Math 417: Homework 6 Solutions

Due Friday, March 9, 2018

1. Goodman 2.7.2. (a) Let $A$, $B$, $C$ be subsets of $G$, and consider $(AB)C$ and $A(BC)$. A general element $x$ of $(AB)C$ is $x = (ab)c$ for some $a \in A, b \in B, c \in C$. By associativity, $x = a(bc)$, so $x$ is an element of $A(BC)$ as well. This logic can be reversed, so $(AB)C = A(BC)$.

(b) Take $N < G$ a normal subgroup. Let $A = aN, B = bN$, and consider the sets $AB = (aN)(bN)$ and $abN$. A general element $x \in abN$ is $x = abn$ for some $n \in N$. Since $a \in aN$ and $bn \in bN$, we have $x = (a)(bn) \in (aN)(bN)$. Thus $abN \subseteq (aN)(bN)$. On the other hand, a general element $x \in (aN)(bN)$ is of the form $x = an_1bn_2$ for some $n_1, n_2 \in N$. We can then write

$$x = an_1bn_2 = abb^{-1}n_1bn_2 = ab(b^{-1}n_1b)n_2$$

Since $N$ is normal, $b^{-1}n_1b \in N$, and so $(b^{-1}n_1b)n_2 \in N$, and $x \in abN$. Thus $(aN)(bN) \subseteq abN$, and $(aN)(bN) = abN$.

(c) The set $G/N$ is the set of cosets of $N$ in $G$, and the equation $(aN)(bN) = abN$ shows that the product of two cosets is indeed another coset. Thus $G/N$ is closed under the operation of “product of sets”. It is associative by part (a).

2. Goodman 2.7.3. (a) We compute

$$T_{A,b} \circ T_{A^{-1},-A^{-1}b}(x) = A(A^{-1}x - A^{-1}b) + b = AA^{-1}x - AA^{-1}b + b = x - b + b = x$$

$$T_{A^{-1},-A^{-1}b} \circ T_{A,b}(x) = A^{-1}(Ax + b) - A^{-1}b = A^{-1}Ax + A^{-1}b - A^{-1}b = x$$

(b) To see that $N = \{T_{E,b} \mid b \in \mathbb{R}^n\}$ is a subgroup, we simply note

$$T_{E,b} \circ T_{E,b'} = T_{E,b+b'}$$

$$T_{E,-b} = T_{E,-b}$$

so $N$ is closed under composition and inverses.

To see that $N$ is normal, we check

$$T_{A,b}T_{E,c}T_{A,b}^{-1}(x) = T_{A,b}T_{E,c}T_{A^{-1},-A^{-1}b}(x) = A(E(A^{-1}x - A^{-1}b) + c) + b$$

$$= AA^{-1}x - AA^{-1}b + Ac + b = x - b + Ac + b = x + Ac = T_{E,Ac}(x)$$

So $T_{A,b}T_{E,c}T_{A,b}^{-1} = T_{E,Ac} \in N$.

3. Goodman 2.7.7. Recall that $D_4$ has eight elements

$$D_4 = \{e, r_{\pi/2}, r_{\pi}, r_{3\pi/2}, j_0, j_{\pi/4}, j_{\pi/2}, j_{3\pi/4}\}$$

The subgroup $N$ of rotations has four elements

$$N = \{e, r_{\pi/2}, r_{\pi}, r_{3\pi/2}\}$$

To see that $N$ is a normal subgroup, note that for any $\theta$ and $\phi$,

$$r_{\theta}r_{\phi}r_{\theta}^{-1} = r_{\theta}r_{\phi}r_{-\theta} = r_{\theta + \phi - \theta} = r_{\phi}$$
5. Goodman 2.7.10. Let $G$ be an abelian group, and let $N \triangleleft G$ be a normal subgroup. Let $aN$ and $bN$ be cosets of $N$. Then

$$(aN)(bN) = abN = baN = (bN)(aN)$$

where we have used the fact that $ab = ba$ because $G$ is abelian. This shows that $G/N$ is abelian as well.
6. Goodman 2.7.11. Suppose that $G/Z(G)$ is a cyclic group, and suppose it is generated by the coset of $a$, so

$$G/Z(G) = \langle aZ(G) \rangle$$

This means that every coset of $Z(G)$ in $G$ equals $a^n Z(G)$ for some $n \in \mathbb{Z}$, and

$$G = \bigcup_{n \in \mathbb{Z}} a^n Z(G)$$

Now let $x$ and $y$ be two elements of $G$. We may write $x = a^k z$ for some $k \in \mathbb{Z}$ and $z \in Z(G)$, and we may write $y = a^\ell w$ for some $\ell \in \mathbb{Z}$ and $w \in Z(G)$. Then

$$xy = a^k z a^\ell w = a^k a^\ell zw = a^{k+\ell} zw$$

where we have used the fact $za^\ell = a^\ell z$ because $z$ is in the center $Z(G)$: $z$ commutes with any other element. Since $w \in Z(G)$, we also have

$$yx = a^\ell w a^k z = a^\ell a^k wz = a^{k+\ell} wz = a^{k+\ell} zw$$

So $xy = yx$, and we have shown that $G$ is abelian.

[Remark: This problem actually implies that, if $G/Z(G)$ is cyclic, then $G/Z(G)$ is trivial. For if $G/Z(G)$ is cyclic, we have just shown that $G$ is abelian. But then $Z(G) = G$, and so $G/Z(G)$ is trivial. Another way to phrase this is that $G/Z(G)$ can never be a non-trivial cyclic group.]