PERTURBATIVE QUANTIZATION AND MASTER EQUATION (AFTER COSTELLO)

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These are notes for talks in the reading group on the "Higher Genus B-Model and Quantization of BCOV Theory" at the Institute for Advanced Study in the Spring term of 2017. The material comes from the book *Renormalization and Effective Field Theory* by Kevin Costello [Cos11]. At some points I have added commentary that helped me to understand the motivation or context. The book of Folland [Fol08], as well as the articles of Huang [Hua13] and in particular Polchinski [Pol84] were helpful in understanding renormalization.

1. Perturbative quantization

1.1. **Action functional.** We begin with a classical field theory: There is a spacetime manifold M, and a space of field configurations

$$\mathcal{B} = \Gamma(M, E) \tag{1}$$

where, as is commonly the case, the fields are sections of a vector bundle $E \to M$. In the classical theory, the physics is described by PDE that we impose on $\phi \in \mathcal{B}$, such as the wave equation or Maxwell's equations. We assume that these PDE are variational, meaning that there is an action functional $S(\phi)$ such that the classical field equations are equivalent to the vanishing of the variational or Euler-Lagrange derivative,

$$\frac{\delta S}{\delta \phi} = 0. \tag{2}$$

This only makes sense if the action S is a *local functional*; this means that S is the integral of a *Lagrangian density*,

$$S(\phi) = \int_M \mathcal{L}(J^{(r)}(\phi)). \tag{3}$$

Here $J^{(r)}(\phi)$ denotes the r-jet of ϕ, and $\mathcal{L}$ is map from r-jets of sections of E to densities on M, with the property that, at each $x \in M$, the value of the integrand at x depends only on the r-jet of ϕ at this same point x. By way of contrast, a nonlocal functional would be something like

$$\int_{\mathbb{R}} \phi(x)\phi(x+1)\, dx, \tag{4}$$

since the integrand at x depends not only on $\phi(x)$ but also the value of ϕ at the distinct point $x + 1$.

1.2. **Perturbation theory.** A way to quantize our classical field theory $(\mathcal{B}, S)$ is as follows. Begin by breaking the classical action S into its quadratic and higher-order parts,

$$S(\phi) = S_{\text{quad}}(\phi) + I(\phi), \tag{5}$$

where $S_{\text{quad}}(\phi)$ is quadratic with respect to $\phi \in \mathcal{B}$, and I is of degree at least 3. (The assumption that there is no constant term is harmless, and the assumption that there is no linear term means that $\phi \equiv 0$ does satisfy the Euler-Lagrange equations, that is, $\phi \equiv 0$ is a critical point of S.)

The functional S_{quad} describes a *free theory*; in this theory the fields evolve by *linear* PDE. The addition of the term I describes the *interaction*, which makes the theory "interesting." One version of perturbation theory involves first quantizing the free theory $(\mathcal{B}, S_{\text{quad}})$, and then deforming in by introducing the "small" interaction I. The quantization of a free theory can be made completely mathematically rigorous using the tools of functional analysis developed in the 20th century; the introduction of the interaction is a different story.

Let us pose the problem this way: we want to compute the partition function of our interacting theory,

$$Z = \int_{\mathcal{B}} \mathcal{D}\phi \, e^{S(\phi)/\hbar}, \tag{6}$$

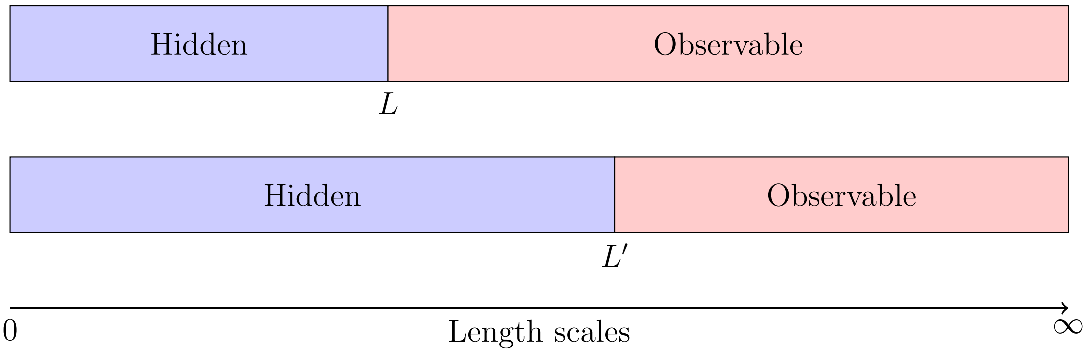
(regarding the phase of the exponent: I am using a Euclidean action with negative-definite quadratic part). This is an infinite-dimesional functional integral, but it will guide us to our answers. The first thing to do is use our decomposition quadratic part and interaction, and then expand the exponential of the interaction,

$$Z = \int_{\mathcal{B}} \mathcal{D}\phi \, e^{S_{\text{quad}}(\phi)/\hbar} e^{I(\phi)/\hbar} = \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\mathcal{B}} \mathcal{D}\phi \, e^{S_{\text{quad}}(\phi)/\hbar} (I(\phi)/\hbar)^n \tag{7}$$

Thus we must compute the integral of $I(\phi)^n$ agains the Gaussian measure $\mathcal{D}\phi \, e^{S_{\text{quad}}(\phi)/\hbar}$. There are combinatorial tricks for doing this, which lead to an expression for this integral in terms of Feynman diagrams. I won't describe these diagrams now because they are a bit different than the class of diagrams we will actually use, and also because this diagrammatic process leads to integrals that are still divergent. This is one way of formulating the problem that the renormalization process was invented to solve.

