Grading on Lagrangian Floer cohomology [Seidel]

We have already made some ad hoc grading arguments in previous lectures, but for the next set of applications it will be important to do this more systematically.

Recall the Lagrangian Grassmannian for $(V, \beta)$ symplectic $V.S.$

$$\Lambda(V, \beta) = \{ \text{ Lag subspaces } \pi V \} = \frac{U(n)}{O(n)}$$

$$\dim V = 2n$$

Recall \( \pi_1(\Lambda(V, \beta)) \cong H_1(\Lambda(V, \beta)) \cong H'(\Lambda(V, \beta)) \cong \mathbb{Z} \)

Curves of $X$ with deck group $\mathbb{Z}/N\mathbb{Z}$ are classified by $H'(X; \mathbb{Z}/N\mathbb{Z})$

let $C(V, \beta)$ denote the generator of $H'(\Lambda(V, \beta); \mathbb{Z})$

Then the $N$-fold quotient $N^N(V, \beta)$ is associated to the image of $C(V, \beta)$ in $H'(\Lambda(V, \beta); \mathbb{Z}/N\mathbb{Z})$. It is also the connected cover that corresponds to the subgroup $N\pi_1(\Lambda(V, \beta))$.

The $\mathbb{Z}/N\mathbb{Z}$ deck group action on $N^N$ is denoted by $\pi$.

If $N = \infty$, we are talking about the universal covering $\tilde{V} = V^{\infty}$.

Now consider a symplectic manifold $(M, \omega)$

let $\Lambda \to M$ be the bundle of Lagrangian Grassmannians

let $\Delta = \Lambda^{\infty} (TM, \omega) \to \mathbb{Z}$ be the square of the complex determinant of the tangent bundle (i.e. $K^{\infty}_M$)
An $N$-fold Muslin covering is a $\mathbb{Z}/N$-cover $\Lambda^N \rightarrow \Lambda$ whose restriction to $x \in \Lambda$ is isomorphic to $\bigwedge^\Lambda M(\mathbb{T}_x M, m_x) \rightarrow \Lambda(\mathbb{T}_x M, m_x)$.

A global mod-$N$ Muslin class is $C^N \in H^1(\Lambda; \mathbb{Z}/N)$ whose restriction to any fiber is the mod-$N$ reduction of $C(\mathbb{T}_x M, m_x)$.

An $N$-th root of $\Delta = K^\otimes(2)$ is a pair $(Z, j)$ where $Z$ is a complex line bundle and $j: \mathbb{C}^N \rightarrow \Delta$ is an iso. Declare $(Z_1, j_1) \sim (Z_2, j_2)$ if $\exists r: Z_1 \cong Z_2$, s.t. $j_1 = j_2 \otimes r$.

If $N = \infty$, an $\infty$-root is a trivialization, equivalence = homotopy.

Prop: $\{N$-fold Muslin coverings $\}/\text{iso} \leftrightarrow \{\text{Global mod } N \text{ Muslin classes} \} \leftrightarrow \{N$-th roots of $\Delta^2/\sim \}.

Prop: $(M, \omega)$ admits an $N$-fold Muslin covering iff $Z\nu(M, \omega)$ goes to zero in $H^2(M; \mathbb{Z}/N)$. Iso-morphism classes are then an affine space over $H^1(M; \mathbb{Z}/N)$.

Now let $L \subseteq M$ be a Lagrangian submanifold. This is a section $s_L: L \rightarrow N|_L : s(x) = T_x L \subseteq N(\mathbb{T}_x M, m_x)$.

An $N$-grading of $L$ is a lift $s^N_L: L \rightarrow N|_L$. $L$ is called $N$-gradable if this exists.

Prop: $L$ is $N$-gradable iff $s^*_L(C^N) \in H^1(L; \mathbb{Z}/N)$ vanishes where $C^N$ is the global Muslin class of $N^1$. 
The set of \( N \)-gradings on connected \( L \) is a \( \mathbb{Z}/N \)-torsor. In particular if \( H^1(L, \mathbb{Z}/N) = 0 \), \( L \) is \( N \)-gradable.

**Automorphisms:** Let \( \phi : M \to L \) be a symplectomorphism.

The action of \( \phi \) naturally lifts to \( \frac{1}{N} \)

An \( N \)-grading of \( \phi \) is a lift \( \tilde{\phi} \in \mathbb{Z}^N \) compatible with \( \phi \circ \).

Not every symplectomorphism is \( N \)-gradable, but the obstruction lives in \( H^1(M; \mathbb{Z}/N) \).

Relation to previous discussion: \( N_\mu = \) minimal Chern number = generator of \( \langle c_1(M), H_2(M) \rangle \subset \mathbb{Z} \)

\( N_L = \) minimalMaslov number = generator of image of Maslov hom \( \mu : H_2(M, L) \to \mathbb{Z} \).

**Prop:** Assume \( H_1(M; \mathbb{Z}) = 0 \).

- \( M \) admits \( N \)-fold Maslov covering \( \tilde{N} \) \( \text{iff} \) \( N | 2N_\mu \) (which is then unique).
- \( L \) admits \( N \)-grading \( \text{iff} \) \( N | N_L \).

**Examples:** Orientations \( \Lambda^2 = \text{oriented Lagrangian Grassmannian}. \)

Since \( 2c_1(M) \mod 2 = 0 \in H^2(M; \mathbb{Z}/2) \), any \( (M, \omega) \) admits a 2-fold Maslov cover. The cover can be twisted by any real line bundle \( \mathcal{E} \), with class \( w_1(\mathcal{E}) \in H^1(M, \mathbb{Z}/2) \)

\( L_{CM} \) is 2-gradable \( \text{iff} \) \( L \) is orientable. If \( L \) then admits exactly two 2-gradings.
$c_1(M)$ is two torsion. If \( 2c_1(M) = 0 \in H^2(M; \mathbb{Z}) \) then \( M \) admits an \( \mathbb{C} \)-grading \( \Lambda^Q \), can be twisted by \( H^1(M; \mathbb{Z}) \).

