The action functional and calculus of variations

First and second variations of the action functional

Consider the local situation near a transverse intersection of two Lagrangians $L_0, L_1$.

By Darboux-Weinstein, we may assume $M = T^*\mathbb{R}^n, \Sigma dp_i dq_i, L_0 = 0$-section $\mathbb{R}^n$.

$L_1 = \text{graph } df$ for some $f : \mathbb{R}^n \to \mathbb{R}$, $df(0) = 0$.

Let $\gamma$ be a path $L_0 \to L_1$:
\[ \gamma : [0, 1] \to M, \quad \gamma(i) \in L_i \text{ for } i = 0, 1. \]

Define the action of $\gamma$ as follows: choose a disk $u : D^2 \to M$ joining $\gamma$ to the origin, with boundary on $L_0 \cup L_1$.

\[ \text{graph } df \]

such that boundary orientation agrees with orientation of $\gamma$.

Set
\[ A(\gamma) = \int_{D^2} u^* w \]

the symplectic area of this disk.

Now $\omega = \sum dp_i dq_i = d\left( \sum p_i dq_i \right) = d\lambda$.

By Stokes' theorem, $A(\gamma) = \int_{D^2} u^* \lambda$. 

By Stokes' theorem, $A(\gamma) = \int_{L_1 \cdot \text{graph } df} \omega$. 

$L_0 = \text{graph } d(0)$. 

Over \( r = \int_{[0,1]} x^* \lambda \),

Over \( l_0 = \text{graph } d(0) \) \( \lambda \) vanishes \( \Rightarrow \) get 0.

Over \( l_1 = \text{graph } d f \)
\[
\int_{\pi(\gamma(1))}^{\pi(\gamma(1))} \lambda = \int_{\pi(\gamma(1))}^{\pi(\gamma(1))} df = f(\pi(\gamma(1))) - f(\pi(\gamma(1)))
\]

Thus \( A(\gamma) = \int_{D^2} u^* w = \int_{[0,1]} x^* \lambda - f(\pi(\gamma(1))) + f(0) \)

First variational formula for \( A \)

Let's use \( A(\gamma) = \int_{D^2} u^* w \) form

A variation of \( \gamma \) is a path \( \gamma_s, s \in (-\varepsilon, \varepsilon) \), \( \gamma_0 = \gamma \)

Note vector fields \( X = \frac{d}{ds} \big|_{s=0} \gamma_s \) which is \( \forall f \) along \( \gamma_s \)

\( X(0) \in T_{\gamma(0)} L_0 \quad X(1) \in T_{\gamma(1)} L_1 \)

We compute \( \frac{dA}{d\gamma} : \frac{d}{ds} \big|_{s=0} A(\gamma_s) \)
Let \( v(s,t) = \gamma_s(t) \) a surface in \( M \)

Differentiation \( A(\gamma_s) - A(\gamma_o) = \int v^*\omega \)

\[
\gamma_0 \quad \gamma_s
\]

\[
t \in [0,1] \\
s' \in [0,s]
\]

\[
= \int \omega \left( \frac{\partial v}{\partial s}, \frac{\partial v}{\partial t} \right) \, ds' \, v \, dt = \int_0^1 \int_0^s \omega \left( \frac{\partial \gamma_s}{\partial s}, \gamma_s' \right) \, ds' \, dt
\]

Apply \( \frac{d}{ds} \mid_{s=0} \) and FTC \( \frac{\delta A}{\delta \gamma} = \int_0^1 \omega(x, \gamma') \, dt \)

Critical points are when \( \frac{\delta A}{\delta \gamma} = 0 \) \( \forall \) variations. This requires \( \gamma' = 0 \).

\( \gamma \) is a constant path. In our local model, this must be the constant path at the origin.

**Second variational formula for \( A \):**

Now suppose we are at a critical point, i.e. \( \gamma = \text{origin} \).

Differentiating \( \frac{d}{ds} \quad A = \int_0^1 \omega \left( \frac{\partial \gamma_s}{\partial s}, \gamma_s' \right) \, dt \) \( \text{wrt. } s \)

we get \( \frac{d^2}{ds^2} \quad A = \int_0^1 \omega \left( \frac{\partial^2 \gamma_s}{\partial s^2}, \gamma_s' \right) \, dt + \int_0^1 \omega \left( \frac{\partial \gamma_s}{\partial s}, \frac{\partial \gamma_s}{\partial s} \right) \, dt \)

Now \( \frac{\partial \gamma_s}{\partial s} \bigg|_{s=0} = 0 \quad \frac{\partial \gamma_s}{\partial s} \bigg|_{s=0} = \gamma' \)
\[
\frac{\delta^2 A}{\delta \gamma^2} = \left. \frac{d^2}{ds^2} A \right|_{s=0} = \int_0^1 \omega(X, \dot{X}) \, dt
\]

where \( \dot{X} = \delta \gamma = \frac{\partial \gamma}{\partial s} \)

We can obtain the Hessian of \( A \) via polarization

\[
\text{Hess} \, A \, (X, Y) = \frac{1}{2} \int_0^1 (\omega(X, \dot{Y}) + \omega(Y, \dot{X})) \, dt
\]

\[
\Rightarrow \text{Hess} \, A \, (X, Y) = \int_0^1 \omega(X, \dot{Y}) \, dt,
\]

which is symmetric

Note \( \omega(X, \dot{Y}) - \omega(Y, \dot{X}) = \omega(X, \dot{Y}) + \omega(Y, \dot{X}) \)

\[
= \frac{d}{dt} \omega(X, Y) \text{ is a total derivative}
\]

so \( \int \omega(X, \dot{Y}) \, dt - \int \omega(Y, \dot{X}) \, dt = \int \frac{d}{dt} \omega(X, Y) \, dt \)

\[
= \omega(X(1), Y(1)) - \omega(X(0), Y(0)) = 0 \text{ since } x(t) \in T \mathcal{L}_1 \text{ and } y(t) \in T \mathcal{L}_0.
\]

