Manifold Refreshment.

Recall \( M \) a smooth manifold.

- Tangent bundle \( TM \) sections are vector fields \( \mathcal{X}(M) \)
- Cotangent bundle \( T^*M \) sections are 1-forms \( \mathcal{A}^1(M) \)
- Wedge powers \( \Lambda^k T^*M \) sections are \( k \)-forms \( \mathcal{A}^k(M) \)

We always work with \( C^\infty \) objects.

Given a vector field \( X \) on \( M \), there is an associated ordinary differential equation

unknown: path \( \gamma(t) : \mathbb{R} \rightarrow M \) equals \( \frac{d\gamma}{dt} = X(\gamma(t)) \) (*)

ODE theory \( \Rightarrow \) given any initial point \( \gamma(0) = p \), we have existence and uniqueness for short time \( E(p) \)

\[ \exists! \gamma : [0, E(p)] \rightarrow M \text{ satisfies } \gamma(t) = p \text{ (*)} \]

If solution curves exist for all points for all times, \( X \) is called a complete vector field. In this case, we get an isotopy \( \phi_t : M \rightarrow M \), called the flow of \( X \)

\[ \phi_t(x) = \left( \text{take } \gamma(t), \text{ where } \gamma \text{ is the solution curve} \right) \]

\[ \text{ (with initial point } \gamma(0) = x \text{) } \]

\( \bullet \) If \( M \) is compact, any vector field is complete.

\( \bullet \) All of the above also holds if the vector field \( X \) has an explicit time dependence \( X_t \). Then the ODE is

\[ \frac{d\gamma}{dt} = X_t(\gamma(t)) \]

The flow is still denoted \( \phi_t \).
Exterior derivative: This is a first order differential operator
\[ d: \Omega^k(M) \to \Omega^{k+1}(M) \]

It is determined by the following properties (which are typically most useful for computation as well):

- \( d \) is natural with respect to smooth maps \( f: M \to N \): consider pull back \( f^*: \Omega^k(N) \to \Omega^k(M) \)
  then \( d(f^* \alpha) = f^*(d\alpha) \)
- \( d^2: \Omega^k(M) \to \Omega^{k+1}(M) \to \Omega^{k+2}(M) \) is zero
- If \( f \in \Omega^0(M) \) is a function, \( df \) is the differential of \( f \):
  \[ df(X) = X \cdot f \] for any vector field \( X \)
- If \( \alpha \in \Omega^k(M) \), \( \beta \in \Omega^0(M) \),
  \[ d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta \]

In an arbitrary local coordinate system \( (x_i)_{i=1}^n \) on \( M \) where \( I \subseteq \{1, \ldots, n\} \) is a subset of size \( k \), we have a \( k \)-form

\[ dx_I = dx_{i_1} \wedge \cdots \wedge dx_{i_k}, \quad i_1 < i_2 < \cdots < i_k \]

Any \( k \)-form has local expression \( \alpha = \sum_{|I|=k} f_I \, dx_I \)

Then \( d\alpha = \sum_{|I|=k} df_I \wedge dx_I \), where \( df_I = \sum_{i=1}^n \frac{\partial f_I}{\partial x_i} \, dx_i \)

Integral: Given a sufficiently smooth \( k \)-chain \( C \) in \( M \) (e.g., an oriented smooth submanifold of \( M \)), there is a well defined integral on \( k \)-forms

\[ \int_C \alpha \quad \text{for} \quad \alpha \in \Omega^k(M) \]
We have Stokes' theorem: \[ \int_C d\alpha = \int_{\partial C} \alpha \]
where \(\partial C\) denotes the boundary of the chain \(C\).

**Contraction:** Let \(X\) be a vector field, \(\alpha\) a \(k\)-form. There is a \((k-1)\)-form \(i_X \alpha\) called the contraction of \(X\) with \(\alpha\). Think of \(\alpha\) as an alternating \(k\)-multilinear form on the tangent spaces, and plug \(X\) into first input.

**Derivation property:** \[ i_X (\alpha \wedge \beta) = (i_X \alpha) \wedge \beta + (-1)^k \alpha \wedge (i_X \beta) \]
if \(\alpha \in \Omega^k(M)\) and \(\beta \in \Omega^1(M)\).