\( L \) admits an \( \mathbb{Q} \)-grading iff \( \mu : H_2(M, L) \to \mathbb{Z} \) vanishes identically.

(\( \text{vanishing Maslov class} \))

Sub example: \( M = \text{Calabi-Yau manifold} \), \( L = \text{Lagrangian fiber} \).

Absolute \( \mathbb{Z}/N \) grading on Floer homology.

In our previous discussion, we associated a Maslov index to a strip \( \xymatrix{ \mathcal{Y} \ar[rr]^{-u} & & \mathcal{X} } \).

Now suppose \( L_0 \) and \( L_1 \) come with lifts \( \tilde{L}_0, \tilde{L}_1 \rightarrow \Lambda^N \).

Theorem: To a pair of Lagrangians \( \tilde{L}_0, \tilde{L}_1 \in \Lambda^N \), we can associate an index \( \tilde{\mu}(\tilde{L}_0, \tilde{L}_1) \in \mathbb{Z}/N \).

With the properties:

- \( \tilde{\mu}(a \tilde{L}_0, b \tilde{L}_1) = a \tilde{\mu}(\tilde{L}_0, \tilde{L}_1) - ab \)
- Graded symplectomorphisms preserve \( \tilde{\mu} \)
- \( \tilde{\mu}(\tilde{L}_1, \tilde{L}_0) = n - \tilde{\mu}(\tilde{L}_0, \tilde{L}_1) \) (mod \( N \))

\( H \) is a strip

\[
\mu(x, y, u) = \tilde{\mu}(\tilde{T}_x L_0, \tilde{T}_x L_1) - \tilde{\mu}(\tilde{T}_y L_0, \tilde{T}_y L_1) \pmod{N}
\]

(Thus the Floer coboundary has degree \(+1\) \( \text{d}y = \sum \omega(x, y) x \))
If \( \phi(L) \) is push of \( L \) by Morse function, \( \text{mod } N \), then

- applications: suppose \( L \) admits a Morse function with only even index critical points.

  Then \( H^*(L) = 0 \Rightarrow L \) orientable \( \Rightarrow L \) is 2-graded.

  Floer complex \( CF^i(L,L) \) concentrated in degree 0, so no differential:

  \[
  HF^i(L,L) = CF^i(L,L) = C_{\text{hom}}^i(L) \cong H^i(L; \mathbb{Z}_2)
  \]

  (Assuming \( HF^i(L,L) \) can be defined)

Here's another: Then (Seidel) Any Lagrangian submanifold \( L \subset \mathbb{CP}^n \) satisfies

\[
H^i(L; \mathbb{Z}/(2n+2)) \neq 0
\]

Since \( N_{\mathbb{CP}^n} = n+1 \), we have Maslov cover for \( N \mid 2n+2 \).

Key idea: \( \mathbb{CP}^n \) admits a Hamiltonian circle action

\( t \in \mathbb{R}/\mathbb{Z} \mapsto -t(t) = \text{deg}(e^{2\pi it}, \ldots, 1) \in U(n+1) \subset \text{Symp}(\mathbb{CP}^n) \)

Since \( H^*(\mathbb{CP}^n; \mathbb{Z}) = 0 \), this loop lifts to a path of graded symplecto morphisms.

\[
\tilde{\sigma} : [0, 1] \to \text{Symp}(\mathbb{CP}^n)
\]

\( \tilde{\sigma}(0) = \text{id} \)

Look at how \( \tilde{\sigma} \) acts at a fixed point on a Lagrangian.

The induced loop has Maslov index 2,

\[
\tilde{\sigma}(1) = \text{shift of grading by } 2 \rightarrow [-2]
\]

Assume \( L \) is \( N \)-graded.

Thus any \( L \) is Hamiltonian isotopic to \( L[-2] \)

By Hamiltonian invariance:

\[
HF^i(L,L) = HF^i(L, L[-2]) = HF^{i-2}(L, L)
\]

So \( HF(L, L) \) must be 2-periodic if \( L \) is \( N \)-graded.
Proof of Theorem: For a contractible $L$, suppose $L$ is a Lagrangian with $H^i(L; \mathbb{Z}/(2n+2)) = 0$.

- Then $L$ is $(2n+2)$-graded and $HF^*(\hat{L}, \hat{L})$ is $\mathbb{Z}/(2n+2)$-graded.
- Implies $(2n+2) | N_L$ so $N_L \geq 2n+2$.

- Also $H^i(L; \mathbb{Q}) = 0$. Thus the monotonicity of $\mathbb{CP}^n$ implies the monotonicity of $L$.

[All of $H_2(M, L)/\text{torsion}$ comes from $H_2(M)$ up to integer multiple.]

Since $L$ is monotone with $N_L \geq 2n+2 \geq 3$

By spectral sequence applies.

The $E_1$ page is

$$E_1^{i,j} = \begin{cases} H^i(L; \mathbb{Z}/2) & \text{if } 0 \leq i \leq n \\ 0 & \text{if } n+1 \leq i \leq 2n+1 \end{cases}$$

The next differential in the Oh SS would have degree $N_L \geq 2n+2$ hence vanishes for degree reasons.

Thus $HF^*(\hat{L}, \hat{L}) \cong H^*(L; \mathbb{Z}/2)$ with grading reduced modulo $2n+2$.

But this is not $2$ periodic! Contradiction.