1.3. **Effective theories.** A modern point of view on renormalization is that we simply cannot expect a quantum theory to be described by a local action functional $S(\phi)$, at least not in so direct a way as suggested by Equation (6). Another slogan one comes upon is that field theory has infinitely degrees of freedom (as opposed to finite-dimensional mechanics), and that the divergences of quantization arise because of the "piling-up" of contributions from all of these degrees of freedom—so, if we can find a way to only consider finitely many degrees of freedom at a time, we can get finite answers. This leads one to consider *effective theories* that describe the physics at certain scales, either at lengths greater than some length threshold, or below some energy threshold.

The theories at different length scales should be related to one another. Consider the following picture:



The horizontal direction represents length scales. The top bar corresponds to the scale L effective theory: the phenomena whose characteristic scale is below L are hidden in the effective theory, while phenomena at scales above L are observable. The bottom bar represents a similar situation, but where we consider the effective theory at some longer scale L'. When $L < L'$, the scale L effective theory should determine the scale L' effective theory: in this passage we must "hide" ("integrate out") all of the phenomena between L and L'. This is called the renormalization group flow. (Although the word "group" is used, my understanding is that it is only a semigroup in general. The passage from L to L' with $L < L'$ involves a loss of information—so it is possible. To reconstruct a scale L theory from the theory at scale L' is not always possible.)

Slogans aside, here is what we will do: fix a free theory $(\mathcal{B}, S_{\text{quad}})$, and consider only interactions that deform this free theory. As an example one could take a single real scalar field on a compact Riemanninan manifold M,

$$\phi \in C^\infty(M) = \mathcal{B}, \tag{8}$$

with quadratic action

$$S_{\text{quad}}(\phi) = -\int_M \phi(\Delta + m^2)\phi \tag{9}$$

where Δ is the nonnegative Laplacian associated to the Riemannian metric.

Let $\mathcal{O}(\mathcal{B}) = \widehat{\text{Sym}}^*(\mathcal{B}^\vee)$ be the completed symmetric algebra of the dual space $\mathcal{B}^\vee$. We regard $\mathcal{O}(\mathcal{B})$ as the space of functionals on $\mathcal{B}$; note that these functionals are allowed to be non-local. We also consider the formal series $\mathcal{O}(\mathcal{B})[[\hbar]]$ and $\mathcal{O}(\mathcal{B})^+[[\hbar]]$, where the superscript $+$ means that we only consider functionals that are at least cubic modulo $\hbar$ (more on this condition later).

We will now define the concept of a *perturbative effective quantum field theory* (or *theory* for short) that is a perturbation of the given free theory $(\mathcal{B}, S_{\text{quad}})$.

1.3.1. *Data.* The data of a theory is a collection

$$I[L] \in \mathcal{O}^+(\mathcal{B})[[\hbar]], \quad 0 < L < \infty \tag{10}$$

of formal series of functionals, at least cubic modulo $\hbar$, indexed by real numbers $L \in (0, \infty)$. The index L is called the *length scale*, and the functional $I[L]$ is called the *scale L effective interaction*.

1.3.2. *Conditions.* The data $I[L]$ are required to satisfy two conditions. First, the functionals $I[L]$ must be related for different values of L by the *renormalization group equation (RGE)*

$$I[L'] = W(P(L, L'), I[L]). \tag{11}$$

Second, the functionals $I[L]$ are required to be *asymptotically local* as $L \to 0$.

Thus, to complete the definition of a perturbative effective quantum field theory, it remains to define the function W, the operator $P(L, L')$, and the notion of asymptotic locality.

1.3.3. *The renormalization group equation.* This is the real core of the definition. The function $W(P, I)$ takes two arguments. The first is an element $P \in \operatorname{Sym}^2 \mathcal{B}$ and is called the *propagator*. The second is a functional $I \in \mathcal{O}(\mathcal{B})[[\hbar]]$. The value of $W(P, I)$ is another such functional, and it is defined as a sum of Feynman graphs. It will be helpful to see the figure at [Cos11, p. 36, fig. 1]. We begin by decomposing

$$I = \sum_{i,k \geq 0} \hbar^i I_{i,k}, \quad I_{i,k} \in \operatorname{Sym}^k(\mathcal{B}^\vee). \tag{12}$$

The index i counts the power of $\hbar$ and is called the *internal loop number*, and the index k counts the homogeneous degree or arity. The graphs we consider have vertices of any valency and any number of loops and external edges. Given a graph, at each vertex of valence k, we choose $I_{i,k}$ for some i at put it at the vertex. At each internal edge, we put P. Call such a decorated graph Γ. By pairing P with the inputs of the vertex functionals $I_{i,k}$, we get a functional whose arity is equal to the number of external edges of the graph; call it $w_\Gamma(P, I)$. A graph so decorated also has a *total loop number*, which is number of loops in the graph plus the sum of the internal loop numbers of the vertices. We then set

$$W(P, I) = \sum_{i,k \geq 0} \hbar^i W_{i,k}(P, I), \tag{13}$$

where $W_{i,k}(P, I)$ is the sum of $w_\Gamma(P, I)/|\operatorname{Aut}(\Gamma)|$ as Γ ranges over all isomorphism classes decorated graphs with total loop number i and k external edges.

