Introduction to the Fukaya Category.

The Floer homology groups can be considered as fitting into a larger algebraic structure called the Fukaya category.

Assume $(\mathcal{M},\omega + d\theta)$ is exact, all Lagrangians below exact.

We take coefficients in $\mathbb{Z}/2$.

$\mathcal{F}(\mathcal{M},\omega)$: Objects = Lagrangian submanifolds (possibly with extra data, e.g., disks)

Morphism Spaces

$\text{Hom}(L_0,L_1) = \text{HF}(L_0,L_1)$

Need to define composition of morphisms $\mu^2(x,y)$

$\mu^2: \text{HF}(L_1,L_2) \otimes \text{HF}(L_0,L_1) \rightarrow \text{HF}(L_0,L_2)$

This counts pseudoholomorphic triangles.
By a "triangle," we mean a map from a disk with three strip-like ends, as depicted above.

This is curvature equivalent to a disk with three punctures on the boundary

\[ \mathcal{S} = \begin{array}{c}
\end{array} \sim \begin{array}{c}
\end{array} \]

To define the moduli space, assume \( L_0, L_1, L_2 \) are pairwise transverse. Pick almost complex structures \( J_{01}, J_{12}, J_{02} \) such that \( HF(L_i, L_j, J_{ij}) \) is well-defined. That is to say, \( J_{ij} \) is a complex structure such that moduli spaces of strips with boundary on \( L_i \) and \( L_j \) are transverse.

(This choice is necessary because \( \mu^2 \) can only be defined for a particular chain-level model of \( HF(L_i, L_j) \)).

Now we pick a family of almost complex structures \( J_0, J_1, J_2 \). (That is, the target space complex structure depends on the point in the domain.)

This family should agree with \( J_{ij} \) over the end corresponding to \( (L_i, L_j) \).
Now let \( x_{ij} \in L_i \cap L_j \) basis element of \( CF(L_i, L_j) \)

Define \( M(x_{01}, x_{12}, x_{02}) = \{ (j, J) \text{-holomorphic maps } u: S \to \mathcal{M} \text{ satisfying boundary conditions asymptotic to } x_{01}, x_{12}, x_{02} \} \)

look at 0-dimensional components

\[
\mu^2(x_{12}, x_{01}) = \sum_{x_{02}} \# M(x_{01}, x_{12}, x_{02})^0 \cdot x_{02}
\]

This actually defines a map

\[
\mu^2: CF(L_1, L_2) \otimes CF(L_0, L_1) \to CF(L_0, L_2)
\]

To see that it is well-defined in homology, we need to prove

\[
\partial(J_{02}) \cdot \mu^2(x, y) = \mu^2(x, \partial(J_{01}) y) + \mu^2(\partial(J_{12}) x, y)
\]

"\( \mu^2 \) is a chain map" or "\( \partial \) is a derivation of \( \mu^2 \)"

(Note we are ignoring signs by working mod 2.)
To prove this, use 2-dimensional components of

\[ \mu^2(y, x, z) \]

Notice how the complex structure on the strip depends in which end

the breaking occurs.

The desired identity holds because the boundary of the one-dimensional

moduli space consists of an even # of points.

In order for these \( \mu^2 \) compositions to define a category,

they must satisfy an associative law

\[ L_0 \xrightarrow{x_{01}} L_1 \xrightarrow{x_{12}} L_2 \xrightarrow{x_{23}} L_3 \]

\[ \mu^2(x_{23}, \mu^2(x_{12}, x_{01})) = \mu^2(\mu^2(x_{23}, x_{12}), x_{01}) \]

as maps

\[ HF(L_2, L_3) \otimes HF(L_1, L_2) \otimes HF(L_0, L_1) \rightarrow HF(L_0, L_3) \]
The way to do this is to show that the difference between 
the two sides is a null homotopic map on the chain level.

We need an operator, say \( P(x,y,z) \), such that

\[
\partial P(x, y, z) + P(x, y, z) + P(x, y, z) + P(x, y, z) = \mu^2(x, \mu^2(y, z)) - \mu^2(\mu^2(x, y), z) \]

This operator \( P \) is actually called \( \mu^3 \).

Think about composing two \( \mu^2 \)'s

\[ \xymatrix{ x \ar[r] & y \\ z \ar[u] & \ar[ur] \} \]

This Riemann surface is equivalent to a disk with 4 
boundary punctures. It has one real modulus.

There is a related picture in terms of graphs (dual complexes)

\[ \xymatrix{ & 1 \ar[dl] \\
2 & & 3 \\
& 1 \ar[ur] } \]
The moduli space of disks with $4$ boundary punctures is an interval

$$
\mathcal{M}^4 = \mathcal{R}^2 \cup \mathcal{N} \cup \mathcal{R}^2
$$

We let $\mathcal{S}^4$ denote the universal family. $S_r$ the fiber over $\mathcal{R}^4$

We allow the target almost complex structure to depend on $z \in \mathcal{S}^4$ (depends on modulus $r$ and point on $S_r$) as well as on the Lagrangians involved $L_0, L_1, L_2, L_3$

The choice of $J$ on $\mathcal{S}^4$ must be compatible with previous choices.

1) Over strip-like ends of $S_r \in \mathcal{S}^4$, say one with labels $L_i, L_j$
   $J$ must coincide with $J_{ij}$.

2) Over boundary points of $\mathcal{R}^4$, $J$ must coincide with the $J$’s previously chosen in the definition of $\mu^2$

Reason: 1) guarantees that when strips break off, the map is homotopic to the correct $J_{ij}$ that defines the differential.

2) guarantees that our $\mu^2$ will actually be a homotopy relating the $\mu^2$ operators defined earlier.
$\mu^3$ is defined by looking at

$$ M = \left\{ (r \in \mathbb{R}^4, u : S^1 \to M) \mid u \text{ is pseudoholomorphic, } \text{ satisfies boundary and asymptotic conditions} \right\} $$

I.e.

$$ \mu^3(x_{23}, x_{12}, x_{01}) = \sum_{x_{03}} \#(M(x_{23}, x_{12}, x_{01}, x_{03})) x_{03} $$

and 0-dimensional amplitudes.

To prove homotopy property, use 1-dimensional amplitudes and cobordism argument.

There are higher operators $\mu^d$, $d \geq 1$ (note $\mu^1 = 0$)

Defined by extending the same idea to $(d+1)$ punctured disks.

These are "higher homotopies" in general, the equivariant line

$$ \partial \mu^d(x, \ldots, x_d) + \mu^d(\partial x, \ldots, x_d) + \cdots + \mu^d(x, \ldots, \partial x_d) $$

= $\sum \text{d-let!}$

The terms in the sum are indexed by trees with "two internal vertices" (volume 2?)
The operators $\{\mu^d\}_{d=1}^2$ make $\tilde{\mathcal{F}}(M,\omega)$, with

$\text{Hom}(L_0, L_1) = CF(L_0, L_1)$ into an $A_\infty$-category.

Moral: The $A_\infty$-relations naturally arise for the moduli spaces of disks we use to define the category.