**Homework:** prove this.

**Lie derivative:** Recall flow \(\varphi_t\) of vector field \(X\) (time independent)

Then we can define \[ L_X \alpha = \frac{d}{dt} (\varphi_t^* \alpha) \bigg|_{t=0} \]

\[ L_X : \Omega^k(M) \to \Omega^k(M) \]
This is an intrinsic notion of derivative that is not dependent on \(X\).

**Cayley's Magic Formula** (or Cayley homotopy formula)

\[ L_X \alpha = d i_X \alpha + i_X d \alpha \quad \text{for } \alpha \in \Omega^k(M) \]

**Derivative at other times:** \[ \frac{d}{dt} \varphi_t^* \alpha = \varphi_t^* L_X \alpha \]

If the vector field is time dependent \(X_t\), and \(\varphi_t\) is the flow, we also have \[ \frac{d}{dt} \varphi_t^* \alpha = \varphi_t^* L_X \alpha \]

In \(L_{X_t} \alpha\), the vector field is first evaluated at time \(t\).
If the $k$-form also has an explicit time dependence

$$\frac{d}{dt} f^*_t \alpha_t = f^*_t (\mathcal{L}_{X_t} \alpha_t + \frac{d\alpha_t}{dt})$$

**Symplectic manifolds** A two form $\omega \in \Omega^2(M)$ is called symplectic if it is nondegenerate (defines a linear symplectic form on each tangent space) and satisfies $d\omega = 0$ ($\omega$ is closed).

By nondegeneracy, all the linear algebra carries over at the level of tangent spaces; can define isotropic, coisotropic, Lagrangian, symplectic submanifolds as those which satisfy the corresponding condition on tangent spaces.

Also we have a Lagrangian grassmannian bundle $U(\omega) / O(n) \rightarrow \Lambda \rightarrow M$

* The closedness condition $d\omega = 0$ is what really holds the geometry together, however, as we shall see.
  (Vaguely: It makes sympletic geometry “locally constant”)

**Symplectic isotopy:** $f_t$ isotopy generated by $X_t$ vector field

When is $f_t$ symplectic (for all $t$)?

$$0 = \frac{d}{dt} f^*_t \omega = f^*_t (\mathcal{L}_{X_t} \omega) = f^*_t (d i_{X_t} \omega + i_{X_t} d\omega) = f^*_t (d i_{X_t} \omega) \quad \text{(since } d\omega = 0)$$

$\iff i_{X_t} \omega$ is closed for all $t$.

We call such $X_t$ a **symplectic vector field**.
If \( i_{X_t} \omega \) is not merely closed, but exact, \( X_t \) is called a Hamiltonian vector field.

**Hamiltonian vector field:** \( f \in C^0(M) \) function, \( df \in \Omega^1(M) \)

define vector field \( X_f \) by

\[
\omega(\cdot, X_f) = -i_{X_f} \omega = df \quad \text{ (uses nondegeneracy)}
\]

\( f_t \) time-dependent family of functions: \( \omega(\cdot, X_{f_t}) = df_t \)

**Poisson bracket:** \( \{f, g\} = \omega(X_f, X_g) \quad (f, g \in C^0(M)) \)

\[
\begin{align*}
\omega(X_f, X_g) &= dg(X_f) = X_f \cdot g \\
&= -\omega(X_g, X_f) = -df(X_g) = -X_g \cdot f
\end{align*}
\]

**Lemma:** If \( X, Y \) are symplectic\( \Rightarrow \) then \( i_{[X,Y]} \omega = d(\omega(Y,X)) \)

since \( \omega \) is closed

\[
\begin{align*}
0 &= \omega(X, Y, Z) = X \cdot \omega(Y, Z) + Y \omega(Z, X) + Z \cdot \omega(X, Y) \\
&= \omega([X,Y], Z) + \omega([Y,Z], X) - \omega([Z,X], Y)
\end{align*}
\]

since \( i_X \omega \) is closed

\[
\begin{align*}
0 &= d(i_X \omega)(Y, Z) = Y \cdot i_X \omega(Z) - Z \cdot i_X \omega(Y) - i_X \omega([Y,Z]) \\
0 &= Y \cdot \omega(X, Z) - Z \cdot \omega(X,Y) - \omega(X, [Y,Z]) \\
0 &= Y \cdot \omega(Z, X) + Z \cdot \omega(X, Y) - \omega([Y,Z], X)
\end{align*}
\]

