Math 417: Midterm 3  
Name: Solutions

Wednesday, April 18, 2018

**Instructions:** Write your name at the top of this page. There are 34 points possible on this exam (plus a 5-point extra credit problem). Take note that the problems are not weighted equally. No books, notes, calculators, or other aids are permitted. The last page is for work that does not fit on other pages.

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1. (5 points) Consider the group $\mathbb{Z} \times \mathbb{Z}$ (the group operation on $\mathbb{Z}$ is addition). There is a unique homomorphism $\alpha : \mathbb{Z}_2 \rightarrow \text{Aut}(\mathbb{Z} \times \mathbb{Z})$, such that $\alpha_{[1]}(a, b) = (b, a)$. Form the semidirect product $G = (\mathbb{Z} \times \mathbb{Z}) \rtimes_{\alpha} \mathbb{Z}_2$. Is $G$ abelian? Prove your answer.

$G$ is not abelian: Counterexample

$g = ((0,0), [1])$ \hspace{1cm} $h = ((1,2), [0])$

$gh = ((0,0), [1])(1,2), [0])$

$= ((0,0) + \alpha_{[1]}(1,2), [1] + [0])$

$= ((0,0) + (2,1), [1]) = (2,1), [1])$

$hg = ((1,2), [0])(0,0), [1])$

$= ((1,2) + \alpha_{[0]}(0,0), [0] + [1])$

$= ((1,2) + (0,0), [1]) = (1,2), [1])$

so $hg \neq gh$

2. (4 points)

(a) If a group $G$ acts on a set $X$, and $y = g \cdot x$, then $G \cdot y = G \cdot x$:  

True \hspace{1cm} False

(b) If $G$ is a group, the action of $G$ on $G$ by left multiplication is transitive:

True \hspace{1cm} False

(c) If $G$ is a group, the action of $G$ on $G$ by conjugation is transitive:

True \hspace{1cm} False

(d) If $G$ is a finite group acting on a set $X$, and $\mathcal{O}$ is an orbit, then $|\mathcal{O}|$ divides $|G|$:  

True \hspace{1cm} False
3. Consider the symmetric group $S_n$ acting on the set $X = \{1, 2, \ldots, n\}$.

(a) (5 points) Let $\text{Stab}(i)$ be the stabilizer of $i \in X$. Let $\pi \in S_n$ be any permutation. Show that $\pi \text{Stab}(i)\pi^{-1} = \text{Stab}(\pi(i))$.

If $g \in \text{Stab}(i)$, then $g(i) = i$ so $(\pi g\pi^{-1})(\pi(i)) = \pi g\pi^{-1}(i) = \pi g(i) = \pi(i)$
so $\pi g\pi^{-1} \in \text{Stab}(\pi(i))$. Thus $\pi \text{Stab}(i)\pi^{-1} \subseteq \text{Stab}(\pi(i))$
if $h \in \text{Stab}(\pi(i))$, $h(\pi(i)) = \pi(i)$. Set $g = \pi^{-1}h\pi$
Then $g(i) = \pi^{-1}h\pi(i) = \pi^{-1}\pi(i) = i$ so $g \in \text{Stab}(i)$
and $h = \pi g\pi^{-1} \in \pi \text{Stab}(i)\pi^{-1}$. So $\text{Stab}(\pi(i)) \subseteq \pi \text{Stab}(i)\pi^{-1}$
and they are equal.

(b) (4 points) Let $H = \text{Stab}(n)$. Show that $N_{S_n}(H) = H$, where $N_{S_n}(H)$ denotes the normalizer of $H$ in $S_n$.

We always have $H \subseteq N_{S_n}(H)$. Let $\pi \in N_{S_n}(H)$
then $\text{Stab}(\pi(n)) = H = \pi H\pi^{-1} = \pi \text{Stab}(\pi(n))\pi^{-1} = \text{Stab}(\pi(n))$
by part (a). So $\text{Stab}(\pi(n)) = \text{Stab}(\pi(n))$
But this implies $\pi(n) = n$: if $\pi(n) \neq n$, pick some $a \neq n, \pi(n)$.
Then the 2-cycle $(a \pi(n))$ fixes $n$ but not $\pi(n)$,
so $\text{Stab}(\pi(n)) \neq \text{Stab}(\pi(n))$.
Since $\pi(n) = n$, $\pi \in \text{Stab}(\pi(n)) = H$.
Thus $N_{S_n}(H) \subseteq H$, and we are done.
4. (7 points) Please read this problem carefully as it is similar to the “necklace” problems but slightly different:

