Almost complex structures on symplectic manifolds

Recall linear structures on $\mathbb{R}^{2n} = \mathbb{C}^n$

Hermitian inner product $h(u,v) = \sum_{i=1}^{n} u_i \overline{v}_i$

$h(u,v) = g(u,v) - i \omega(u,v)$

$-i h(u,v) = h(u,iv) = g(u,iv) - i \omega(u,iv)$

$-i g(u,v) = \omega(u,v)$

So $\omega(u,v) = -g(u,iv) \text{ and } g(u,v) = \omega(u,iv)$

$g$ symmetric $\Rightarrow$

$\omega(iv,iv) = -g(iu,v) = -g(v,iv) = -\omega(v,u) = \omega(u,v)$

Abstract these properties: $(V, \omega)$ symplectic. $J: V \rightarrow V$ is a complex structure if $J^2 = -Id$

$J$ is tame (tamed by $\omega$) if $\omega(x, Jx) > 0 \forall x \neq 0$

$J$ is compatible (with $\omega$) if $J$ is tame and $\omega(x, Jy)$ is symmetric.

When $J$ is compatible with $\omega$, $g(x,y) = \omega(x, Jy)$ is a metric on $V$. Then $(V, \omega, J, g)$ is isomorphic to $\mathbb{C}^n$ with its standard structures.

When $J$ is merely tame, the symmetrization

$g(x,y) = \frac{1}{2} [\omega(x, Jy) + \omega(Jx, y)]$ is a metric on $V$. 
Reason for considering both. Often, we need to perturb $J$ to a “generic” situation. Positivity of $c_1(x, Jx)$ is extremely important, but symmetry is less important considering time. ACS’s gives us more freedom.

We use the same terminology for structures on a manifold $(M, \omega)$ symplectic manifold. $J: TM \to TM$ is called an almost complex structure if $J^2 = -\text{Id}$ (at each pt). It is tame/compatible if it is so at each point.

Reason for “almost”: A “true” complex manifold has local holomorphic coordinate charts $(z_1, \ldots, z_n) \in U \subseteq \mathbb{C}^n$.

The tangent spaces are also identified with $\mathbb{C}^n$, and we the standard structure $J = i$ there. The coordinate transitions preserve this structure, so we get a well-defined $J$ on all of $M$. The $J$ obtained this way satisfies an extra condition called Integrability.

The Nijenhuis tensor

$$N(x, y) = [Jx, Jy] - J[x, Jy] - J[y, Jx] - [x, y]$$

vanishes for all vector fields $x, y$.

Almost complex = We don't impose this condition.

Vanishing of $N$ is sufficient for existence of local holomorphic coordinates (Newlander-Nirenberg).

In dimension 2, $N$ vanishes for dimensional reasons. ∴ Any almost complex surface is a Riemann surface.
Homework: if $(M, J)$ is almost-complex, and $f: M \to C$ is holomorphic, meaning $df \circ J = i \cdot df$, then $\nabla (N(x,y)) = 0$

for all vector fields $X$ and $Y$.

So in general an almost complex manifold may have no holomorphic functions. But it still has many holomorphic curves $u: (\Sigma,j) \to (M,J)$, $du \circ j = J \circ du$ $(\Sigma,j)$ a Riemann surface

**Contractibility of space of tame/compatible almost complex structures.**

**Linear theory:**

$(C^n, J_0 = i, g, \omega)$ standard complex vector space

$J_+ (\omega) = \text{tame ACS}$ \hspace{1cm} $J_c (\omega) = \text{compatible ACS}$

**Proposition.** The map $J \mapsto S := (J + J_0)^{-1} (J - J_0)$ is a diffeomorphism of $J_+ (\omega)$ onto the unit ball (wrt. $g$) in space of matrices satisfying the linear equation $J_0 S + SJ_0 = 0$.

The subspace $J_c (\omega)$ maps diffeomorphically onto the set of symmetric matrices with $i$ this space

$J_+ (\omega) \leftrightarrow \left\{ S \mid 1 \leq |S| < 1 \text{ and } J_0 S + SJ_0 = 0 \right\}$

$J_c (\omega) \leftrightarrow \left\{ S \mid uu^T \text{ and } S^T = S \right\}$
Proof: let \( J \in J_+(\omega) \) then

\[
\omega(x, (J + J_0)x) = \omega(x, Jx) + \omega(x, J_0x) \geq 0
\]

so \( J + J_0 \) cannot have a kernel.

Thus \( S = (J + J_0)^{-1}(J - J_0) \) is well defined.

