"Group" is an abstract concept, but many examples are
"Groups of symmetry transformations" like $\text{Sym}(X)$, $\text{D}_n$, $\text{GL}(n, \mathbb{R})$, and so on.

Given a group $G$, it is then natural to ask if we can think of $G$ as symmetries of something (in an abstract sense).

Let $X$ be a set, and let $G$ be a group.

**Definition:** An action of $G$ on $X$ is a function

$$G \times X \rightarrow X \quad \text{denoted} \quad (g, x) \mapsto g \cdot x$$

Satisfying

1. $e \cdot x = x$ for all $x \in X$, where $e$ is the identity element of $G$.
2. $(gh) \cdot x = (g \cdot (h \cdot x))$ for all $g, h \in G$, $x \in X$.

There is another way to think about actions, which is as a homomorphism $\alpha : G \rightarrow \text{Sym}(X)$, where $\text{Sym}(X) = \{f : X \rightarrow X \mid f \text{ bijective}\}$ is the symmetric group of $X$.

**Lemma** An action $G \times X \rightarrow X$ of a group $G$ on a set $X$
determines and is determined by a homomorphism $\alpha : G \rightarrow \text{Sym}(X)$

$$g \mapsto \alpha_g$$

where $\alpha_g(x) = g \cdot x$.
Proof: Let \( G \times X \to X \) \((g, x) \mapsto g \cdot x\) be a group action.

For each \( g \in G \), let \( \alpha_g : X \to X \) be the function
\[
\alpha_g(x) = g \cdot x.
\]
We claim \( \alpha_g \) is bijective. In fact, its inverse is \( \alpha^{-1}_g \) for \( \alpha^{-1}_g(\alpha_g(x)) = g^{-1} \cdot (g \cdot x) = (g^{-1}g) \cdot x = e \cdot x = x \) by axioms of a group action.

So for all \( g \in G \), \( \alpha_g \in \text{Sym}(X) \), and \( \alpha : G \to \text{Sym}(X) \) is a function.

Last, we check \( \alpha \) is a homomorphism:
\[
\alpha_{gh}(x) = (gh) \cdot x = g \cdot (h \cdot x) = \alpha_g(\alpha_h(x)) \quad (\text{all } x \in X)
\]
so \( \alpha_{gh} = \alpha_g \circ \alpha_h \), as desired.

* Conversely, suppose \( \alpha : G \to \text{Sym}(X) \) is a homomorphism.

Define a function \( G \times X \to X \) by declaring \( g \cdot x = \alpha_g(x) \).

We must check the axioms of a group action.

(i) since \( \alpha \) is a homomorphism, it take identity to identity.

Thus \( \alpha_e = \text{Id}_X \) where \( \text{Id}_X : X \to X \) is the identity function.

so \( e \cdot x = \alpha_e(x) = \text{Id}_X(x) = x \), as desired.

(ii) since \( \alpha \) is a homomorphism, \( \alpha_{gh} = \alpha_g \circ \alpha_h \)

so \( (gh) \cdot x = \alpha_{gh}(x) = \alpha_g(\alpha_h(x)) = g \cdot (h \cdot x) \), as desired.

Due to this lemma, we will often switch between the two “pictures” of a group action: (a) A map \( G \times X \to X \)
(b) a map \( G \to \text{Sym}(X) \)

Definition 5.1.1 in the book corresponds to picture (b).
Examples

(i) \(X\) any set, \(G = \text{Sym}(X)\) symmetric group.
\(\text{Sym}(X)\) acts on \(X\).
\(\text{Sym}(X) \times X \rightarrow X\)
\((\sigma, x) \rightarrow \sigma(x)\) (apply function \(\sigma\) to \(x\))
The corresponding homomorphism \(\chi: \text{Sym}(X) \rightarrow \text{Sym}(X)\)
is the identity.

(ii) Let \(H \leq \text{Sym}(X)\) be a subgroup. Then \(H\) acts on \(X\)
similarly to example (i).

(iii) \(X = \mathbb{R}^n\) \(G = \text{GL}(n, \mathbb{R})\)
\(\text{GL}(n, \mathbb{R}) \times \mathbb{R}^n \rightarrow \mathbb{R}^n\)
\((A, \mathbf{x}) \mapsto A\mathbf{x}\) multiply vector and matrix.
Also any subgroup of \(\text{GL}(n, \mathbb{R})\) acts on \(\mathbb{R}^n\).

(4) Given a group \(G\), we can take \(X = G\), and find several
actions of \(G\) on itself.

(4a) recall left multiplication \(L_g: G \rightarrow G\)
\(L_g(h) = gh\).
We saw in lecture 10 that this gives a homomorphism
\(L: G \rightarrow \text{Sym}(G)\)
\(g \mapsto L_g\)
So this is a group action of \(G\) on \(G\), called left multiplication.
(The corresponding map \(G \times G \rightarrow G\) is just multiplication.)

(4b) For \(g \in G\), we have conjugation by \(g: C_g: G \rightarrow G\)
\(C_g(h) = ghg^{-1}\). We saw in lecture 22
that the function \(C: G \rightarrow \text{Aut}(G)\), \(g \mapsto C_g\)
is a homomorphism. Now \(\text{Aut}(G)\) is a subgroup of \(\text{Sym}(G)\),
so we can also regard conjugation as a homomorphism.
\(C: G \rightarrow \text{Sym}(G)\)
This is called the conjugation action.
\(G \times G \rightarrow G\)
\((g, h) \mapsto ghg^{-1}\)
What about right multiplication? \( R_g : G \rightarrow G \quad R_g(h) = hg \)

This does define a function \( G \rightarrow \text{Sym}(G) \)

\[
g \mapsto R_g
\]

But unless \( G \) is abelian, this function is NOT a HOMOMORPHISM.

For \( R_{gh}(x) = xgh = R_hR_g(x) \), so \( R_{gh} = R_h \circ R_g \), and \( R_{gh} \neq R_g \circ R_h \) unless \( gh = hg \).

On the other hand, \( \alpha : G \rightarrow \text{Sym}(G) \)

\[
\alpha_g(x) = xg^{-1}
\]
\[
\alpha_g = R_{g^{-1}}
\]

is a homomorphism!

\[
\alpha_{gh}(x) = x(gh)^{-1} = xh^{-1}g^{-1} = R_g(R_h(x)) = \alpha_g(\alpha_h(x))
\]

So "right multiplication by the inverse" is a group action of \( G \) on \( G \).

Some basic sets associated to a group action.

**Definition:** Let \( G \times X \rightarrow X \) be a group action.

1. For \( x \in X \), the set \( G \cdot x = \{ g \cdot x \mid g \in G \} \) is called the **orbit** of \( x \) (the book uses \( \Theta(x) \) for this.)

2. The action is **transitive** if there is an \( x \in X \) such that \( G \cdot x = X \).

3. For \( x \in X \), the **stabilizer** of \( x \) is

\[
\text{Stab}(x) = \{ g \in G \mid g \cdot x = x \}
\]

it is a subset of \( G \).

4. The **kernel** of the action is \( \ker(\alpha : G \rightarrow \text{Sym}(X)) \)

\[
\text{it equals } \bigcap_{x \in X} \text{Stab}(x)
\]