Invariance of Floer homology in the exact case
Application to cotangent bundles.

For this lecture we assume that \((M, \omega)\) is exact
and we fix a choice of 1-form \(\Theta\) such that \(\omega = d\Theta\).

With respect to the choice of \(\Theta\), a Lagrangian \(L\) is exact if
\(\Theta|_L = df_L\) for some function \(f_L: L \rightarrow \mathbb{R}\).

The exactness assumptions imply the asphericity conditions
\(\langle \omega, \pi_2(M) \rangle = 0\)
\(\langle \omega, \pi_2(M, L) \rangle = 0\)
So the arguments from the last lecture show that \(\partial \circ \partial = 0\)
on the Floer complex \(CF(L_0, L_1)\) if \(L_0\) and \(L_1\) are exact.

The main example for today: \(M = T^*Q\)
\(\Theta = \sum -p_i dq_i\)
\(\omega = \sum dq_i \wedge dp_i\)
\(L_0 = Q (0\text{-section})\)
\(L_1 = \phi_H^1(Q), \text{where}\)
\(\phi_H^1\) is the time-1 flow of a Hamiltonian \(H: M \rightarrow \mathbb{R}\).

Then (Floer) \(HF(Q, \phi_H^1(Q)) \simeq H^*(Q; \mathbb{Z}_2)\)

The proof relies on
(1) Invariance of HF with respect to change in complex structure
(2) Invariance of HF with respect to change of \(H\).
(3) Computation of \(HF(Q, \phi_H^1(Q))\) for particular convenient choices of \(J\) and \(H\).
To prove invariance: Continuation maps and homotopy operators.

Assume $J_0$ and $J_1$ are two almost complex structures for which the Floer complex $CF(0, l_1)$ can be defined. Call the corresponding Floer boundary operators $\mathcal{D}(J_0)$ and $\mathcal{D}(J_1)$.

We want to show $(CF(0, l_1), \mathcal{D}(J_0))$ is chain homotopy equivalent to $(CF(0, l_1), \mathcal{D}(J_1))$.

We define a chain map by looking at strips with $J$ depending on the point in the domain.

\[
\begin{array}{c}
\text{use } J_0 \\
\text{here}
\end{array}
\begin{array}{c}
\text{interpolate } J_0 \rightarrow J_1
\end{array}
\begin{array}{c}
\text{use } J_1 \\
\text{here}
\end{array}
\Rightarrow M
\]

use front space of compatible $J$ is connected.

Define continuation map $\chi_{0,1}: (CF(0, l_1), \mathcal{D}(J_0)) \rightarrow (CF(0, l_1), \mathcal{D}(J_1))$ by counting 0-dimensional components of the moduli space

\[
\chi_{0,1}(x) = \sum \# \text{moduli space}
\]

Note that, unlike $\mathcal{D}$, there is no IR-symmetry, so we can expect to get 0-dimensional components.
Why is $\mathcal{X}_{0,1}$ a chain map?

Consider 1-dimensional components of $\overline{\mathcal{M}}\{T_{s} \mid s \in \mathbb{R}\}$

The boundary is

\[ \partial(J_0) \sim \mathcal{X}_{0,1} \]

\[ + \mathcal{X}_{0,1} \sim \partial(J_1) \]

So $\partial(J_1) \circ \mathcal{X}_{0,1} + \mathcal{X}_{0,1} \circ \partial(J_0) = 0$

Swapping roles of $J_0$ and $J_1$, we get a continuation map $\mathcal{X}_{1,0} : (CF(L_{0,1}), \partial(J_1)) \to (CF(L_{0,1}), \partial(J_0))$

We want to show $\mathcal{X}_{0,1} \circ \mathcal{X}_{1,0}$ and $\mathcal{X}_{1,0} \circ \mathcal{X}_{0,1}$ are homotopic to identity maps. This means finding a map $\mathcal{P} \in (CF(L_{0,1}), \partial(J_0))$ such that

\[ \partial(J_0) \circ \mathcal{P} + \mathcal{P} \circ \partial(J_0) = \text{Id} - \mathcal{X}_{1,0} \circ \mathcal{X}_{0,1} \]

For this we consider a 1-parameter family of 1-parameter families of $\{T_{s} \mid s \in \mathbb{R}\}$.