1.3.4. *Propagator.* We need to define the specific propagator $P(L, L') \in \operatorname{Sym}^2 \mathcal{B}$ that is used in the renormalization group equation. This is the only point in the definition of a perturbative effective theory where the given free theory S_{quad} enters. We will define $P(L, L')$ for the free scalar field and refer to Costello for the general definition [Cos11, pp. 71–72, Definition 13.1.1]. Recall that in this case we have

$$S_{\text{quad}}(\phi) = -\int_M \phi(\Delta + m^2)\phi, \tag{14}$$

where Δ is the nonnegative Riemannian Laplacian. The propagator is built from the elliptic theory of the operator $\Delta + m^2$. Since M is a compact manifold, we have a collection of eigenvalues

$$m^2 = \lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \tag{15}$$

and an orthonormal basis $\{\phi_n\}_{n \geq 0}$ of $L^2(M)$ consisting of eigenfunctions,

$$(\Delta + m^2)\phi_n = \lambda_n \phi_n. \tag{16}$$

We then let $K_t \in \operatorname{Sym}^2 \mathcal{B} = \operatorname{Sym}^2 C^\infty(M)$ be the heat kernel

$$K_t(x,y) = \sum_{n=0}^{\infty} e^{-\lambda_n t} \phi_n(x) \otimes \phi_n(y). \tag{17}$$

As long as $t > 0$, $K_t(x,y)$ can be interpreted as a smooth function[1] on $M \times M$, while at $t = 0$ (which we exclude) it becomes the delta distribution along the diagonal in $M \times M$. Now given two positive numbers L and L', define

$$P(L,L') = \int_L^{L'} K_t \, dt \in \operatorname{Sym}^2 \mathcal{B}, \tag{18}$$

and this is the propagator we use in the renormalization group equation.

Note that if we set $L = 0$, then $P(0,L')$ is a parametrix for the operator $\Delta + m^2$. Let us emphasize that we are excluding this case: $P(0,L')$ is not a smooth function on $M \times M$, it is a distribution. If we start putting $P(0,L')$ in our Feynman diagrams we will precisely reproduce the UV divergences that frustrated the development of quantum field theory until the Nobel Prize-winning work of Feynman, Schwinger, and Tomonaga.

1.3.5. *Asymptotic locality.* The condition of asymptotic locality means that the effective interactions $I[L]$ have an asymptotic expansion as $L \to 0$ in terms of local functionals. The condition is that, for each $i \geq 0$, $k \geq 0$, and $r \geq 0$ there is a function $g_{i,k,r}(L) \in C^\infty((0,\infty))$ and a *local* functional $\Phi_{i,k,r}$ such that for every i and k, we have

$$I_{i,k}[L] \sim \sum_{r \geq 0} g_{i,k,r}(L) \Phi_{i,k,r}. \tag{19}$$

The asymptotic equivalence symbol $\sim$ means that there is a sequence of numbers d_R indexed by $R \geq 0$ such that $\lim_{R\to\infty} d_R = \infty$ and such that for every $a \in \mathcal{B}$, and every $R \geq 0$,

$$\lim_{L \to 0} L^{-d_R} \left(I_{i,k}[L](a) - \sum_{r=0}^{R} g_{i,k,r}(L) \Phi_{i,k,r}(a) \right) = 0. \tag{20}$$

This completes the definition of a perturbative effective quantum field theory.

1.3.6. *Remark on reversibility of RGE.* We remarked above that the renormalization group may only be a semigroup in general. But note the following two facts regarding the renormalization function W:

(1) $W(0, I) = I$,
(2) $W(P_1, W(P_2, I)) = W(P_1 + P_2, I)$.

Thus the inverse of the operation $W(P(L,L'),\cdot)$ is just $W(-P(L,L'),\cdot)$, and so the renormalization group equation for $L < L'$,

$$I[L'] = W(P(L,L'), I[L]), \tag{21}$$

is invertible. Thus, according to our definition, the scale L effective interaction $I[L]$ for *any single value* of L determines $I[L]$ for *all other values* of L. Put another way, all of the effective theories contain the *same* amount of information.

Polchinski [Pol84] has remarked on this fact: it seems to be an artifact of our restriction to the perturbative regime. Namely, given a theory at some scale $L > 0$, it makes sense to

[1] The clarification of the sense in which one can say $C^\infty(M \times M) \cong C^\infty(M) \otimes C^\infty(M)$ is the subject of Costello's appendix on nuclear spaces.

calculate the effective interaction at an arbitrarily small length scale ϵ *as a power series in the perturbation parameter*, which for us is $\hbar$. If, hypothetically, there is some scale ℓ at which the continuum structure of spacetime breaks down into a lattice, we can still compute this power series at scales $\epsilon < \ell$, but presumably this series will have "bad" analytic properties. Since the present discussion is in terms of formal power series, we won't consider this further.

1.4. **Main theorems.** Now we can state the main theorems that provide a classication of effective theories. Fix a free theory $(\mathcal{B}, S_{\text{quad}})$, for example the scalar field theory $\mathcal{B} = C^\infty(M)$, $S_{\text{quad}}(\phi) = -\int_M \phi(\Delta + m^2)\phi$. We consider (perturbative effective quantum field) theories deforming $(\mathcal{B}, S_{\text{quad}})$.

