Math 417: Midterm 3 Practice problems

1. Let $G$ be the set of 3-by-3 matrices with the property that there is exactly one nonzero entry in each row, exactly one nonzero entry in each column, and the nonzero entries are always $+1$ or $-1$. Prove that $G$ is isomorphic to the semidirect product of $S_3$ and $H$, where $H$ is the group of 3-by-3 matrices that are diagonal with $\pm 1$ along the diagonal.

2. Suppose that $G \cong \mathbb{Z}_5 \rtimes \mathbb{Z}_3$ is a semidirect product of $\mathbb{Z}_5$ and $\mathbb{Z}_3$ with respect to a homomorphism $\alpha : \mathbb{Z}_3 \to \text{Aut}(\mathbb{Z}_5)$. Show that $\alpha$ is trivial and that $G \cong \mathbb{Z}_5 \times \mathbb{Z}_3$. Is $G$ a cyclic group?

3. Consider the vector space $\mathbb{R}^n$. Let $G = \mathbb{R}^\times = \mathbb{R} \setminus \{0\}$ be the group of nonzero real numbers with multiplication. Show that the multiplication of vectors by scalars $G \times \mathbb{R}^n \to \mathbb{R}^n$, $(\lambda, v) \mapsto \lambda v$
defines an action of $G$ on $\mathbb{R}^n$.

4. Consider the group $D_4$, the symmetries of a square. Let $V$ be the set of vertices of the square, and let $E$ be the set of edges of the square. Go through each of the 8 elements of $D_4$ and answer the questions: How many elements of $V$ does it fix? How many elements of $E$ does it fix?

5. Let a group $G$ act on a set $X$. Let $Y \subseteq X$ be a subset, and define $G_Y = \{ g \in G | \forall y \in Y, \ g \cdot y = y \}$
to be the set of group elements that fix every element of $Y$. Show that $G_Y$ is a subgroup of $G$.

6. Let a group $G$ act on itself by conjugation. Show from the definitions that the kernel of this action equals the center of $G$.

7. Find the number of orbits in $\{1, 2, 3, 4, 5, 6, 7, 8\}$ under the action of the subgroup of $S_8$ generated by $(13)$ and $(247)$.

8. How many ways are there to divide a set of 10 people into two sets of 5?

9. How many ways are there to seat 7 people around a round table, if we regard two arrangements that differ by a rotation as the same?

10. How many ways are there to color the edges of a square with 4 colors (if we regard colorings that differ by the action of an element of $D_4$ as being the same)?

11. Write out the conjugacy classes in $S_4$. Write out the class equation for $S_4$.

12. Let $G$ be a finite group, and let $p$ be a prime number dividing $|G|$. Let $P$ be a subgroup of $G$ whose order is a power of $p$, and which is normal. Show that any $p$-Sylow subgroup of $G$ must contain $P$.

13. Show that every group of order 45 has a normal subgroup of order 9.