Lecture 19  More Theorems about quotient groups.

Main

**Theorem:** If \( \varphi: G \to \overline{G} \) is a surjective homomorphism with kernel \( N \), then \( \tilde{\varphi}: G/N \to \overline{G} \) with \( \tilde{\varphi}(aN) = \varphi(a) \) is an isomorphism.

There is a correspondence of subgroups of \( G/N \) and \( \overline{G} \), which amounts to

**Prop. 27.1:** Let \( \varphi: G \to \overline{G} \) be a surjective homomorphism, with kernel \( N \).

(a) There is a bijective correspondence

\[
\{ \text{subgroups of } \overline{G} \} \leftrightarrow \{ \text{subgroups of } G \text{ containing } N \}
\]

given by

\[
B \leftrightarrow \varphi^{-1}(B) = \{ g \in G \mid \varphi(g) \in B \}
\]

(b) This bijection preserves the property of being normal.

i.e., \( B \) is normal in \( \overline{G} \) \( \iff \) \( \varphi^{-1}(B) \) is normal in \( G \).

**Proof:** Let \( B \leq \overline{G} \). Then \( \varphi^{-1}(B) \leq G \).

Since \( e \in B \), \( \varphi^{-1}(e) = \ker \varphi = N \) is contained in \( \varphi^{-1}(B) \).

So \( \varphi^{-1}(B) \) is indeed a subgroup containing \( N \).

Conversely, if \( A \leq G \) is a subgroup containing \( N \) (\( N \leq A \)) then \( \varphi(A) \) is a subgroup of \( \overline{G} \).

So

\[
\{ \text{subgroups of } \overline{G} \} \leftrightarrow \{ \text{subgroups of } G \text{ containing } N \}
\]

are two maps. We must check they are inverses.

**Claim:** \( \varphi(\varphi^{-1}(B)) = B \)

**Proof:** \( \varphi(\varphi^{-1}(B)) = \{ \varphi(a) \mid a \in \varphi^{-1}(B) \} \)

\[= \{ \varphi(a) \mid \varphi(a)eB \} = B. \]
Claim: $\varphi^{-1}(\varphi(A)) = A$, provided $N \leq A$.

Let $x \in \varphi^{-1}(\varphi(A))$ then $\varphi(x) \in \varphi(A)$

so there is $a \in A$ such that $\varphi(x) = \varphi(a)$

thus $\varphi(a^{-1}x) = e$, $\varphi(a^{-1}x)e = e$, so $a^{-1}x \in N \leq A$

so $a^{-1}x = a'$ for some $a' \in A$. Then $x = aa' \in A$

this shows $\varphi^{-1}(\varphi(A)) = A$

Conversely, if $a \in A$, then $\varphi(a) \in \varphi(A)$, so $a \in \varphi^{-1}(\varphi(A))$

thus $A \leq \varphi^{-1}(\varphi(A))$ as well, proving the claim.

This completes the proof of (a).

For (b), let $K$ be a normal subgroup of $G$ containing $N$ ($N \leq K \triangleleft G$)

want to show $\varphi(K) \triangleleft \overline{G}$. Let $g \in \overline{G}$ be any element,
and let $\varphi(k) \in \varphi(K)$. Need $g\varphi(k)g^{-1} \in \varphi(K)$

Since $\varphi$ is surjective, $g = \varphi(g')$ for some $g' \in G$

so $g\varphi(k)g^{-1} = \varphi(g')\varphi(k)\varphi(g'^{-1}) = \varphi(g'kg'^{-1})$

and $g'kg'^{-1} \in K$ since $K$ is normal, so $\varphi(g'kg'^{-1}) \in \varphi(K)$, as was to be shown.

Similarly let $\overline{K} \triangleleft \overline{G}$ be a normal subgroup.

Let $a \in \varphi^{-1}(K)$ and $g \in G$. Need $gag^{-1} \in \varphi^{-1}(K)$

$\varphi(gag^{-1}) = \varphi(g)\varphi(a)\varphi(g'^{-1})$, and this is in $\overline{K}$ since $\varphi(a) \in \overline{K}$

and $\overline{K}$ is normal. So $\varphi(gag^{-1}) \in \overline{K}$ and $gag^{-1} \in \varphi^{-1}(K)$

Example: What are the subgroups of $\mathbb{Z}_n$?

