Homework Assignment 1

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1 Hamiltonian and symplectic diffeomorphisms

Let (M, ω) be a symplectic manifold of dimension $2n$. A symplectic diffeomorphism is a map $\rho : M \to M$ such that $\rho^*\omega = \omega$. The group of symplectic diffeomorphisms is denoted $\mathrm{Symp}(M, \omega)$, and the connected component containing the identity element is denoted $\mathrm{Symp}_0(M, \omega)$. A symplectic isotopy is a path ρ_t of symplectic diffeomorphisms such that $\rho_0 = \mathrm{Id}$. As we discussed in lecture, a symplectic isotopy is generated by a time-dependent symplectic vector field, that is, a vector field X_t such that $d(i_{X_t}\omega) = 0$ for each t. Evidently, any element $\phi \in \mathrm{Symp}_0(M, \omega)$ is obtained as $\phi = \rho_1$ for some symplectic isotopy $(\rho_t)_{t\in[0,1]}$.

Of particular interest for us is the subgroup $\mathrm{Ham}(M, \omega) \subset \mathrm{Symp}_0(M, \omega)$ of Hamiltonian diffeomorphisms. This should contain the Hamiltonian flow of any (time-independent) function $H : M \to \mathbb{R}$, however, that set is (apparently) not closed under composition. For this reason we allow time-dependent Hamiltonians.

For the precise definition, let $H : \mathbb{R} \times M \to \mathbb{R}$ be a time-dependent function. We write H_t for the function on M at fixed $t \in \mathbb{R}$. Let X_t be the Hamiltonian vector field characterized by $-i_{X_t}\omega = dH_t$, and let ρ_t be the flow of X_t. Then ρ_t is called a Hamiltonian isotopy. The correspondence $H \mapsto \rho_1$ defines a map from time-dependent functions to $\mathrm{Symp}_0(M, \omega)$, and $\mathrm{Ham}(M, \omega)$ is by definition the image of this map.

Problem 1. Show that, for any $t \in \mathbb{R}$, $\rho_t \in \mathrm{Ham}(M, \omega)$. The point is that this is true by definition for $t = 1$, so what about $t = -14$? Hint: rescale.

Problem 2. Show that $\mathrm{Ham}(M, \omega)$ is a group.

1. Identity: should be obvious.

2. Composition: Let ρ_t be generated by H_t, and let σ_t be generated by K_t. Show that $\rho_t \circ \sigma_t$ is a Hamiltonian isotopy. Hint: add the functions, but with a twist.

3. Inverses: Let ρ_t be generated by H_t. Show that ρ_t^{-1} is a Hamiltonian isotopy. Hint: negate the function, but with a twist.

The groups $\mathrm{Symp}_0(M,\omega)$ and $\mathrm{Ham}(M,\omega)$ may be regarded as infinite-dimensional Lie groups. The corresponding Lie algebras may be identified as the spaces of symplectic and Hamiltonian vector fields respectively.

$$\mathfrak{symp}(M,\omega) = \{X \mid i_X\omega \text{ is closed}\} \cong \Omega^1_{\text{closed}}(M) \tag{1}$$

$$\mathfrak{ham}(M,\omega) = \{X \mid i_X\omega \text{ is exact}\} \cong \Omega^1_{\text{exact}}(M) \cong C^\infty(M)/\mathbb{R} \tag{2}$$

Problem 3. Show that $\mathfrak{ham}(M,\omega)$ is a Lie ideal in $\mathfrak{symp}(M,\omega)$. Thus, the quotient, which is identifiable as $H^1(M;\mathbb{R})$, is a Lie algebra. What is the induced Lie bracket on $H^1(M;\mathbb{R})$?

Problem 4. Show that if $H^1(M;\mathbb{R}) = 0$, then $\mathrm{Ham}(M,\omega) = \mathrm{Symp}_0(M,\omega)$.