Let $R \in [0, \infty)$ be the new parameter.

At $R=0$ $\mathcal{J} \equiv \mathcal{J}_0$

\[ \text{constantly} \]

$\mathcal{J}_0$
As \( R \rightarrow 1 \) the constant path \( J_0 \) is defined to a path that interpolates from \( J_0 \) to \( J_1 \) via the same \( \{ J_s \} \) used to define \( X_{0,1} \), then back to \( J_0 \) via the path used to define \( X_{1,0} \).

\[
\begin{array}{cccccc}
J_0 & J_5 & J_1 & J_5 & J_0 \\
\downarrow & & & & \\
1 & \leq R & \leq 1
\end{array}
\]

Then as \( R \rightarrow \infty \), the region where \( T = J_1 \) increases in length with \( R \) bounded.

This defines a 1 parameter family of strip changing problems parameterized by \( R \in [0, \infty) \).

Define \( P \) by counting 0-dimensional components of the parametrized moduli space. This means we count "exceptional" strips: those that a rigid even with repeat to the variation of the \( R \)-parameter.

In the graded situation, where \( \mathcal{T} \) has degree \(-1\), \( P \) will have degree \(+1\).

Now to prove the homotopy formula

\[
2P + PD = \text{Id} - X_{1,0} \circ X_{0,1}
\]

we use the Gromov-Compactification of the 1-dimensional components of the \( R \)-parametric moduli space.

The Gromov boundary consists of

- \( R = 0 \) end
- \( R = \infty \) end
- Floer differential strip breaking
The \( R=0 \) end counts
strips with constant \( J = J_0 \), like the differential.
But since we are looking at index 0 strips, they
must be constant. Thus the \( R=0 \) end contributes
the identity map \( \text{Id} \).

At the \( R=\infty \) and, we have compactified by adding
a stratum corresponding to pairs of strips

\[
J_0 \quad | \quad J_3 \quad | \quad J_1 \quad \rightarrow \quad \cdots \quad | \quad J_1 \quad | \quad J_3 \quad | \quad J_0
\]

This is precisely the moduli space defined
\( \mathcal{X}_{J_0 \circ J_0} \).

Floor differential breaking will result in boundaries
consisting of a floor differential (constant \( J \))
joined to an exceptional strip in the \( R \)-family.

\[
\begin{aligned}
\frac{J_{0}}{\text{Contributes to } \mathcal{X}(J_0)} & \quad \left\{ \begin{array}{c}
J_{0} \quad | \quad J_{3} \quad | \quad J_{1} \quad \mid_{R=1} \quad \cdots \\
\text{For some particular value of } R
\end{array} \right. \\
\frac{J_{0}}{\text{Contributes to } \mathcal{X}(J_0)} & \quad \left\{ \begin{array}{c}
P \quad \mid_{R=1} \quad \cdots \\
\end{array} \right.
\end{aligned}
\]

\[
\begin{aligned}
\frac{J_{0} \quad | \quad J_{3} \quad | \quad J_{1} \quad \mid_{R=1} \quad \cdots}{P} & \quad \left\{ \begin{array}{c}
\mathcal{X}(J_0) \cdot P
\end{array} \right.
\end{aligned}
\]
Counting the boundary points modulo 2, we obtain

\[ \text{Id} + \chi_{1,0} \circ \chi_{0,1} + \chi(J_0) \circ \rho + \rho_0 \chi(J_0) = 0 \quad \text{as desired} \]

Swapping the roles of \( J_0 \) and \( J_1 \), we get a homotopy between \( \text{Id} \) and \( \chi_{0,1} \circ \chi_{1,0} \).

Thus both \( \chi_{0,1} \) and \( \chi_{1,0} \) are chain homotopy equivalences.

We can use a similar continuation map argument to show that \( HF(L_0, L_1) \) is invariant under Hamiltonian deformation of \( L_0 \) or \( L_1 \). Let \( \phi \) be the Hamiltonian diffeomorphism generated by a time-dependent Hamiltonian \( H_t \). We assume that \( L_0 \cap \phi_1 L_1 \) and \( L_0 \cap \phi(t) L_1 \).

