The Oh Spectral sequence.

We want to weaken Floer's hypothesis $\langle \omega, \pi_2(M,L)\rangle = 0$

A useful generalization is the <u>monotone</u> condition

There are two homomorphisms defined on $\pi_2(M,L)$

$\mu : \pi_2(M,L) \longrightarrow \mathbb{Z}$ Maslov index

$\int \omega : \pi_2(M,L) \to \mathbb{R}$ symplectic area.

We say that $L \subset M$ is <u>Monotone</u> if

$$\exists \lambda \geq 0 : \int_D \omega = \lambda \mu(D) \quad \text{for all } D \in \pi_2(M,L).$$

<u>Note</u>: this condition includes the case of spheres $S \in \pi_2(M)$, where it reads

$$\int_S \omega = \lambda \cdot 2 \langle c_1(TM), S \rangle$$

(We call M monotone if this holds)

This is related to the Fano condition in algebraic geometry

<u>Prop</u> If M is a Fano algebraic manifold, then M admits a monotone Kähler form.

<u>Proof</u> Take a projective embedding corresponding to $(K_M)^{\otimes(-\ell)}$ $\ell \gg 0$
The class of the pull back of Fubini-Study form is

$$[i^*\omega_{FS}] = \ell \, c_1(TM)$$

Note M monotone and $H^1(L;\mathbb{R}) = 0 \Rightarrow L$ monotone.

The purpose of this condition is that it implies a correlation between area and index that obstructs certain types of bubbling.

Define the minimal Maslov # of L:

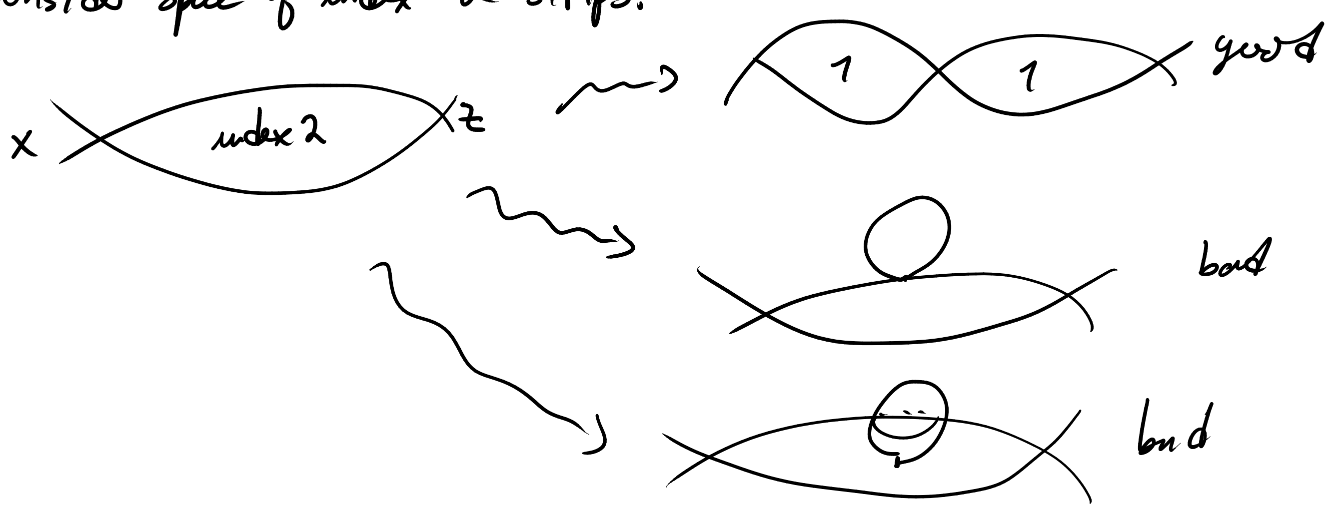
N_L = non negative generator of $\mathrm{Im}(\mu : \pi_2(M,L) \to \mathbb{Z}) \subseteq \mathbb{Z}$

- $N_L = 0$ means μ vanishes identically.
- If L is orientable, N_L is even.

