Index theorems for Pseudoholomorphic curves

We know that the inverse image of a regular value of a Fredholm map is a smooth manifold whose dimension is the index of the linearization \( \text{ind}(D_u\bar{\partial}_J) \).

But what is this number? The index theory of Atiyah-Singer provides a framework for answering such questions.

Background: we saw that \( \text{ind}(D) \) is unchanged by non-continuous deformations of \( D \) staying within the space of Fredholm operators. It was observed that homotopy of the geometric data defining \( D \) (e.g. change in \( J \), metric, etc.) therefore do not change the index. So Gelfand proposed to find a topological formula for \( \text{ind}(D) \) in terms of the geometry of the problem.

Atiyah-Singer solved this. The result could also be cast as a vast generalization of the Riemann-Roch Theorem.

**Thm (Riemann-Roch)** Let \( (S,j) \) be a compact Riemann surface \( u: S \to M \) a map to the almost complex manifold \( (M,J) \). The index of the linearized operator \( D_u\bar{\partial}_J \) is

\[
\text{ind}_R(D_u\bar{\partial}_J) = (\dim \mathcal{E}_M) \chi(S) + 2 \left< c_1(TM), u_*[S] \right>
\]

(If \( D_u\bar{\partial}_J \) is \( C \)-linear: \( \text{ind}_C = (\dim \mathcal{E}_M)(1-g) + \deg(u^*TM) \) )
We want a generalization that will cover holomorphic curves with boundary on Legendrian submanifolds, and also with asymptotic caustics at intersection points.

Assume $S$ is compact with boundary $\partial S = \bigsqcup_i B_i$

![Diagram of boundary components $B_1$, $B_2$, and $B_3$]

and each boundary $B_i$ is mapped to a Legendrian $L_i$.

The linearized operator at $u: S \to M$ acts on sections of $u^*TM$ satisfying the restriction that sections along $B_i$ are null in $u^*TL_i$.

Thus we have a complex vector bundle $u^*TM$ over $S$ with subbundles $u^*TL_i$ along $B_i$.

Since $S$ is not closed, $u^*TM$ can be trivialized over $S$. But the subbundles $u^*TL_i$ "go along for the ride", we cannot trivialize them simultaneously.

The non-triviality of the subbundle $u^*TL_i$ is measured by the Maslov index $\mu: \pi_1(U_n/O_n) \to \mathbb{Z}$.

Each $B_i$ with its induced boundary orientation determines a loop in the Legendrian Grassmannian of $C^n$, where we trivialize $u^*TM \cong S \times C^n$. 
Thus (Riemann–Roch with boundary) In the set up just described

$$\tilde{\mu} \cdot \delta_J (\partial \tilde{\nu}^S) = (\partial \tilde{\mu} \cdot \nu^S) \chi(S) + \sum \mu(B_i)$$

Now we consider the case of strips

$$u: \quad \text{strips} \quad \rightarrow \quad \text{loop with corners}$$

$$S = \mathbb{R} \times [0, 1] \times (0, t)$$

The boundary is not a loop, and the image has “corners” at intersection points \(x, y \in L_0 \cap L_1\).

We will generalize the Maslov index to this situation.
Consider the complex vector bundle \(u^* TM\) over \(S\)

Once again we may trivialize \(\tilde{A}\). Thus we get subbundles \(u^* T_{L_0}\) over \(\mathbb{R} \times \{0\}\) and \(u^* T_{L_1}\) over \(\mathbb{R} \times \{1\}\).

We assume that

(i) \(L_0\) and \(L_1\) intersect transversely at \(x\) and \(y\)

(ii) The complex structure \(J\) on \(M\) is such that

$$J \cdot T_x L_0 = T_x L_1 \quad \text{in} \quad T_x M$$

$$J \cdot T_y L_0 = T_y L_1 \quad \text{in} \quad T_y M.$$  

(iii) For \(|S| > R\), the subbundles \(u^* T_{L_0}\) and \(u^* T_{L_1}\) are constant w.r.t. the chosen trivialization of \(u^* TM\).
Now we define a loop in the Lagrangian Grassmannian.

\[ \text{follow } u^*T_{L_1} C C^n \]

\[ \text{go from } T_{xL_1} \text{ to } T_{xL_0} \]

\[ \text{by } e^{ \frac{it}{2g} T_{xL_1} } \]

\[ \text{follow } u^*T_{L_0} C C^n \]

\[ \text{go from } T_y L_0 \text{ to } T_y L_1 \]

\[ \text{by path } e^{ \frac{it}{2g} T_y L_0 } \]

The nodal index \( \mu(x, y, u) \) is \( \mu \) of this loop.

Fleiger's Riemann-Roch theorem is:

**Theorem**: In the above situation: \( D^d_o \) has a Fredholm extension

\[ L^2(S, u^*TM, u^*T_{L_0}, u^*T_{L_1}) \to L^2 - S^0(S, u^*TM) \]

and

\[ \text{Ind } (D^d_o) = \mu(x, y, u) \]

**Remarks**: Assumption (i) is necessary for the Fredholm property.

(Otherwise, use weighted Sobolev spaces.)

The term \((\dim V) \chi(S)\) is not present because of the convention defining \( \mu(x, y, u) \)

**Dependence on the homotopy class of \( u \):**

In general, \( \mu(x, y, u) \) does not depend only on \( x \) and \( y \).

For example, if there are homotopically nontrivial maps

\[ u: (D^2, S^1) \to (M, L) \]

or

\[ u: S^2 \to M \]
Then taking connected sum of $w$ with such surfaces will change the homotopy class of $w$, and (possibly) the index.

Using a linear grading analysis, one can show that these connected sum operations change the index by the formulas:

**Disk $v$:** $\mu(x, y, w \# v) = \mu(x, y, w) + \mu(v)$

**Sphere $w$:** $\mu(x, y, w \# w) = \mu(x, y, w) + 2 \langle c_1(TM), w^* [S^2] \rangle$

**Consequences for the grading on Floer homology $HF^*(L_0, L_1)$**

The differential counts strips in classes such that $\mu(x, y, w, u) = 1$. To make $CF^*(L_0, L_1)$ $\mathbb{Z}$-graded, we need to introduce function $\mu(x)$ such that $\mu(x, y, u) = \mu(x) - \mu(y)$. 
The ambiguity makes this impossible, unless we put in some conditions on $c_1(TM)$ and $\mu(V)$ for $V : (D^3, s^1) \to (M, L_i)$.

E.g. assume $2c_1(TM) = 0$ (similar to Calabi-Yau).

This means that there is a well defined homomorphism

$\mu_L : H_1(L) \to \mathbb{Z}$

such that (Maslov class of $L$)

$\mu(L) = \mu_L(\partial V)$

for any disk $V$.

Then assume $\mu_L = 0$ and $\mu_L = 0 \Rightarrow$ get $\mathbb{Z}$-graded Floer homology.

In general we can only get a grading in $\mathbb{Z}/N\mathbb{Z}$ where $N\mathbb{Z}$ is the subgroup generated by the ambiguity terms $2\langle c_1(TM), w_*[S^2] \rangle$ and $\mu(V)$.

If $L_0$ and $L_1$ are oriented, $\mu(V) \in 2\mathbb{Z}$ for $V$ of course $2\langle c_1(TM), w_*[S^2] \rangle \in 2\mathbb{Z}$

So $2|N$ in this case, and Floer homology $CF_*(L_0, L_1)$ is at least $\mathbb{Z}/2\mathbb{Z}$ graded.