Then \( \phi = \phi_1 \), where \( \{ \phi_t \}_{t \in [0, 1]} \) is the isotopy generated by the time-dependent vector field \( X_t \)

\[ X_t : c(-, X_t) = d H_t \]

The sort of "continuation strips" we look at now have a fixed almost complex structure \( J \), but a moving Lagrangian boundary condition along the edge corresponding to \( L_1 \)

\[
\begin{array}{ccc}
L_1 & \phi_t(L_1) & \phi(L_1) \\
\hline
\vdots & \vdots & \vdots \\
L_0 & \chi_0 : (CF(L_0, L_1), \partial(J)) \rightarrow (CF(L_0, \phi(L_1), \partial(J))
\end{array}
\]
There is similarly $\phi_{\gamma^1} : CF(L_0, \phi(L_1)) \to CF(L_0, L_1)$

Similar arguments as before show that $\phi_L$ & $\phi_{\gamma^1}$ are chain maps that are mutually homotopy-inverse homotopy equivalences.

Now we want to compute the "self-Floer homology"

$HF(L,L)$ for $L \subset \mathbb{R}^4$

Since $L$ and $\mathbb{R}^4$ are not transverse, this symbol has no meaning so far. But we can define it to be

$HF(L, \phi(L))$

where $\phi(L)$ is a transverse Hamiltonian push-off of $L$, and the group is defined using any regular $J$. This is meaningful because we now know that the result is independent of these choices.

Moreover, since we can take any $J$ and $\phi$ we want (as long as they satisfy the regularity conditions) the game is now to find a particularly clever choice that allows us to compute.

Here we follow Floer, "Witten's complex and co-dimension Morse theory."

To start choose a metric $g$ on $L$ and a Morse function $f : L \to \mathbb{R}$

We assume $(g, f)$ is Morse-Smale, so the Morse complex is defined. $C^\infty(L, f) \triangleright D(g, f)$
On $T^*L$, these same data $(g, f)$ give us a hamiltonian pushoff of $L$, and an almost complex structure as follows:

The metric $g$ on $L$ induces a metric $\tilde{g}$ on $T^*L$ such that the splitting $T(T^*L) = T^\text{vert}(T^*L) \oplus T^\text{horiz}(T^*L)$ is $g$-orthogonal, where $T^\text{horiz}(T^*L)$ is defined by the Levi-Civita connection of $g$ and

$$T^\text{vert}_x(T^*L) = T^*_x L$$ and $T^\text{horiz}_x(T^*L) = T^*_x L$

have metrics induced by $g$.

Let $H = f \circ \pi$ (time-independent), $\phi_t$ its flow. Then $\phi_t(L) = \text{graph of } f$.

Let $0$

Define $J_t = (\phi_t^*)_* J (\phi_t^*)^*$

We consider $CFL, \phi_t(L) \in \mathcal{E}(\{J_t\}_{t \in [0,1]})$

- Generators $\leftrightarrow$ critical points of $f$

There is a relationship between gradient flux lines for the more small pair $(g, f)$ on $L$ and $J_t$-strips in this Floer complex.

$$\begin{align*}
\gamma : R &\rightarrow L \\
u (s, t) &= \phi_t(\gamma(s))
\end{align*}$$

$$\begin{align*}
\frac{d}{ds} \gamma(s) + (\text{grad}_g f)(\gamma(s)) &= 0 \\
\frac{\partial u}{\partial s} + J_t \frac{\partial u}{\partial t} &= 0.
\end{align*}$$
Floor shows that if $f$ is $C^2$-small, this is a bijection between thin lines and strips.

Thus \((\mathcal{CF}(L, \phi, (L)), \mathcal{O}(\{F\})) \rightleftharpoons (C_*(\omega^{\text{morse}}(L, g), \mathcal{O}(g, f)))\)
are isomorphic chain complexes.

Thus \(\text{HF}(L, \phi, (L)) \rightleftharpoons H_*(\omega^{\text{morse}}(L)) \rightleftharpoons H_{\#}(L)\)