Argument that $HF(L,L)$ can be defined if $N_L \geq 3$

$CF(L, \phi(L)) \circlearrowleft \partial$ Need $\partial \circ \partial = 0$.

Consider space of index 2 strips.



in each of the bad cases, the bubble(s) must carry non zero energy.

$$0 < \int_{\text{Bubble}} \omega = \lambda \, \mu(\text{Bubble}) \qquad \lambda \geq 0$$

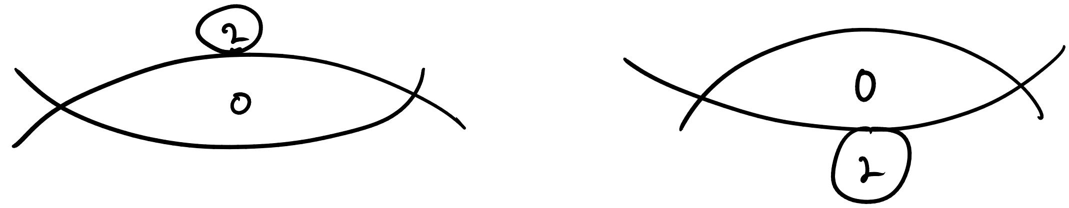
so $\mu(\text{Bubble}) > 0$ But $\mu(\text{Bubble}) \in N_L \mathbb{Z}$ so

$\mu(\text{Bubble}) = k N_L$, $k \geq 1$, $\mu(\text{Bubble}) \geq 3$ by hypothesis

But $\mu(\text{Bubble})$ is the index carried by the bubble. Since the total index is 2, the other component has index $2 - \mu(\text{Bubble}) < 0$. For generic J such a negative dimensional strip with marked point will not exist. (Since the main component is somewhere injective)

So the bad bubbling doesn't occur and $\partial \circ \partial = 0$.

If $N_L = 2$, the situation is more complicated, as bubbling may occur. In some situations it can be argued that the boundaries



cancel independently, so $\partial \circ \partial$ is still zero.

Let L be monotone with $N_L \geq 3$, let ϕ be a C^1-small Hamiltonian diffeo generated by a C^2-small Hamiltonian (as in proof of Floer's theorem). For example let f be a C^2-small Morse function on L, and extend f to M using a Darboux-Weinstein neighborhood and cutting off.

The Floer complex $CF(L, \phi(L))$ has generators corresponding to critical points.

Two critical points are connected by a strip if

 $\mu(x, y, u) = 1$

For any pair x, y, the possible values of $\mu(x,y,u)$ are constrained to lie in a coset of $N_L\mathbb{Z} \subset \mathbb{Z}$

Critical points of morse index difference 1 have "small" strips between them according to Floer's argument. So generators can only be connected if

$$\text{morse ind}(x) - \text{morse ind}(y) - 1 = kN_L \qquad k \geq 0$$

Thus the Floer boundary operator splits as a sum

$$\partial = \partial_0 + \partial_1 + \partial_2 + \cdots$$

(Homological Convention) $\quad \partial_k$ has degree $kN_L - 1$

In fact ∂_0 corresponds to the Morse homology differential as in Floer's theorem.

So, filtering $CF(L, \phi(L))$ by morse index, we obtain a spectral sequence whose E^1 term is

$$H_*(L; \mathbb{Z}_2)$$

If $N_L \geq n+2 \Rightarrow$ degenerates at E^1

If $N_L = n+1 \Rightarrow$ only 1 possible differential.

If $HF(L, \phi(L)) = 0 \Rightarrow N_L \leq n+1$

If additionally : $H^*(L; \mathbb{Z}_2) \neq H^*(S^n, \mathbb{Z}_2) \Rightarrow N_L \leq n$

Oh computed $HF(\mathbb{RP}^2 \subset \mathbb{CP}^2) = H^*(\mathbb{RP}^2; \mathbb{Z}_2)$

example