Consider $\widetilde{\mathrm{Symp}}_0(M,\omega)$, the universal cover of the identity component of the symplectic diffeomorphism group. There is a homomorphism from this group to $H^1(M;\mathbb{R})$, called flux, defined as follows. A point in the universal cover is a symplectic isotopy $(\phi_t)_{t\in[0,1]}$ considered up to homotopy relative the endpoints. Working with a particular representative, let X_t be the corresponding symplectic vector field. Define

$$\mathrm{flux}((\phi_t)_{t\in[0,1]}) = \int_0^1 [i_{X_t}\omega]\, dt \in H^1(M;\mathbb{R}) \tag{3}$$

Problem 5. Let $\gamma : S^1 \to M$ be a smooth map, representing a 1-cycle $[\gamma] \in H_1(M;\mathbb{R})$, and let $(\phi_t)_{t\in[0,1]}$ be a symplectic isotopy. Define a map $\phi(\gamma) : [0,1] \times S^1 \to M$ by

$$\phi(\gamma)(t,\theta) = \phi_t(\gamma(\theta)) \tag{4}$$

This map defines a 2-chain in M also denoted $\phi(\gamma)$. Show that

$$\langle \mathrm{flux}((\phi_t)_{t\in[0,1]}), [\gamma]\rangle = \int_{\phi(\gamma)} \omega \tag{5}$$

Deduce that the flux homomorphism is well-defined on $\widetilde{\mathrm{Symp}}_0(M,\omega)$, that is, show that it is invariant under homotopy rel endpoints.

Problem 6. Show that flux is indeed a homomorphism.

The obstruction to descending the flux homomorphism from $\widetilde{\mathrm{Symp}}_0(M,\omega)$ to $\mathrm{Symp}_0(M,\omega)$ is evidently the nontriviality of flux on the kernel of the covering map, that is, $\pi_1(\mathrm{Symp}_0(M,\omega))$. Let $\Gamma \subset H^1(M;\mathbb{R})$ be the image of the fundamental group under flux. Then there is a well-defined homomorphism $\mathrm{Symp}_0(M,\omega) \to H^1(M;\mathbb{R})/\Gamma$. It is a theorem due to Banyaga that the kernel is precisely $\mathrm{Ham}(M,\omega)$.

$$\mathrm{Symp}_0(M,\omega)/\,\mathrm{Ham}(M,\omega) \cong H^1(M;\mathbb{R})/\Gamma \tag{6}$$

2 Examples of symplectic manifolds

2.1 Cotangent bundles

Let Q be a smooth manifold of dimension n. The total space of the cotangent bundle T^*Q is a manifold of dimension $2n$. It has a canonical symplectic structure, which can be seen in a few ways.

Let $(q^i)_{i=1}^n$ be an arbitrary local coordinate system on Q. This yields a coordinate coframe (basis of 1-forms at every point) $(dq^i)_{i=1}^n$. An arbitrary 1-form may then be represented as $\sum_{i=1}^n p_i\,dq^i$, where p_i are the cooeficients of the representation in the coframe. Thus $(q^i, p_i)_{i=1}^n$ are local coordinates on T^*Q. But now, $(dq^i, dp_i)_{i=1}^n$ is a coordinate coframe on T^*Q. Define a 1-form on T^*Q

$$\lambda = \sum_{i=1}^n p_i\,dq^i \tag{7}$$

The symplectic form on T^*Q is

$$\omega = d\lambda = \sum_{i=1}^n dp_i \wedge dq^i \tag{8}$$

Note that ω not merely closed by actually exact.

Problem 7. Verify by tensor calculus that λ and ω do not dependent on the chosen local coordinate system on Q.

Here is an alternative definition. The cotangent bundle T^*Q comes with a structure map $\pi : T^*Q \to Q$. Let $D\pi : T(T^*Q) \to TQ$ denote its differential. Define a 1-form on T^*Q as follows. A point $x \in T^*Q$ may be regarded as a pair (q,α), where $q \in Q$ and $\alpha \in \mathrm{Hom}(T_qQ, \mathbb{R})$. Set, for any vector $X \in T_{(q,\alpha)}(T^*Q)$,

$$\lambda_{(q,\alpha)}(X) = \alpha(D\pi(X)) \tag{9}$$

that is, project X down to Q and evaluate α on it. This defines a 1-form λ on T^*Q.