Denote by $\mathcal{T}^{(n)}$ the set of theories that are defined modulo $\hbar^{n+1}$. Denote by $\mathcal{T}^{(\infty)}$ the set of theories that defined to all orders. By definition, $\mathcal{T}^{(\infty)} = \lim \mathcal{T}^{(n)}$.

Theorem 1. *The projection $\mathcal{T}^{(n+1)} \to \mathcal{T}^{(n)}$ canonically has the structure of a principal bundle for the abelian group $\mathcal{O}_{\text{loc}}(\mathcal{B})$ of local functionals. Also, $\mathcal{T}^{(0)}$ is canonically isomorphic to the abelian group $\mathcal{O}^+_{\text{loc}}(\mathcal{B})$ of local functionals that are at least cubic.*

Theorem 2. *There are noncanonical bijections*

$$\mathcal{T}^{(n)} \longleftrightarrow \mathcal{O}^+_{\text{loc}}(\mathcal{B})[\hbar]/(\hbar^{n+1}) \tag{22}$$

$$\mathcal{T}^{(\infty)} \longleftrightarrow \mathcal{O}^+_{\text{loc}}(\mathcal{B})[[\hbar]] \tag{23}$$

where on the right-hand sides, we are considering local functionals that are at least cubic modulo $\hbar$.

1.4.1. *General remarks.* It is clear that Theorem 1 implies Theorem 2. One can simply choose sections of all of the maps $\mathcal{T}^{(n+1)} \to \mathcal{T}^{(n)}$, which clearly exist since the fibers are contractible. However, the strategy of proof is to first prove Theorem 2 by constructing a noncanonical bijection depending on a certain auxiliary datum, and then deducing Theorem 1 by analyzing the dependence on this auxiliary datum.

1.4.2. *Auxiliary datum: "Renormalization Scheme".* A more detailed version of Theorem 2 says that the bijections are induced by a choice of what is called a *renormalization scheme*. This use of the term "renormalization" is to be distinguished from the renormalization group equation.

You have probably heard that the renormalization process involves "subtracting off infinities" and introducing "counterterms." Well, the proof of Theorem 2 is where that happens. The point is that, while the renormalization group equation allows us to pass between different positive length scales, a local functional lives at length scale $L = 0$, as interactions can only happen at points. Given a local functional I, we can't get into the effective theory game unless we can pass from $L = 0$ to some positive value of L. The natural thing to do would be to try

$$I[L] = W(P(0, L), I) \tag{24}$$

but as we have said, the whole point is that this expression doesn't make sense. Instead, we introduce a counterterm $I^{\text{CT}}(\epsilon)$ such that

$$I[L] = \lim_{\epsilon \to 0} W(P(\epsilon, L), I - I^{\text{CT}}(\epsilon)) \tag{25}$$

exists. For a given I, there are many counterterms $I^{\text{CT}}(\epsilon)$ that have this property, and they give different effective interactions $I[L]$. The renormalization scheme is a precise recipe for constructing the counterterm $I^{\mathrm{CT}}(\epsilon)$ from I. It can be encoded as a choice of a complement, inside $C^\infty((0,\infty))$, to the subspace of functions whose limit at zero exists.

The process for constructing the counter terms goes something like this: First consider $W(P(\epsilon,L),I)$ for small ϵ. This expression has no reason to be meaningful because all phenomena below scale ϵ are simply *omitted* rather properly *hidden*. In this situation ϵ is called the *cutoff*. By analyzing the singularity of this function as $\epsilon \to 0$, and working order-by-order in $\hbar$, we cook up a local counterterm $I^{\mathrm{CT}}(\epsilon)$ depending on the cutoff that has the same singularity as $W(P(\epsilon,L),I)$, so that when we take $I - I^{\mathrm{CT}}(\epsilon)$, the singularity is canceled. Making the counterterm local involves some asymptotic analysis since the functional $W(P(\epsilon,L),I)$ is nonlocal.

1.4.3. *Remark on cubic condition.* The reason we only consider interactions that are at least cubic modulo $\hbar$ is as follows. If we allow I to contain a term like $\int_M \mu\phi^2$, then our total action will look like

$$S = S_{\mathrm{quad}} + I = -\int_M \phi(\Delta + m^2)\phi + \int_M \mu\phi^2 + \cdots, \tag{26}$$

and we can absorb this interaction into the quadratic part,

$$S = -\int_M \phi(\Delta + (m^2 - \mu))\phi + \cdots. \tag{27}$$

Thus the quadratic part of the action changes, and so the propagator would have to change as well, and so we start to move away from the connection to the original free theory.

On the other hand, a term like $\hbar\int_M \mu\phi^2$ is perfectly fine. In fact, it corresponds to the very important phenomenon called *mass renormalization*. In this case, the squared mass changes from m^2 to $m^2 - \hbar\mu$, and $-\hbar\mu$ would be called the one-loop correction to the squared mass of the ϕ particle.

1.5. **Classifying theories.** Theorems 1 and 2 are interesting because they say that many quantum theories exist, and in fact the space of theories is infinite-dimensional, even at each order in $\hbar$. But now there are too many theories—which of them are of interest? So far the only condition we have imposed is that the classical quadratic action is the given S_{quad}. What other conditions can we sensibly impose?