since \( i_Y \omega \) closed

\[
\begin{align*}
0 &= X \cdot \omega(Z, Y) + Z \cdot \omega(Y, X) - \omega([X,Z], Y) \\
0 &= X \cdot \omega(Y, Z) + Z \cdot \omega(X, Y) - \omega([Z,X], Y)
\end{align*}
\]
Combine: \[ \omega = -z \cdot \omega(x,y) - \omega([x,y], z) \]
\[ \omega([x,y], z) = z \cdot \omega(y, x) \]

Cor: \( X \cdot x, y \text{ are symplectic } \Leftrightarrow \) \([x,y] \text{ is hamiltonian } \Rightarrow \)

Prop: \[ [X_f, X_g] = \{f, g\} \]

Proof: \[-i[X_f, X_g] \omega = d \left( \omega(X_f, X_g) \right) = d \{ f, g \} \]

Prop: Poisson bracket satisfies \( \{ f, gh \} = \{ f, g \} h + g \{ f, h \} \)
and \( \{ f, \{ f_2, f_3 \} \} + \{ f_2, \{ f_3, f \} \} + \{ f_3, \{ f, f_2 \} \} = 0 \)
(Jacobi identity)

Proof: \[ \{ f, gh \} = X_f(gh) = X_f(g) h + g X_f(h) \]
\[ = \{ f, g \} h + g \{ f, h \} \]

Jacobi: \[ 0 = d \omega(\{ X_1, X_2, X_3 \}) \]
\[ = X_1 \omega(X_2, X_3) + X_2 \omega(X_3, X_1) + X_3 \omega(X_1, X_2) \]
\[ - \omega([X_1, X_2], X_3) - \omega([X_2, X_3], X_1) - \omega([X_3, X_1], X_2) \]
\[ = X_1 \{ f_2, f_3 \} + X_2 \{ f_3, f_1 \} + X_3 \{ f_1, f_2 \} \]
\[ - \omega(X_1 f_2, f_3) - \omega(X_2 f_3, f_1) - \omega(X_3 f_1, f_2) \]
\[ = \{ f_1 \{ f_2, f_3 \} \} + \{ f_2 \{ f_3, f_1 \} \} + \{ f_3 \{ f_1, f_2 \} \} \]
\[ - \{ f_2 \{ f_2, f_3 \} f_1 \} - \{ f_3 \{ f_3, f_1 \} f_2 \} - \{ f_1 \{ f_1, f_2 \} f_3 \} \]
\[ = 2 \cdot (\text{Jacobi expression}) \]

Point: \( dw = 0 \Rightarrow \text{Jacobi identity} \).
Conservation of energy: If $X_M$ is vector field of $H$, then $X_M \cdot H = 0$, i.e., $H$ is constant along trajectories of $X_M$.

**Proof:** $X_M \cdot H = \omega(X_M, X_M) = 0$ by skew symmetry.

Now we justify why the symplectic geometry is "locally constant" in some senses.

**Moser theorem:** Let $M$ be compact, and let $\omega_t$ be a family of symplectic forms such that $\frac{d\omega_t}{dt}$ is exact for all $t$.

Then there exists an isotopy $f_t : M \to M$ such that $f_t^* \omega_t = \omega_0$ for all $t$.

**Investigate using Lie-Cartan calculus:**

Want: for vector field $X_t$ generating $f_t$:

$$0 = \frac{d}{dt} f_t^* \omega_t = f_t^* (\mathcal{L}_{X_t} \omega_t + \frac{d\omega_t}{dt})$$

$$0 = \mathcal{L}_{X_t} \omega_t + \frac{d\omega_t}{dt}$$

$$0 = d i_{X_t} \omega_t + i_{X_t} dw + \frac{d\omega_t}{dt} \quad \text{by Cartan Magic}$$

$$0 = d i_{X_t} \omega_t + \frac{d\omega_t}{dt} \quad \text{since } dw = 0$$

Now since $\frac{d\omega_t}{dt}$ is exact, we can choose $\beta_t$ such that $\frac{d\omega_t}{dt} = d\beta_t$.