Problem 8. Verify that the two definitions of λ agree. This gives another proof that λ is canonical.

Let g be a Riemannian metric on the manifold Q. Thus g defines a symmetric bilinear form on each tangent space T_qQ. Using the metric to identify T_qQ with T_q^*Q, we obtain a symmetric bilinear form on T_q^*Q. Regarding this as a quadratic function, we get a function H on the cotangent bundle. In the local coordinates (q^i, p_i), this is

$$H(q,p) = \sum_{i,j=1}^n g^{ij}(q) p_i p_j \tag{10}$$

Problem 9. Show that the Hamiltonian flow of H coincides with the geodesic flow for the metric g.

2.2 Surfaces

Let M denote a two-dimensional manifold.

Problem 10. Show that M admits a symplectic form iff it is orientable.

Problem 11. Classify closed symplectic two-manifolds up to symplectic diffeomorphism. (Starting from the smooth classification of course, and using Moser's theorem.)

2.3 Complex projective spaces

This a fundamental example, which is well-covered in many references. Consider $M = \mathbb{C}^n$, with the symplectic form coming from the standard Hermitian inner product $h(z, w) = z \cdot \overline{w}$. If $(z_i)_{i=1}^n$ denote complex coordinates with real and imaginary parts $z_i = x_i + \sqrt{-1}y_i$, then $\omega = \sum_{i=1}^n dx_i \wedge dy_i$.

The complex projective space $\mathbb{CP}^{n-1}$ is the space of lines in $\mathbb{C}^n$ (through the origin). One way to construct this is to consider nonzero vectors modulo rescaling by nonzero complex numbers. Alternatively, one may consider vectors of norm 1 modulo rescaling by complex numbers of norm 1, and in this way we can see the symplectic form. The set of vectors of norm 1 is the sphere

$$S^{2n-1} = \{x_1^2 + y_1^2 + \cdots + x_n^2 + y_n^2 = 1\} \tag{11}$$

the unit complex numbers form the group $U(1) = \{e^{i\theta}\}$, and $\mathbb{CP}^{n-1}$ is the quotient space $S^{2n-1}/U(1)$.

The Fubini-Study symplectic form ω_{FS} on $\mathbb{CP}^{n-1}$ is characterized as follows. Suppose that X and Y are two vectors at a point $p \in \mathbb{CP}^{n-1}$. To compute their symplectic pairing $\omega_{\mathrm{FS}}(X, Y)$, pick a point $\tilde{p} \in S^{2n-1}$ mapping to p, lift X and Y to vectors $\tilde{X}$ and $\tilde{Y}$ tangent to S^{2n-1} at p, and compute the symplectic pairing $\omega(\tilde{X}, \tilde{Y})$ with respect to the symplectic form on $\mathbb{C}^n$. When you say it like this, it seems surprising that this is even close to being well-defined.

Problem 12. Show that ω_{FS} is well-defined, and is a symplectic form, using the following steps.

1. Consider the function $H = \sum_{i=1}^n |z_i|^2$. Show that the Hamiltonian flow generated by H is (after a suitable reparametrization) the same as the $U(1)$ action rescaling all the coordinates by a unit complex number. This shows that the $U(1)$ action preserves the symplectic form and the hypersurface $S^{2n-1} = H^{-1}(1)$. (Though these facts can be readily checked in other ways.)

2. Consider the restriction $i^*\omega$, where $i : S^{2n-1} \to \mathbb{C}^n$ is inclusion map. This is a two-form on the odd-dimensional manifold S^{2n-1}, so it must have a degenerate direction. Show that the degenerate subspace is spanned by X_H, the Hamiltonian vector field for H, which by the previous part also spans the tangent space to the $U(1)$ orbit.