1.5.1. *Correctness of the classical limit.* Suppose there is some classical action $S = S_{\mathrm{quad}} + I_0$ that we are interested in, and we want to say which quantum theories quantize *this particular* interaction I_0. To do this, we use Theorem 1: we say that a given theory $T \in \mathcal{T}^{(\infty)}$ *quantizes* I_0, or *has classical limit* I_0, if the projection of T to $\mathcal{T}^{(0)}$ corresponds to I_0 under the canonical bijection furnished by Theorem 1.

Let us say a little more about why the condition is phrased this way. We can pick a renormalization scheme and consider the theory $\{I[L]\}_{L\in(0,\infty)}$ that corresponds to I_0 by Theorem 2. The limit $\lim_{L\to 0} I[L]$ may not exist, since it is built from $I_0 - I^{\mathrm{CT}}(\epsilon)$, and $I^{\mathrm{CT}}(\epsilon)$ has a singularity as $\epsilon \to 0$. However, the counterterms are only introduced because of diagrams with positive loop number, and so $I^{\mathrm{CT}}(\epsilon)$ is divisible by $\hbar$. The right thing to consider is

$$\lim_{L\to 0}(I[L] \mod \hbar) \in \mathcal{O}^+_{\mathrm{loc}}(\mathcal{B}). \tag{28}$$

This is the classical limit of our effective interactions $I[L]$.

1.5.2. *Symmetry.* Another natural condition is to ask that symmetries of the original classical theory are preserved by the quantization. For instance, on Minkowski space physicists typically want theories that are invariant under Lorentzian isometries (the Poincaré group). The simplest way to impose this condition is to assume that all of the effective interactions $I[L]$ are invariant under the relevant group action.

However, there is one very important symmetry that cannot be treated by requiring the interactions to be invariant. This is the gauge symmetry of a gauge theory, and we turn to it next.

2. Quantization of gauge theories

2.1. **Motivation.** In general, a gauge symmetry is a local symmetry, meaning that one choose different transformations at different points of spacetime. For instance, we might have "gauge fields" $\mathcal{A} = \mathrm{Map}(M, V)$ that are maps to a vector space V, and the gauge group could be $\mathcal{G} = \mathrm{Map}(M, G)$, where G is a group that acts on V, and we could assume that the classical action $S : \mathcal{A} \to \mathbb{R}$ is $\mathcal{G}$-invariant, and we could seek to quantize this theory.

There is an important point here, and a variety of reasons for it, but the bottom line is the following slogan:

$$\text{Gauge symmetry is not a symmetry.} \tag{29}$$

This sounds absurd: we just said that $S : \mathcal{A} \to \mathbb{R}$ is $\mathcal{G}$-invariant! While the gauge symmetry is a symmetry of the classical action, it is not a symmetry of the *physical theory.* Rather, it is a symmetry that must be taken into account to define what the theory even is. Some things to think about, in no particular order:

(1) In the presence of gauge symmetry, the "physical" space of field configurations is not $\mathcal{A}$ but $\mathcal{B} = \mathcal{A}/\mathcal{G}$.
(2) Because the action is $\mathcal{G}$-invariant, the $\mathcal{G}$-action takes a solution of the Euler-Lagrange equations to another solution. Suppose we want to consider a boundary-value problem for this PDE on some domain. Because the symmetry is local, we can modify a solution only in the interior of the domain. This means that none of the trandi-tional boundary value problems (Dirichlet/Neumann in Euclidean signature, Cauchy in Lorentzian) can be well-posed. Yet, they should be well-posed modulo $\mathcal{G}$.
(3) The field $A \in \mathcal{A}$ has several modes of oscillation: some are tangent to the $\mathcal{G}$-orbits: these are unphysical and we need to get rid of them. Others are transverse to the $\mathcal{G}$-orbits and those are the ones we want to keep. There is no canonical splitting, but we can, at least locally on $\mathcal{A}$, try to construct a transversal submanifold to the $\mathcal{G}$-orbits, called a slice, and only consider fields configurations in the slice. This process is called *choosing a gauge-fixing condition.*

The perturbative quantization of a gauge theory introduces a variety of difficulties. If the space of fields is $\mathcal{B} = \mathcal{A}/\mathcal{G}$ then it has no linear structure and our previous analysis about propagators and whatnot does not apply. So we would rather work on $\mathcal{A}$ directly preserving $\mathcal{G}$-invariance. But there are a couple of problems with that. Because the Euler-Lagrange equations are degenerate, there is no Greens function or propagator. Even if we could get around that, it is very difficult to see how $\mathcal{G}$-invariance could ever be compatible with effective theories and renormalization. The $\mathcal{G}$-action is local, but the effective interaction $I[L]$ is nonlocal. One could think that perhaps there would be something like $\mathcal{G}[L]$, a group of a gauge transformations that are L-local, but it seems that if one composes such things one would get transformations that are completely global.

What we will end up using is called the Batalin-Vilkovisky formalism. In modern terms, it involves reformulating the quotient $\mathcal{B} = \mathcal{A}/\mathcal{G}$ as a derived quotient, and then taking the derived critical locus of the classical action S. The output will be a situation where one can say that gauge invariance is not preserved *strictly* but only *homotopically*.

2.2. **Batalin-Vilkovisky formalism.** The traditional explanation of the Batalin-Vilkovisky (BV) formalism involves starting with gauge fields, introducing the ghost fields (derived quotient) and then the antifields and antighosts (derived critical locus). But we need not explain it that way, we can just assemble all of the fields into a single struture from the beginning.

2.2.1. *Free theories.*

Definition 1. A *free BV theory* on a compact manifold M consists the following.