(That $\beta_t$ may be chosen to depend smoothly in $t$, may be deduced from Hodge theory, for example.)
\[ o = d i_{X_t} \omega + d\beta_t. \] It suffices to solve
\[ 0 = i_{X_t} \omega + \beta_t \tag{Moser's equation} \]

This equation is uniquely solvable for \( X_t \) since \( \omega \) is nondegenerate. This completes the proof.

Moser's theorem states that a 1-parameter family of symplectic manifolds with constant cohomology class of symplectic forms is trivial.

**Theorem (Darboux–Weinstein)** Let \( N \subset M \) be a submanifold and let \( \omega_0 \) and \( \omega_1 \) be symplectic forms on a tubular neighborhood of \( N \) such that \( \omega_0|_N = \omega_1|_N \). Then there exists a smaller tubular neighborhood \( U_0 \) and \( U_1 \) and a diffeomorphism \( \phi : U_0 \rightarrow U_1 \) such that \( \phi|_N = id_N \) and \( \phi^*\omega_1 = \omega_0 \).

**Proof** \( \omega_1 - \omega_0 \) is a form which vanishes when restricted to \( N \). By the “relative Poincaré lemma” on a tubular neighborhood of \( N \), there is a 1-form \( \beta \) on a tubular neighborhood such that \( d\beta = \omega_1 - \omega_0 \) and \( \beta|_N = 0 \).

Consider the family of forms \( \omega_t = \omega_0 + t d\beta \). There is a possibly smaller tubular neighborhood of \( N \) such that all these forms are symplectic there. In such a neighborhood, we solve the Moser equation
\[ i_{X_t} \omega_t + \beta = 0 \]

Then the flow of \( X_t \) will satisfy \( g_t^*\omega_t = \omega_0 \).
Note that \( X_t \) vanishes on \( N \) since \( p \) does, so \( g_t \) is identity on \( N \).

Cor (Darboux Theorem) Let \( p \in M \) be a point, then there exists a neighborhood \( U \) of \( p \) and a neighborhood \( V \) of \( 0 \in \mathbb{R}^n \) and a diffeomorphism \( \phi : V \to U \) such that \( \phi^*\omega_M = \omega_{std} \).

Proof. Use linearization firm to construct \( \phi : V \to U \)
\((D\phi_p)^*\omega_M, p = \omega_{std,0}\) holds at the level of the tangent space to \( p \). Then apply Darboux-Weinstein.

Another perspective on the Moser theorem:
What is a family of symplectic manifolds parametrized by a base \( B \)? (Assume all manifolds are diffeomorphic to a fixed compact manifold \( M \)).

One answer: As above, we could have a continuous/smooth map \( B \to \{\text{Symplectic forms on } M\} \).
Such families are slightly "wild".

Another "tamer" answer: A family of symplectic structures on \( M \) parametrized by \( B \) consists of
(a) a fibration \( M \to B \)
   (Assume it's differentiably locally trivial)
(b) a two-form \( \omega \in \Omega^2(M) \) such that
   (i) \( d\omega = 0 \) on \( M \)
   (ii) \( \forall b \in B \omega_b = \omega_{\pi^{-1}(b)} \) is a symplectic form on \( M_b = \pi^{-1}(b) \).
Theorem Consider the case $B = I = [0,1]$ Any family of symplectic structures (in the "tame" sense) over $I$ is trivial. The trivialization is constructed canonically from $\Sigma$.

Proof Let $(M, \omega)$ be such a family.

Let $t \in I = [0,1]$ be a coordinate, and let $\pi^{-1}(t) = \frac{\partial}{\partial t}$ be the standard vector field on $I$.

Claim There is a unique vector field $\tilde{\chi}$ on $M$ such that

(a) $i_{\tilde{\chi}} \omega = 0$

(b) $D\pi(p)(\tilde{\chi}) = X_{\pi(p)} \quad \forall p \in M$

Proof of claim: Linear algebra. $\omega$ is a 2-form on a $(2n+1)$-dimensional space. Since $\omega \mid \pi^{-1}(t)$ is always non-degenerate, rank $\omega = 2n$.

