The Oh Spectral sequence.

We want to weaken Floer's hypothesis \( \langle w, \pi_2(M,L) \rangle = 0 \). A useful generalization is the **monotone** condition:

There are two homomorphisms defined on \( \pi_2(M,L) \):

\[
\mu : \pi_2(M,L) \to \mathbb{Z} \quad \text{Maslov index}
\]

\[
\int w : \pi_2(M,L) \to \mathbb{R} \quad \text{symplectic area.}
\]

We say that \( L \subset M \) is **monotone** if

\[
\exists \lambda > 0 : \int_D w = \lambda \mu(D) \quad \text{for all } D \in \pi_2(M,L).
\]

**Note:** This condition includes the case of spheres \( S \in \pi_2(M) \), where it reads

\[
\int_S w = \lambda \cdot 2 \langle c_1(TM), S \rangle \quad (\text{we call } M \text{ monotone if this holds})
\]

This is related to the Fano condition in algebraic geometry:

**Prop:** If \( M \) is a Fano algebraic manifold, then \( M \) admits a monotone Kähler form.

**Proof:** Take a projective embedding corresponding to \((K_M)^{\otimes 2}\) for \( l > 0 \). The class of the pull-back of the Fubini–Study form is

\[
[i^*\omega_G] = l c_1(TM)
\]

Note: \( M \) monotone and \( H^1(L;\mathbb{R}) = 0 \) \( \Rightarrow \) \( L \) monotone.
The purpose of this condition is that it implies a correlation between area and index that obstructs certain types of bubbling.

Define the minimal Maslov # of \( L \):
\[
N_L = \text{nonnegative generator of } \text{Im} (\mu: \pi_2(M_L) \to \mathbb{Z}) \subseteq \mathbb{Z}
\]
- \( N_L = 0 \) means \( \mu \) vanishes identically.
- If \( L \) is orientable, \( N_L \) is even.

Argument that \( HF(L,L) \) can be defined if \( \mu(L) \geq 3 \)
\[
\text{CF}(L, \phi(L)) \supseteq \mathbb{Z} \quad \text{Need } \omega = 0.
\]

Consider space of index 2 strips.

In each of the bad cases, the bubble(s) must carry non-zero energy:
\[
0 < \int_{\text{Bubble}} \omega = \lambda \mu(\text{Bubble}) \quad \lambda > 0
\]
so \( \mu(\text{Bubble}) > 0 \) but \( \mu(\text{Bubble}) \notin N_L \mathbb{Z} \) so \( \mu(\text{Bubble}) = kN_L \), \( k \geq 1 \), \( \mu(\text{Bubble}) \geq 3 \) by hypothesis.
But \( \mu(\text{Bubble}) \) is the index correct by the bubble since the total index is 2, the other component has index \( 2 - \mu(\text{Bubble}) \leq 0 \). For generic \( J \) such a negative dimension strip with nubked point will not exist. (Since the main component is somewhere injective)

So the bad bubbling doesn't occur and \( E \cap d = 0 \).

If \( N_L = 2 \), the situation is more complicated, as bubbling may occur. In some situations it can be argued that the boundaries

\[ \begin{array}{ccc}
\circ & \circ & \circ \\
\circ & \circ & \circ \\
\end{array} \]

Can be independently, so \( \mathcal{E} \cap d \) is still zero.

Let \( L \) be nontrivial with \( N_L \geq 3 \), let \( \phi \) be a \( C^1 \)-small Hamiltonian diffeo generated by a \( C^2 \)-small Hamiltonian (as in proof of Floer's theorem). For example let \( f \) be a \( C^2 \)-small Morse function on \( L \), and extend \( f \) to \( M \) using a Darboux-Weinstein neighborhood and cutting off.

The Floer complex \( CF(L, \phi(L)) \) has generators corresponding to critical points.

Two critical points are connected by a strip if

\[ \mu(x, y, u) = 1 \]
For any pair \( x, y \), the possible values of \( \mu(x, y, n) \) are constrained to lie in a coset of \( \mathbb{Z} \) in \( \mathbb{Z} \).

Critical points of morse index difference 1 have "small" strips between them according to Floer's argument. So generators can only be connected if
\[
\text{morse ind}(x) - \text{morse ind}(y) - 1 = k N_L \quad k \geq 0
\]

Thus the Floer boundary operator splits as a sum
\[
\partial = \partial_0 + \partial_1 + \partial_2 + \cdots
\]

(Homological Convention) \( \partial_k \) has degree \( k N_L - 1 \)

In fact \( \partial_0 \) corresponds to the Morse homology differential as in Floer's theorem.

So, filtering \( CF(L, \phi(L)) \) by morse index, we obtain a spectral sequence whose \( E^1 \) term is
\[
H_*(L; \mathbb{Z}_2)
\]

If \( N_L \geq n+2 \) \( \Rightarrow \) degenerates at \( E^1 \)
If \( N_L = n+1 \) \( \Rightarrow \) only 1 possible differential.

If \( HF(L, \phi(L)) = 0 \) \( \Rightarrow \) \( N_L \leq n+1 \)

Additionally, \( H^*(L; \mathbb{Z}_2) \neq H^*(S^n, \mathbb{Z}_2) \) \( \Rightarrow \) \( N_L \leq n \)
Oh computed  \( HF(\mathbb{RP}^2 \times \mathbb{CP}^2) = H^*(\mathbb{RP}^2; \mathbb{Z}_2) \)

example