1. (1) A $\mathbb{Z}$-graded vector bundle $E \to M$, whose sections are denoted $\mathcal{E} = \Gamma(M, E)$.
2. (2) A local pairing $\langle\ ,\ \rangle_{\mathrm{loc}} : E \otimes E \to \mathrm{Dens}(M)$ with values in densities. It is required to be skew-symmetric of cohomological degree -1 and nondegenerate fiberwise. By integration it induces a pairing $\langle\ ,\ \rangle : \mathcal{E} \otimes \mathcal{E} \to \mathbb{C}$.
3. (3) A differential operator $Q : \mathcal{E} \to \mathcal{E}$ of cohomological degree 1, satisfying $Q^2 = 0$, and which is skew-self-adjoint for $\langle\ ,\ \rangle$. We require that $(\mathcal{E}, Q)$ is an elliptic complex.

Recall that a complex is called *elliptic* if the complex $(\pi^* E, \sigma(Q))$ of vector bundles on $T^*M \setminus M$ is acyclic, where $\pi : (T^*M \setminus M) \to M$ is the projection, and $\sigma(Q)$ denotes the principal symbol of Q.

Definition 2. A *gauge fixing operator* on a free BV theory (as in the previous definition) is a differential operator $Q^{\mathrm{GF}} : \mathcal{E} \to \mathcal{E}$ of cohomological degree -1, satisfying $(Q^{\mathrm{GF}})^2 = 0$, and which is self-adjoint for $\langle\ ,\ \rangle$. We require that $D = [Q, Q^{\mathrm{GF}}]$ is a generalized Laplacian.

Recall that an second-order operator is called a generalized Laplacian if its principal symbol defines a Riemannian metric.

2.2.2. *Odd symplectic heat kernel and the propagator.* Take a free BV theory and gauge fixing operator as above. The heat kernel K_t for D generates the heat semigroup e^{-tD}. The only difference between usual (metric) and odd symplectic cases is that we use the odd symplectic form rather than a metric to define the convolution operation.

The pairing $\langle\ ,\ \rangle : \mathcal{E} \otimes \mathcal{E} \to \mathbb{C}$ yields a map $1 \otimes \langle\ ,\ \rangle : \mathcal{E} \otimes \mathcal{E} \otimes \mathcal{E} \to \mathcal{E}$. Given $K \in \mathcal{E} \otimes \mathcal{E}$ and $\phi \in \mathcal{E}$, define

$$K \star \phi = (-1)^{|K|}(1 \otimes \langle\ ,\ \rangle)(K \otimes \phi) \tag{30}$$

In words, we contract the second tensor factor of K with ϕ using the odd symplectic pairing. The *odd symplectic heat kernel* $K_t \in \mathcal{E} \otimes \mathcal{E}$ is defined by the property that

$$K_t \star \phi = e^{-tD}\phi \tag{31}$$

The *propagator* of the free BV theory with gauge fixing condition is

$$P(L, L') = \int_L^{L'} (Q^{\mathrm{GF}} \otimes 1) K_t \, dt. \tag{32}$$

The reason for this definition is that the quadratic action for a free BV theory is

$$S_{\text{quad}}(\phi) = \frac{1}{2}\langle \phi, Q\phi \rangle, \tag{33}$$

and indeed $P(0, L)$ is a parametrix for Q, in the sense that

$$\mathrm{Id} - [P(0, L), Q] = e^{-LD}, \tag{34}$$

and e^{-LD} is a smoothing operator.

2.2.3. *Antibracket.* The odd skew pairing $\langle\ ,\ \rangle : \mathcal{E} \otimes \mathcal{E} \to \mathbb{C}$ may be regarded as an odd symplectic form on the field space $\mathcal{E}$. Therefore, the algebra of functionals $\mathcal{O}(\mathcal{E})$ carries an odd Poisson bracket, which we denote

$$\{\ ,\ \} : \mathcal{O}(\mathcal{E}) \otimes \mathcal{O}(\mathcal{E}) \to \mathcal{O}(\mathcal{E}) \tag{35}$$

Batalin and Vilkovisky call this Poisson bracket the *antibracket*. Thus the field space $\mathcal{E}$ is to be regarded as an odd symplectic or Poisson manifold.

2.2.4. *BV operator.* We are going to do some odd symplectic geometry on $\mathcal{E}$. One might expect that odd symplectic geometry is the same as even symplectic geometry, with "some signs scattered around." For the most part this is true, but there is one place where the symmetry properties of the symplectic form really make a difference: namely, there is *no canonical measure* on an odd symplectic manifold. One can pick a Darboux coordinate system, and write down what would appear to be the Liouville measure, but the measure one gets depends on the choice of coordinates.

A reformulation is this: pick a Darboux coordinate system, and let μ denote the natural volume form in these coordinates. Take a Hamiltonian function $H \in \mathcal{O}(\mathcal{E})$, and let X_H be its Hamiltonian vector field. The flow of X_H does not necessarily preserve μ. We can measure the defect by taking the divergence with respect to μ, and this defines the *BV operator*,

$$\Delta H = \mathrm{div}_\mu X_H. \tag{36}$$

Observe that ΔH is a second-order operator of H: one derivative to construct X_H, and a second for the divergence.

Another characterization of the BV operator is that it is the second-order differential operator whose principal symbol is the inverse of the symplectic form (Poisson bivector). In even symplectic geometry, this can't exist, because the principal symbol of an operator is always symmetric.