$\varphi: \mathbb{Z} \to \mathbb{Z}_n$ is a surjective homomorphism with kernel $\langle n \rangle$

$\varphi(x) = [x]_n$

$\{\text{subgroups of } \mathbb{Z} \text{ containing } \langle n \rangle \}$

$\{\langle d \rangle \mid d \text{ divides } n \}$

so $\{\text{subgroups of } \mathbb{Z}_n \} = \{\langle [d] \rangle \mid d \text{ divides } n \}$
Proposition 2.7.14: Let $\varphi : G \to \overline{G}$ be a surjective homomorphism with kernel $N$. Let $K \triangleleft \overline{G}$ be normal and let $K = \varphi^{-1}((R))$ then $G/K \cong \overline{G}/K$.

Since $\overline{G} \cong G/N$, and $\overline{K} \cong K/N$, we can also write this as $G/K \cong (G/N)/(K/N)$.

Proof: Define a homomorphism $\Psi: G \to \overline{G}/\overline{K}$ as $\Psi = \pi \circ \varphi$ where $G \xrightarrow{\varphi} G \xrightarrow{\pi} \overline{G}/\overline{K}$.

Then $\Psi$ is surjective since both $\varphi$ and $\pi$ are surjective. Now $\ker(\Psi) = \{x \in G \mid \Psi(x) = e\} = \{x \in G \mid \pi(\varphi(x)) = \overline{K}\}$

$= \{x \in G \mid \varphi(x) \in K\} = \varphi^{-1}(K) = K$

So by the main theorem, there is an isomorphism $\tilde{\Psi}: G/K \to \overline{G}/\overline{K}$

$\tilde{\Psi}(x) = \Psi(x)$

Proposition 2.7.15: Let $N \triangleleft G$ and $\varphi : G \to \overline{G}$ a homomorphism with kernel $K$. If $N \triangleleft K$, there is a homomorphism $\tilde{\varphi} : G/N \to \overline{G}$ such that $\tilde{\varphi} \circ \pi = \varphi$

Try to prove this yourself, or see textbook. See also Cor. 2.7.16.

Next problem: If $A \trianglelefteq G$, $B \trianglelefteq G$, is $AB = \{ab \mid a \in A, b \in B\}$ a subgroup of $G$? Not necessarily.
Ex: \( G = S_4 \) \( A = \langle (12) \rangle = \{ e, (12) \} \)
\( B = \langle (234) \rangle = \{ e, (234), (243) \} \)

\( AB = \{ e, (234), (243), (12), (1234), (1423) \} \)

Not a subgroup since \((234)(12) = (1342)\) is not in \( AB \)

But if \( N \triangleleft G \) is normal, and \( A \trianglelefteq G \), then \( AN \trianglelefteq G \):

Take \( a_1 n_1, a_2 n_2 \in AN \). Then
\[ a_1 n_1 a_2 n_2 = \underbrace{a_1 a_2}_{CA} (a_2^{-1} n_1 a_2) n_2 \in AN \]
\[ \underbrace{eA}_{EN} \underbrace{eN}_{EN} \]
Since \( N \) is normal

If \( an \in AN \) then \((an)^{-1} = n^{-1} a^{-1} = a^{-1} (an^{-1} a^{-1}) \in AN \)
\[ \underbrace{eA}_{EN} \underbrace{eN}_{EN} \]

Proposition 2.7.19 (Diamond isomorphism theorem)

Let \( N \triangleleft G \), \( A \trianglelefteq G \). Then \( AN \trianglelefteq A \) and \( N \trianglelefteq AN \)
and \( A/AN \cong AN/N \)

Proof: \( n \in AN \) and \( a \in A \)

Then \( an^{-1} a^{-1} \in AN \) since \( A \) is a subgroup

and \( an^{-1} eN \) since \( N \) is normal. So \( An \trianglelefteq A \)

If \( n \in N \) and \( g^{-1} \in N \) since \( N \triangleleft G \).

Since \( N \triangleleft AN \) there is a surjective homomorphism
\[ \pi : AN \rightarrow AN/N. \]

There is also a homomorphism \( \iota : A \rightarrow AN \), \( \iota(a) = a \).
Then \( \varphi : A \rightarrow AN/N \)
\[ \varphi = \pi \circ \iota \]
is a homomorphism.
It is surjective \((an)N = aN = \varphi(a) \).
For any \( an \in AN/N \),
The kernel of \( \varphi \) is \( \{ a \in A \mid an = N \} = aN \). \( \{ a \in A \mid a \in N \} = AN/N \)
So by the main theorem there is an isomorphism \( \varphi : A/AN \rightarrow AN/N. \)