklaces are the same if they lie in the same orbit. There are \( \binom{7}{3} = \frac{7!}{3!4!} = 35 \) choices for step (1).

We need to find the number of orbits of \( D_7 \) on this set:

\[
\text{(#orbits)} = \frac{1}{|D_7|} \sum_{g \in D_7} |\text{Fix}(g)| = \frac{1}{14} \sum_{g \in D_7} |\text{Fix}(g)|
\]

e fixes everything \( |\text{Fix}(e)| = 35 \)
rotation \( r \) fixes nothing, always a pair of adjacent red and blue.
rotation \( r^2 \) fixes nothing. Whatever color 1 is, 3 would have to be same, then 5, then 7, then 2, then 4, then 6, so all would be same color.
Similarly, \( r^k \) fixes nothing for \( 1 \leq k \leq 6 \).
(this may not always be true if you change the number of beads)

Now \( j \), the flip about x-axis, fixes 3 things.

\[
\text{#orbits} = \frac{1}{|D_7|} \sum_{g \in D_7} |\text{Fix}(g)| = \frac{1}{14} \left( 35 + 7 \cdot 3 \right)
\]

\[
= \frac{1}{14} (56) = 4 \quad \text{There are 4 possible necklaces.}
\]