There is another wrinkle in the story, which is that, on an infinite-dimensional odd symplectic manifold such as $\mathcal{E}$, the BV operator Δ is rather singular. The idea is that an odd symplectic form behaves more like a metric, and the BV operator is the Laplacian for this metric. A nice thought experiment in a simplified situation is the following. Let

$$\mathcal{H} = \ell^2 = \left\{ (x_n)_{n=1}^{\infty} \,\middle|\, \sum_{n=1}^{\infty} x_n^2 < \infty \right\} \tag{37}$$

denote Hilbert space. The metric on $\mathcal{H}$ allows us to regard $\mathcal{H}$ as an infinite-dimensional Riemannian manifold. We ask, what is the Laplacian Δ on this manifold? It is nothing but

$$\Delta = \sum_{n=1}^{\infty} \frac{\partial^2}{\partial x_i^2}. \tag{38}$$

Now there is a perfectly nice smooth function on $\mathcal{H}$ given by

$$f(x) = \|x\|^2 = \sum_{n=1}^{\infty} x_i^2. \tag{39}$$

Now observe

$$\Delta f = \sum_{n=1}^{\infty} 2, \tag{40}$$

which is clearly a problem.

We will eventually see that the singularity of the naive BV operator is of a one-loop nature, and that we can regularize it using the same tools as for Feynman integrals. This will fit nicely with the philosophy of effective theories, since we will get a BV algebra structure at each length scale.

2.2.5. *Regularized BV operator and antibracket.* Fix $L \in (0, \infty)$. The regularized BV operator Δ_L applied to a functional $I \in \mathcal{O}(\mathcal{E})[[\hbar]]$ is defined by a sum over diagrams with one vertex and one loop, where the vertex is labeled by I and the loop is labeled by the odd symplectic heat kernel K_L. Thus, if $I = \sum_{i,k\geq 0} \hbar^i I_{i,k}$ with $I_{i,k} \in \mathrm{Sym}^k(\mathcal{E}^\vee)$, then $(\Delta_L I)_{i,k}$ is given by taking $I_{i,k+2}$ and contracting two inputs using K_L. See the figure at [Cos11, p. 176, fig. 1].

Since this operation introduced a loop, we should increase the loop number by one and add a factor of $\hbar$ in the definition of Δ_L, but, following Costello, we will choose to write this $\hbar$ factor explicitly. For this reason Δ_L usually does not appear by itself in a meaningful formula, but the combination $\hbar\Delta_L$ does.

The regularization can also be applied to the antibracket. We define $\{I, J\}_L$ as a sum over diagrams with two vertices and one internal edge. The two verticies are labeled by $I_{r,m}$ and $J_{s,n}$, the internal edge is labeled by K_L; this diagram contributes to $(\{I, J\}_L)_{r+s,m+n-2}$ (loop numbers add). From these definitions, it is fun to check the BV identity:

$$\{I, J\}_L = \Delta_L(IJ) - \Delta_L(I)J - (-1)^{|I|} I \Delta_L(J). \tag{41}$$

2.3. **Quantum master equation.** Formally, the quantum master equation for the action S is

$$\Delta(\exp(S/\hbar)) = 0, \tag{42}$$

saying that the functional integral measure is invariant. This equation does not really fit into our framework, first because we are working in terms of effective interactions $I[L]$, and second because the BV operator needs to be regularized. Fortunately, these requirements line up exactly, and our version of the quantum master equation is, at scale L,

$$\Delta_L \left[\exp\left(\frac{1}{2\hbar} \langle \phi, Q\phi \rangle + \frac{1}{\hbar} I[L](\phi) \right) \right] = 0. \tag{43}$$

This is equivalent to the equation

$$QI[L] + \frac{1}{2}\{I[L], I[L]\}_L + \hbar \Delta_L I[L] = 0. \tag{44}$$

To see this, write the original equation as a sum over disconnected diagrams with any number of vertices and a single internal edge carrying the kernel K_L. By collecting the terms in the right way one gets the second equation times $\exp(S[L]/\hbar)$. Equation (44) is the form we shall usually use, and we call it the *scale L quantum master equation*.

Now we have a lemma that says that the quantum master equation is preserved under renormalization group flow.

Lemma 1. *A functional $I[L] \in \mathcal{O}^+(\mathcal{E})[[\hbar]]$ satisfies the scale L quantum master equation if and only if $I[L']$ satisfies the scale L' quantum master equation, where $I[L'] = W(P(L, L'), I[L])$.*

2.3.1. *Remarks.* In terms of the general problem of quantization of gauge theories, the quantum master equation is the way gauge invariance is formulated for the effective interaction. We had to choose a gauge fixing operator Q^{GF} in order to construct the propagator and run the whole program, but how does the theory we get depend on this choice? If the quantization process is not to spoil gauge invariance, then the theory we get should not depend much on the choice of gauge fixing operator. The quantum master equation implies that it does not depend on this choice up to homotopy, though explaining this fully is rather involved.

2.3.2. *Pre-theories and theories.* Let $(E, \langle\ ,\ \rangle, Q, Q^{\mathrm{GF}})$ be a free BV theory with gauge fixing operator. Let $P(L, L')$ be the associated propagator. What was previously known as a theory will now be called a pre-theory.

