Math 417: Ring theory practice problems

Note: These practice problems only cover ring theory, corresponding to lectures 32-39. The final exam covers everything in the course, so you should also review the lectures, homeworks, previous exams, and previous practice problems for the other material.

1. Consider the ring \( \mathbb{Q}[x]/(f) \), where \( f = x^2 + x + 1 \). Compute the product of two elements of the form \( a + bx + (f) \), and write the result in the same form.

2. Show that the ring \( \mathbb{Q}[x]/(x^2 + x + 1) \) in the previous problem is a field.

3. Consider \( S = \mathbb{R} \times \mathbb{R} \) with component-wise operations: \( (a, b) + (a', b') = (a + a', b + b') \) and \( (a, b)(a', b') = (aa', bb') \). You may take for granted that this is a ring. Answer the following questions.
   - Does \( S \) have a multiplicative identity?
   - Is \( S \) an integral domain? Is \( S \) a field?
   - How many elements \( x \in S \) are there that satisfy the equation \( x^2 = x \)?
   - Find an ideal \( J \subset S \) such that \( S/J \cong \mathbb{R} \).

4. Find the greatest common divisor of the polynomials \( f = 2 - 3x + x^2 \) and \( g = -1 + x - x^2 + x^3 \) (working in \( \mathbb{R}[x] \)).


6. Consider the ring \( \mathbb{Z}_2[x] \). Show that the polynomial \( x^2 + x + [1] \) is irreducible in this ring.

7. In \( \mathbb{C}[x] \), find the factorization of \( x^3 - 1 \) into irreducible factors. Hint: to get started, notice that 1 is a root of this polynomial.

8. There is a unique homomorphism \( \varphi_{3i} : \mathbb{Q}[x] \to \mathbb{C} \) such that \( \varphi_{3i}(x) = 3i \) and \( \varphi_{3i}(q) = q \) for any \( q \in \mathbb{Q} \). Compute \( \varphi_{3i}(x^3 + x^2 + 1) \). Extra credit: Is \( \varphi_{3i} \) surjective?

9. In \( \mathbb{Q}[x] \), consider the subset \( \mathbb{Z}[x] \) consisting of all polynomials whose coefficients are integers. Is \( \mathbb{Z}[x] \) a subring of \( \mathbb{Q}[x] \)? Is \( \mathbb{Z}[x] \) an ideal in \( \mathbb{Q}[x] \)?

10. Let \( R \) be a ring. An element \( x \in R \) is called nilpotent if \( x^k = 0 \) for some integer \( k > 0 \). Suppose that \( x \in R \) is nilpotent, and \( P \subset R \) is a prime ideal. Show that \( x \in P \).

11. Consider the ring \( \mathbb{R}[x, y] \) of polynomials in two variables. Elements of \( \mathbb{R}[x, y] \) are expressions of the form
   
   \[
   f = \sum_{i=0}^{n} \sum_{j=0}^{m} a_{ij} x^i y^j = a_{00} + a_{10}x + a_{01}y + a_{20}x^2 + a_{11}xy + a_{02}y^2 + \cdots,
   \]

   with \( a_{ij} \in \mathbb{R} \); multiplication and addition work as in ordinary algebra. Consider the set
   
   \[ M = \{sx + ty \mid s, t \in \mathbb{R}[x, y] \} \]

   Show that \( M \) is a maximal ideal in \( \mathbb{R}[x, y] \).

12. In the ring of \( R = \text{Mat}_2(\mathbb{R}) \) of \( 2 \times 2 \) matrices with real entries, consider the element \( a = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \).

   Suppose that \( I \subset R \) is an ideal that contains \( a \). Show that \( I = R \). Note: Since \( R \) is noncommutative, the requirement that \( I \subset R \) is an ideal means that both \( ra \) and \( ar \) are in \( I \) for every \( r \in R \) (and that \( I \) is a subgroup under addition).