Consider a regular hexagon in the $xy$-plane. At each of the six vertices of the hexagon, we place an electron. The electron is allowed to have two states: spin up (U) or spin down (D). (You can imagine that each particle has a little arrow that points in either the positive or negative $z$-direction.) A possible configuration looks like:

Let $X$ be the set of all possible configurations of electrons. The dihedral group $D_6$ acts on $X$, by permuting the electrons at the vertices of the hexagon, but the flip symmetries also have the effect of reversing the direction of the spin; for instance, the action of the flip about the $x$-axis on the configuration above is:

Find the number of orbits of $D_6$ acting on $X$.

Thus are $2^6$ configurations
e fixes everyth : $|\text{Fix}(e)| = 2^6 = 64$
$r = \frac{2\pi}{3}$ fixes 2 configs, all up/all down
$|\text{Fix}(r)| = 2$
$r^2$ fixes $2^2$ configs
2 sets of 3 that must be same
$|\text{Fix}(r^2)| = 2^2 = 4$
Use this space for work.

\[ r^3 \text{ fixes } 2^3 \text{ configs} \]
\[ 3 \text{ sets of 2 that must be same} \]
\[ |\text{Fix}(r^3)| = 2^3 = 8 \]

\[ r^4, \text{ same as } r^2 \quad \text{Fix}(r^4) = 2^2 = 4 \]
\[ r^5, \text{ same as } r \quad \text{Fix}(r^5) = 2 \]

Flips whose axis passes through 2 vertices fix nothing. (Spin at vertex a axis always gets flipped.)

Flips whose axis passes through 2 edges fix \( 2^3 = 8 \)

3 pairs that must be opposite.

There are 3 such flips.

So \( \# \text{orbits} = \frac{1}{12} (64 + 2 + 4 + 8 + 4 + 2 + 3 \cdot 8) \)

\[ = \frac{1}{12} (108) = 9 \]
5. Assume that $n$ is odd, and consider the dihedral group $D_n$.

(a) (5 points) Determine, with proof, the number of 2-Sylow subgroups of $D_n$.

| $|D_n| = 2n$. Since $n$ is odd, the highest power of 2 dividing $|D_n|$ is 2. So 2-Sylow subgroups have order 2. A group of order 2 is always of the form $\langle e, g \rangle$ where $g$ has order 2.

In $D_n$, rotations have order dividing $n$, so not 2 since $n$ is odd. All flips have order 2. There are $n$ flips: $\{ r^i, r^i_j, r^i_j... r^{n-1}i \}.$
Each of these flips generates a 2-Sylow subgroup.
So there are $n$ 2-Sylow subgroups in $D_n$.

(b) (4 points) Verify that your answer to the previous part satisfies the conclusions of the third Sylow theorem.

3rd Sylow theorem implies #2-Sylow subgroups $|1D_n/2|

and #2-Sylow subgroups $\equiv 1 \pmod{2}$

Indeed $|D_n/2| = n$ and $n/n$ is true.
Also $n \equiv 1 \pmod{2}$ since $n$ is odd.
6. **(5 points) Extra Credit.** Show that a group of order 40 has exactly 4 elements of order 5.

Let $G$ be a group, $|G| = 40$. $40 = 2^3 \cdot 5$

so 5-Sylow subgroups have order 5.

Let $n_5$ be the number of 5-Sylow subgroups.

3rd Sylow theorem $\Rightarrow n_5 \mid 2^3$ and $n_5 \equiv 1 \pmod{5}$

so $n_5 \in \{1, 2, 4, 8\}$ and $n_5 \equiv 1 \pmod{5}$

$2 \not\equiv 1 \pmod{5}$ $\quad 4 \not\equiv 1 \pmod{5} \quad 8 \not\equiv 1 \pmod{5}$

so we must have $n_5 = 1$. Call the unique 5-Sylow subgroup $H$.

On the other hand, any $g \in G$ of order 5 generates a subgroup $\langle g \rangle$ of order 5. Thus $\langle g \rangle$ is a 5-Sylow subgroup.

So $\langle g \rangle = H$ and $g \in H$. Thus $H$ contains all elements of order 5 that exist in $G$.

$H \cong \mathbb{Z}_5$, and $H$ contains 4 elements of order 5. So $G$ does as well.
Use this page for work that does not fit on other pages.