Summary \( dA \gamma(X) = \int_0^1 \omega(X, \dot{Y}) \, dt \)

Crit \( A \) = constant paths at intersection points

Hess \( \gamma \, A \, (X, Y) = \int_0^1 \omega(X, \dot{Y}) \, dt \)
Now we get the metric and complex structure involved.

\[ g(x, y) = \omega(x, Jy) \]

on vector fields along a path \( X(t), Y(t) \)

\[ \langle X, Y \rangle = \int_0^1 g(X, Y) \, dt \]

\[ = \int_0^1 \omega(x, Jy) \, dt \]

\[ dA_\gamma(x) = \langle X, \nabla A_\gamma \rangle = \int_0^1 \omega(x, J \nabla A_\gamma) \, dt \]

\[ = \int_0^1 \omega(x, \dot{\gamma}) \, dt \]

\[ \nabla A_\gamma = -J \dot{\gamma} \]

\[ J \nabla A_\gamma = \dot{\gamma} \]

Formally, positive gradient flow \( \frac{\partial}{\partial s} \gamma_s = \nabla A_{\gamma_s} = -J \dot{\gamma} \)

\[ \frac{\partial \gamma_s}{\partial s} + J \dot{\gamma} = 0 \]

if \( u(st) = \gamma_s(t) \) \( \frac{\partial u}{\partial s} + J \frac{\partial u}{\partial t} = 0 \) \( J \)-holomorphic.
\[
\text{Hessian} \quad \text{Hess} \ A (x, y) = \langle x, \partial_y \rangle = \int g(x, \partial_y) \, dt \\
= \int \omega(x, \partial_y) \, dt \\

\text{so} \quad \dot{y} = J \partial y \quad \text{or} \quad \partial y = -J \dot{y}
\]

\(\partial\) is the 1-d Dirac operator \quad \Rightarrow \quad \partial^2 = -\frac{d^2}{dt^2} = -\Delta

This self-adjoint operator represents the Hessian of \(A\) w.r.t. \(\langle \cdot, \cdot \rangle\)

Now we want to understand spectral theory of

\[D = -J \frac{d}{dt} \quad \text{acting on} \quad \left\{ X(t) : [0, 1] \to T_p M \mid X(0) \in T_p L_0, X(1) \in T_p L_1 \right\}
\]

The spectrum depends on the relative position of \(L_0\) and \(L_1\)

Since \(\partial y = \lambda y\) is an ODE, we use the standard method. Look for a solution of the form \(y(t) = e^{At}y_0\)

\[\partial y = -JA e^{At} y_0 = \lambda e^{At} y_0 \Rightarrow -JA = \lambda \quad \text{so} \quad A = \lambda J
\]

\[y(t) = e^{\lambda J t} y_0 = (\cos (\lambda t) \text{Id} + \sin (\lambda t) J) y_0
\]

We need \(y(0) = y_0 \in T_p L_0\) and \(y(1) = e^{\lambda J} y_0 \in T_p L_1\)

Thinking of \(T_p M\) as \(C\)-vector-space via \(J\), \(e^{\lambda J}\) is a \(C\)-scalar.
So we ask, are there vectors $y_0 \in T_p L_0$ and scalar $\mu \in U(1)$ such that $y(1) = \mu y_0 \in T_p L_1$?

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Linear algebra:  
$T_p M \rightarrow \mathbb{C}^n$  
$T_p L_0 \rightarrow \mathbb{R}^n$  
$T_p L_1 \rightarrow \mathbb{R}^n$  
$A \in U(n)/O(n)$
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Seek $\nu \in \mathbb{R}^n$ such that $\mu \nu \in L_1 = A \mathbb{R}^n \iff \mu A^{-1} \nu \in \mathbb{R}^n$

Same as asking that $\mu L_0$ and $L_1$ are not transverse.

The number of solutions is the Maslov index of the loop $\mu \mapsto \mu L_0$ in the Lagrangian Grassmannian $\Lambda_n$.

$= \text{degree of } \det^2: U(n)/O(n) \rightarrow U(1)$

$\det^2(\mu I) = \mu^{2n}$

So are $2n$ solutions.

Now, each $\mu$ as above gives infinitely many eigenvalues $\lambda$ for $D = -J \frac{d}{dt}$.

Since $\mu = e^{i \lambda}$, the spectrum has a symmetry $\lambda \mapsto \lambda + 2\pi k$.

We conclude that $D$ has $2\pi k$-families of eigenvalues, with the number of families being $2n$.

In particular, $P$ has $\infty$-ly many positive-negative eigenvalues.
\[ \lambda = 0 \text{ is eigenvalue } \Leftrightarrow \mu = 1 \text{ works } \Leftrightarrow \]

\[ L_0 \text{ and } L_1 \text{ are not transverse at } p. \]

\[ n = 1 \]

\[ L_1 = e^{i\theta} L_0 \]
\[ \text{or} \quad L_1 = e^{i(\theta + \pi)} L_0 \]
\[ \text{or} \quad L_1 = e^{i(\theta + 2\pi)} L_0 \]

\[ \text{Spec } D = \left\{ \theta - 2\pi, \theta, \theta + 2\pi, \theta + 4\pi, \ldots \right\} \]
\[ \left\{ \ldots, \theta - \pi, \theta + \pi, \theta + 3\pi, \theta + 5\pi, \ldots \right\} \]

We see \( 2 = 2n \) families with \( 2\pi \mathbb{Z} \)-symmetry.