Since the rank is even, rank $\omega = 2n$, and so $\omega$ has a one-dimensional null space at each point. Thus there is a one-dimensional space of $\tilde{\chi}$ such that $i_{\tilde{\chi}} \omega = 0$.

Such $\tilde{\chi}$ cannot be tangent to the fiber, since $\omega$ is non-degenerate on the fiber. Thus $i_{\tilde{\chi}} \omega = 0$ $\Rightarrow$ $D\pi(\tilde{\chi}) = 0$.

Since target is 1-dim, $D\pi(\tilde{\chi}) = \alpha X$ for some $\alpha \neq 0$.

Rescale $\tilde{\chi}$ if necessary to achieve property (b).

End proof of claim.

Now let $s_\epsilon$ be the flow of $\tilde{\chi}$ on $M$. Because $D\pi(\tilde{\chi}) = X$

$s_\epsilon$ covers $\sigma_\epsilon := (\text{flow of } X \text{ on } I)$

$M \xrightarrow{s_\epsilon} M$

$\pi \downarrow \downarrow \pi$

$I \xrightarrow{\sigma_\epsilon} I$

Hence, $s_\epsilon$ maps $M_t = \pi^{-1}(t)$ to $M_{t+s} = \pi^{-1}(t+s)$

(take obvious precautions about flow running off ends of interval)
(Note: If fibers are not compact or have boundary, more care is needed.)

Now note that $f_\ast$ preserves $\Omega$!

\[ L^*_\chi \Omega = d i^*_\chi \Omega + i^*_\chi d \Omega = d i^*_\chi \Omega = 0 \]

So $f_\ast \Omega = \Omega$

\[ M_0 = \pi^{-1}(0) \xrightarrow{i_0} M \quad \qquad i_0^* f_\ast \Omega = i_0^* \Omega = \omega_0 \text{ on } M_0 \]

\[ \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \]

\[ M_s = \pi^{-1}(s) \xrightarrow{i_s} M \quad (f_s|_{M_0})^* i_s^* \Omega = (f_s|_{M_0})^* \omega_s \]

\[ \Rightarrow \quad \omega_s = (f_s|_{M_0})^* \omega_s \]

Conclude $f_s|_{M_0}: (M_0, \omega_0) \to (M_s, \omega_s)$ is a symplectic diffeomorphism for all $s \in [0, 1]$.

Comparison with first proof:

Given: $\omega_t$ family such that $\frac{d \omega_t}{dt}$ is exact.

What we need is to say that we can convert these data into a symplectic fibration (i.e., family in the “famly” sense).

Indeed, define a two-form $\omega$ on $M \times I$. Naively:

For $(p, t) \in M \times I$:

\[ T_{(p, t)}(M \times I) = T_p M \times \mathbb{R} \]

let $\omega_{p, t}$ on $T_p M \times \mathbb{R}$ be $(\omega_t)_p$ on $T_p M$ and zero on $\mathbb{R}$ factor.

Clearly $\omega$ restricted to $M \times \{t\}$ is $\omega_t$. 
Then \( \omega \) is not closed!

\[
\text{ext. d}r. \text{ on } M \times I \quad \text{ext. d}r. \text{ on } M
\]

3-form \( \omega \) on \( M \times I \)

Correcting term: as before, solve \( d_M \beta_t = \frac{d\omega_t}{dt} \) on \( M \)

\( \beta_t \) yields a 1-form \( \beta \) on \( M \times I \).

Define \( \sigma = \omega + dt \wedge \beta \)

\[
d\sigma = d\omega + d(dt \wedge \beta) = dt \wedge \frac{d\omega_t}{dt} - dt \wedge d\beta
\]

\[
(\text{Now } d\beta = d_M \beta_t + dt \wedge \frac{d\beta_t}{dt} = \frac{d\omega_t}{dt} + \omega_t \wedge \frac{d\beta_t}{dt})
\]

\[
d\sigma = dt \wedge \frac{d\omega_t}{dt} - dt \wedge \frac{d\omega_t}{dt} - dt \wedge dt \wedge \frac{d\beta_t}{dt} = 0
\]

Lastly, \( \sigma \) restricted to \( M \times \{ t \} \) is \( \omega_t \), since \( dt \wedge \beta \) restricts trivially