Definition 3. A *pre-theory* quantizing the given free BV theory is a collection of effective interactions $I[L] \in \mathcal{O}^{+,0}(\mathcal{E})[[\hbar]]$ that have cohomological degree 0 and are at least cubic modulo $\hbar$, which satisfy the renormalization group equation

$$I[L] = W(P(L, L'), I[L']) \tag{45}$$

and the asymptotic locality condition. We denote by by $\widetilde{\mathcal{T}}^{(\infty)}$ set of theories, and by $\widetilde{\mathcal{T}}^{(n)}$ the set of pre-theories defined modulo $\hbar^{n+1}$.

Theorems 1 and 2 have analogs in this context, where they say pre-theories can be parametrized, at each order in $\hbar$, by degree zero local functionals $\mathcal{O}^0_{\mathrm{loc}}(\mathcal{E})$, with the restriction that at order zero the functional is at least cubic.

We reserve the term theory for a pre-theory that satisfies the quantum master equation.

Definition 4. A *theory* is a pre-theory that satisfies the quantum master equation. That is, we require that

$$QI[L] + \frac{1}{2}\{I[L], I[L]\}_L + \hbar\Delta_L I[L] = 0 \tag{46}$$

for some L, and hence for every L by Lemma 1. The set of theories is denoted $\mathcal{T}^{(\infty)}$ and the set of theories defined modulo $\hbar^{n+1}$ is denoted $\mathcal{T}^{(n)}$.

2.4. **Obstruction theory.** By theorems 1 and 2, we know that there are plenty of pre-theories deforming a given free BV theory. It remains to understand which of these pre-theories, if any, are theories (satisfying quantum master equation). The form of the quantum master equation (44) is a type of Maurer-Cartan equation, and so standard ideas from deformation-obstruction theory can be used to understand this problem.

2.4.1. *Order zero and classical master equation.* Take some pre-theory $\{I[L]\}$. At order zero in $\hbar$, the quantum master equation reads

$$QI_0[L] + \frac{1}{2}\{I_0[L], I_0[L]\}_L = 0 \tag{47}$$

where $I_0[L]$ is the effective interaction modulo $\hbar$; it defines an element of $\widetilde{\mathcal{T}}^{(0)}$. If we take $L \to 0$ in this equation, we get

$$QI_0 + \frac{1}{2}\{I_0, I_0\} = 0 \tag{48}$$

where $I_0 = \lim_{L\to 0} I_0[L]$. This I_0 is precisely the classical limit of our pre-theory. It is the local functional $I_0 \in \mathcal{O}^+_{\mathrm{loc}}(\mathcal{E})$ that corresponds to $\{I_0[L]\} \in \widetilde{\mathcal{T}}^{(0)}$ under the canonical bijection furnished by Theorem 1. Equation (48) is called the *classical master equation*, and so we have

Theorem 3. *The canonical bijection $\widetilde{\mathcal{T}}^{(0)} \leftrightarrow \mathcal{O}^+_{\mathrm{loc}}(\mathcal{E})$ restricts to a bijection between $\mathcal{T}^{(0)}$ and the set of solutions of the classical master equation* (48).

2.4.2. *From order n to order $n+1$.* Suppose we are given an order n theory $\{I[L]\} \in \mathcal{T}^{(n)}$. We want to know if it lifts to an order $n+1$ theory. By Theorem 1, we know that it at least lifts to an order $n+1$ pre-theory; choose such an lift and call it $\{\widetilde{I}[L]\} \in \widetilde{\mathcal{T}}^{(n+1)}$. This lift may not satisfy the quantum master equation, but we know that by Theorem 1, there is a canonical way to alter the lift by addition of a local functional. So we want to see if there is a local functional we can add to make the quantum master equation hold true.

Consider the expression

$$O_{n+1}[L] = \hbar^{-(n+1)}\left(Q\widetilde{I}[L] + \frac{1}{2}\{\widetilde{I}[L], \widetilde{I}[L]\}_L + \hbar\Delta_L\widetilde{I}[L]\right). \tag{49}$$

The expression in parentheses is precisely the failure of the quantum master equation to be satisfied, and it is divisible by $\hbar^{n+1}$ because the equation is satisfied to order n.

Theorem 4. *The limit $O_{n+1} = \lim_{L\to 0} O_{n+1}[L]$ exists and gives an element of $\mathcal{O}_{\mathrm{loc}}(\mathcal{E})$. It satisfies the equation*

$$QO_{n+1} + \{I_0, O_{n+1}\} = 0. \tag{50}$$

The set of lifts of $\{I[L]\} \in \mathcal{T}^{(n)}$ to $\mathcal{T}^{(n+1)}$ are in bijection with the set of local functionals $J \in \mathcal{O}_{\mathrm{loc}}(\mathcal{E})$ that satisfy

$$QJ + \{I_0, J\} = O_{n+1}. \tag{51}$$

It is evident from this theorem that the complex controlling the deformation-obstruction problem is nothing but $(\mathcal{O}_{\mathrm{loc}}(\mathcal{E}), Q + \{I_0, \ \})$. The obstruction O_{n+1} has cohomological degree 1, while the interactions themselves have cohomological degree 0. Thus the relevant obstruction space is $H^1(\mathcal{O}_{\mathrm{loc}}(\mathcal{E}), Q + \{I_0, \ \})$. If the cohomology class of O_{n+1} vanishes, then the set of lifts is in bijection with degree zero local functionals satisfying

$$QJ + \{I_0, J\} = 0. \tag{52}$$

There is actually a further equivalence relation on such solutions, namely homotopy.

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