1. Goodman 5.2.1. We describe our necklaces by a collection of beads of three possible colors sitting at the vertices of a regular hexagon in the plane. Since each vertex may have one of 3 possible colors, there are $3^6$ possible colorings of the vertices of the hexagon. The group $D_6$ acts on the set of colorings, and the essentially distinct necklaces correspond to the orbits of the action. We denote by $r = r_{2\pi/6}$ the smallest counterclockwise rotation of the hexagon, and by $j$ the flip about the $x$-axis. We number the vertices of the hexagon counterclockwise starting at 1.

$|\text{Fix}(e)| = 3^6$ because the identity fixes everything.

$|\text{Fix}(r)| = 3$: If a coloring is fixed under $r$, the color at vertex 1 must be the same as the color at vertex 2, which must be the same as the color at vertex 3, and so on, so the only choice is the color at vertex 1.

$|\text{Fix}(r^2)| = 3^2$: If a coloring is fixed under $r^2$, the color at vertex 1 must be the same as the color at vertex 3, which must be the same as the color at vertex 5. Then we can pick possibly different color at vertex 2, and we must have the same color at vertices 4 and 6. So the only choice is the colors at vertices 1 and 2.

$|\text{Fix}(r^3)| = 3^3$: If a coloring is fixed under $r^3$, the colors at vertices 1 and 4 must be the same, the colors at 2 and 5 must be the same, and the colors at 3 and 6 must be the same. So the only choice is the colors of the vertices at 1, 2, and 3.

$|\text{Fix}(r^4)| = 3^2$: Since $r^4 = r^{-2}$, this is the same as the case of $r^2$.

$|\text{Fix}(r^5)| = 3$: Since $r^5 = r^{-1}$, this is the same as the case of $r$.

Now there are two more cases:

Flips whose axis passes through two vertices: there are 3 of these ($j, r^2j, r^4j$), and the have $3^4$ fixed points: This is because the two vertices on the axis can have any colors, while among the 4 vertices not on the axis, there are two pairs that must have the same color.

Flips whose axis passes through two edges: there are 3 of these ($rj, r^3j, r^5j$), and they have $3^3$ fixed points: This is because the six vertices are broken into 3 pairs that must have the same color.

The orbit stabilizer theorem gives the number $N$ of orbits, hence of distinct necklaces as:

$$N = \frac{1}{12} (3^6 + 3^2 + 3^3 + 3^2 + 3 + 3\cdot 3^4 + 3\cdot 3^3) = 92$$

2. Goodman 5.2.2. We use the notation from the previous problem. Here the total number of colorings is $6!/(2!)^3 = 90$ (number of ways to partition a set of size six into three subsets of size 2).

$|\text{Fix}(e)| = 90$.

$|\text{Fix}(r)| = 0$, because the fixed points of $r$ have all beads of the same color, which is impossible if we are required to use 2 beads of each color.

$|\text{Fix}(r^2)| = 0$, because the fixed points of $r^2$ always have at least 3 beads of the same color.
4. Goodman 5.4.5. First we recall that the group \( D_n \) is not abelian for \( n > 2 \) (since \( ry = yr \)). Thus none of the groups \( D_{15}, Z_3 \times D_5, Z_5 \times D_3 \) are abelian. But \( Z_{30} \) is abelian. Therefore \( Z_{30} \) is not isomorphic to any of the other groups on the list.

To distinguish between \( D_{15}, Z_3 \times D_5, \) and \( Z_5 \times D_3, \) we count elements of order 2. Recall that in \( D_n, \) the rotations have order dividing \( n, \) while the flips all have order 2. Thus in \( D_{15} \) the rotations do not have order 2, and there are 15 flips of order 2. So \( D_{15} \) has 15 elements of order 2.

In \( Z_3 \times D_5, \) an element of order 2 must have first coordinate equal to \( [0] \in Z_3, \) so the elements of order 2 in this group correspond to the elements of order 2 in \( D_5, \) of which there are 5.

In \( Z_5 \times D_3, \) an element of order 2 must have first coordinate equal to \( [0] \in Z_5, \) so the elements of order 2 in this group correspond to the elements of order 2 in \( D_3, \) of which there are 3.

Thus, the groups \( D_{15}, Z_3 \times D_5, \) and \( Z_5 \times D_3 \) have different numbers of elements of order 2, and are pairwise non-isomorphic.

5. Goodman 5.4.11. Since \( |D_n| = 2n, \) we see immediately that \( D_{14} \) and \( D_7 \times Z_2 \) both have order 28. Since \( 28 = 2^2 \cdot 7, \) a 2-Sylow subgroup has \( 2^2 = 4 \) elements. Let us denote by \( r = r_{2\pi/14} \) the smallest counter-clockwise rotation of the 14-gon, and by \( \rho = r_{2\pi/7} \) the smallest counter-clockwise rotation.
rotation of the 7-gon. (As rotations of the plane, \( \rho = r^{2} \)). We denote by \( j \) the flip over the x-axis. Then
\[
D_{14} = \{ e, r, r^{2}, \ldots, r^{13}, j, r j, r^{2} j, \ldots, r^{13} j \}
\]
\[
D_{7} = \{ e, \rho, \rho^{2}, \ldots, \rho^{6}, j, \rho j, \rho^{2} j, \ldots, \rho^{6} j \}
\]
In \( D_{14} \), we have a subgroup \( P = \{ e, r^{7}, j, r^{7} j \} \): we compute the multiplication table:

\[
\begin{array}{c|ccc}
  & e & r^{7} & j & r^{7} j \\
\hline
  e & e & r^{7} & j & r^{7} j \\
  r^{7} & r^{7} & e & r^{7} j & j \\
  j & j & j r^{7} = r^{-7} j = r^{7} j & e & j r^{7} j = r^{-7} j^{2} = r^{7} \\
  r^{7} j & r^{7} j & r^{7} j r^{7} = r^{7} r^{-7} j = j & r^{7} j^{2} = r^{7} & e
\end{array}
\]