3. Using the previous two parts, show that ω_{FS} is well defined and nondegenerate.

4. If π denotes the projection $S^{2n-1} \to \mathbb{CP}^{n-1}$, show that $\pi^* \omega_{\mathrm{FS}} = i^* \omega$ as differential forms on S^{2n-1}. Use this to show $\pi^* d\omega_{\mathrm{FS}} = 0$ on S^{2n-1}, and show how to conclude that $d\omega_{\mathrm{FS}} = 0$ on $\mathbb{CP}^{n-1}$.

It is fact that any complex submanifold of $\mathbb{CP}^{n-1}$ is also a symplectic submanifold with respect to the Fubini-Study symplectic form.

3 Examples of Lagrangian submanifolds

3.1 Cotangent bundles

Consider the cotangent bundle T^*Q with its standard symplectic structure introduced above. There are many interesting Lagrangian submanifolds in T^*Q.

Problem 13. Verify that the base Q sitting inside T^*Q as the zero-section, is Lagrangian. Verify that for $q \in Q$, the fiber of the cotangent bundle T_q^*Q is Lagrangian.

Let $\alpha \in \Omega^1(Q)$ be a one-form on Q. One normally thinks of this as a tensor field on Q, but it also defines a submanifold of T^*Q in an tautological way: define $\mathrm{graph}(\alpha)$ to be the set of pairs $(q, \alpha_q) \in T^*Q$ where q varies over Q. This has the characteristic property that the projection $\pi : T^*Q \to Q$ induces a diffeomorphism $\mathrm{graph}(\alpha) \to Q$.

Problem 14. Show that $\mathrm{graph}(\alpha)$ is Lagrangian iff α is a closed as a one-form on Q.

Problem 15. If α is exact as a one-form on Q, show that there is a Hamiltonian diffeomorphism ϕ of T^*Q such that $\phi(Q) = \mathrm{graph}(\alpha)$.

The Liouville class of a Lagrangian embedding $i : L \to T^*Q$ is the class $[i^*\lambda] \in H^1(L;\mathbb{R})$, where $\lambda = p\,dq$ is the primitive of the symplectic form (also known as the Liouville one-form).

Problem 16. Find the Liouville class of $\mathrm{graph}(\alpha)$. Investigate how the Liouville class behaves under symplectic and Hamiltonian isotopies. Deduce that the "moduli space" of Lagrangian submanifolds which are graphs over the zero section—modulo Hamiltonian isotopy—may by identified with $H^1(Q;\mathbb{R})$.

3.2 Graph of a symplectic diffeomorphism

This is example is of great theoretical significance as it allows us to translate questions about symplectic diffeomorphisms into questions about Lagrangian submanifolds. Equip the Cartesian product $M \times M$ with the symplectic form $\omega \oplus -\omega$. That is to say, with respect to the splitting $T(M \times M) = \pi_1^* TM \oplus \pi_2^* TM$, the symplectic form is ω on the first factor and $-\omega$ on the second.

Problem 17. Let $f : M \to M$ be a diffeomorphism. Prove that the graph

$$\mathrm{graph}(f) = \{(x, f(x)) \mid x \in M\} \tag{12}$$

is a Lagrangian submanifold of $M \times M$ if and only if f is a symplectic diffeomorphism.

Observe that the intersection $\mathrm{graph}(f) \cap \mathrm{graph}(g)$ is the set of x such that $f(x) = g(x)$. If g is the identity map, we get the fixed points of f.

3.3 Lagrangian suspension

Let (M, ω) be symplectic. Let $L \subset M$ be a Lagrangian submanifold, and consider a *loop* of Hamiltonian diffeomorphisms $\phi : S^1 \to \mathrm{Ham}(M, \omega)$, which we assume is generated by a time-dependent Hamiltonian $H(t, x)$ which is periodic in t:

$$H : S^1 \times M \to \mathbb{R} \tag{13}$$

Now consider the manifold $M \times \mathbb{R} \times S^1$, with the form $\sigma = \omega + dr \wedge dt$. Consider the embedding

$$L \times S^1 \to M \times \mathbb{R} \times S^1 \tag{14}$$

$$(x, t) \mapsto (\phi_t(x), -H_t(\phi_t(x)), t) \tag{15}$$

Problem 18. Show that the image of $L \times S^1$ is Lagrangian.