This table shows that \( P \) is indeed a subgroup, and by comparing the table to that of \( \mathbb{Z}_{2} \times \mathbb{Z}_{2} \), we see that \( P \cong \mathbb{Z}_{2} \times \mathbb{Z}_{2} \). (An explicit isomorphism is \( \phi : e \mapsto (0, 0), r^{7} \mapsto (0, 1), j \mapsto (1, 0), r^{7} j \mapsto (1, 1) \).) Since \( P \) has order 4, \( P \) is a 2-Sylow subgroup of \( D_{14} \).

In \( D_{7} \), we have a subgroup \( H = \{ e, j \} \), since \( j^{2} = e \). It is clear that \( H \cong \mathbb{Z}_{2} \). Then \( H \times \mathbb{Z}_{2} \leq D_{7} \times \mathbb{Z}_{2} \) is a subgroup, and it is isomorphic to \( \mathbb{Z}_{2} \times \mathbb{Z}_{2} \). Since it has order 4, \( H \times \mathbb{Z}_{2} \) is a 2-Sylow subgroup of \( D_{7} \times \mathbb{Z}_{2} \).

We can define a function \( \psi : D_{14} \to D_{7} \times \mathbb{Z}_{2} \) by the formula
\[
\psi(r^{k} j^{a}) = (\rho^{k} j^{a}, [k]) \in D_{7} \times \mathbb{Z}_{2}.
\]
where \( k \in \{0, \ldots, 13\} \) and \( a \in \{0, 1\} \). We check that \( \psi \) is a homomorphism; let’s break this into 4 cases depending on whether the power of \( j \) is 0 or 1.

\[
\psi(r^{k} r^{\ell}) = \psi(r^{k + \ell})
\]
\[
= (\rho^{k + \ell}, [k + \ell])
\]
\[
= (\rho^{k}, [k])(\rho^{\ell}, [\ell])
\]
\[
= \psi(r^{k})\psi(r^{\ell})
\]

\[
\psi(r^{k}(r^{\ell} j)) = \psi(r^{k + \ell} j)
\]
\[
= (\rho^{k + \ell} j, [k + \ell])
\]
\[
= (\rho^{k}, [k])(\rho^{\ell} j, [\ell])
\]
\[
= \psi(r^{k})\psi(r^{\ell} j)
\]

\[
\psi((r^{k} j)r^{\ell}) = \psi(r^{k - \ell} j)
\]
\[
= (\rho^{k - \ell} j, [k - \ell])
\]
\[
= (\rho^{k} j \rho^{\ell}, [k + [\ell]])
\]
\[
= (\rho^{k}, [k])(\rho^{\ell}, [\ell])
\]
\[
= \psi(r^{k} j)\psi(r^{\ell})
\]
where we have used the fact that $-\ell = \ell \mod 2$. And similarly

$$ \psi((r^k j)(r^\ell j)) = \psi(r^{k-\ell}) $$

$$ = (\rho^{k-\ell}, [k-\ell]) $$

$$ = (\rho^{k} j \rho^{\ell} j, [k] + [\ell]) $$

$$ = (\rho^{k} j, [k]) (\rho^{\ell} j, [\ell]) $$

$$ = \psi(r^{k}) \psi(r^{\ell}) $$

Now that we know $\psi$ is a homomorphism, we compute its kernel. So take $r^k j^a \in D_{14}$ with $k \in \{0,1,\ldots,13\}$ and $a \in \{0,1\}$. Suppose that $\psi(r^k j^a) = (\rho^{k} j^a, [k]) = (e, [0])$. Then $[k] = [0]$ in $\mathbb{Z}_2$, so $k$ is even. Then $\rho^{k} j^a = e$ implies that $a = 0$ and $k$ is divisible by 7. Since $k$ is divisible by both 2 and 7, the Chinese remainder theorem implies that $k$ is divisible by 14. Since $0 \leq k \leq 13$, we get $k = 0$. Thus the kernel of $\psi$ is trivial.

Since $\psi : D_{14} \rightarrow D_7 \times \mathbb{Z}_2$ is a homomorphism with trivial kernel, it is injective. Since both the domain and the range have the same number of elements (namely, 28 elements), $\psi$ is surjective as well. Thus $\psi$ is an isomorphism.

6. Goodman 5.4.18. Let $G = S_4$, the group of permutations of 4 elements. Then $|S_4| = 4! = 4 \cdot 3 \cdot 2 = 2^3 \cdot 3$, and the largest prime dividing the order of $S_4$ is 3. A 3-Sylow subgroup has 3 elements in this case, and an example is given by the cyclic subgroup generated by a 3-cycle, for instance

$$ P = \langle (123) \rangle = \{e, (123), (132)\} $$

However, this subgroup is not normal. In fact, if we let $a = (34)$, we find

$$ aPa^{-1} = (34)P(34) = \{e, (124), (142)\} \neq P $$