Write \( A = J_0^{-1}J \)

Then \( S = [J_0(A + I)]^{-1}[J_0(A - I)] = (A + I)^{-1}(A - I) \)

To show \( \|S\| < 1 \), show \( \|Ax - x\|^2 < \|Ax + x\|^2 \) \( \forall x \)

\[
\|Ax + x\|^2 - \|Ax - x\|^2 = g(Ax + x, Ax + x) - g(Ax - x, Ax - x)
\]

\[
= g(Ax, Ax) + g(Ax, x) + g(x, Ax) + g(x, x)
\]

\[
- [g(Ax, Ax) - g(Ax, x) - g(x, Ax) + g(x, x)]
\]

\[
= 4g(x, Ax) = 4g(x, J_0^{-1}Jx) = 4\omega(x, J_0^{-1}Jx)
\]

\[
= 4\omega(x, Jx) > 0
\]

Thus \( \|S\| < 1 \)

Conversely, if \( \|S\| < 1 \), \( I - S \) is invertible.

Solve \( S = (J + J_0)^{-1}(J - J_0) \) for \( J \)

\[
(J + J_0)S = J - J_0
\]

\[
JS + J_0S = J - J_0
\]

\[
JS - J = -J_0S - J_0
\]

\[
- J(I - S) = - J_0(I + S)
\]

\[
J = J_0(I + S)(I - S)^{-1}
\]
And conversely, \( J \) satisfies \( \omega(x, Jx) > 0 \).

To see this, \( \omega(x, Jx) = g(x, J_{0}^{-1} Jx) = g(x, (I+S)(I-S)^{-1}x) = g((I-S)y, (I+S)y) = g(y, y) - g(Sy, y) = \|y\|^2 - \|Sy\|^2 > 0 \) since \( \|S\| < 1 \).

Now we check \( J^2 = -I \iff J_{0} S + SJ_{0} = 0 \).

\[
J = J_{0} (I+S)(I-S)^{-1}
\]

\[
J^2 = -I \iff J_{0} (I+S)(I-S)^{-1} J_{0} (I+S)(I-S)^{-1} = -I
\]

\[
\iff (I+S) \left( \sum_{k=0}^{\infty} S^k \right) J_{0} (I+S) = J_{0} (I-S)
\]

\[
\iff (I+2 \sum_{k=1}^{\infty} S^k) J_{0} (I+S) = J_{0} (I-S)
\]

\[
\iff J_{0} + J_{0} S + 2 \sum_{k=1}^{\infty} S^k J_{0} + 2 \sum_{k=1}^{\infty} S^k J_{0} S = J_{0} (I-S)
\]

\[
\iff 2 \sum_{k=1}^{\infty} S^k J_{0} + 2 \sum_{k=1}^{\infty} S^k J_{0} S = -2 J_{0} S
\]

\[
\iff \sum_{k=0}^{\infty} (S^k J_{0} + S^k J_{0} S) = -J_{0} S
\]

\[
\iff \sum_{k=0}^{\infty} S^k J_{0} + \sum_{k=0}^{\infty} S^k J_{0} S = 0
\]

\[
\iff \sum_{k=0}^{\infty} S^k J_{0} + (I-S)^{-1} J_{0} S = 0
\]
\[(I - S)^{-1} - I) J_0 + (I - S)^{-1} J_0 S = 0 \]

\[(I - (I - S)) J_0 + J_0 S = 0 \]

\[SJ_0 + J_0 S = 0 \]

**Simplification:**

\[(I + S)(I - S)^{-1} = I + 2(I - S)^{-1} - I\]

\[-2(I - S)^{-1} - I\]

\[J_0 (I + S)(I - S)^{-1} = J_0 \left(2(I - S)^{-1} - I\right)\]

\[J^2 = J_0 \left(2(I - S)^{-1} - I\right) J_0 \left(2(I - S)^{-1} - I\right)\]

\[= 4 J_0 (I - S)^{-1} J_0 (I - S)^{-1} - 2 J_0 (I - S)^{-1} J_0 - 2 J_0^2 (I - S)^{-1} + J_0^2 = -I\]

\[J^2 = -I \iff 4 J_0 (I - S)^{-1} J_0 (I - S)^{-1} - 2 J_0 (I - S)^{-1} J_0 - 2 J_0^2 (I - S)^{-1} = 0\]

\[2(I - S)^{-1} J_0 (I - S)^{-1} - (I - S)^{-1} J_0 - J_0 (I - S)^{-1} = 0\]

\[2(I - S)^{-1} J_0 - (I - S)^{-1} J_0 (I - S) - J_0 = 0\]

\[2 J_0 - J_0 (I - S) - (I - S) J_0 = 0\]

\[2 J_0 - J_0 + J_0 S - J_0 + SJ_0 = 0\]

\[J_0 S + SJ_0 = 0\]

**Last we check:** \(w(x, J_0 y)\) symmetric \(\iff S\) symmetric

Let \(b(x, y) = w(x, J_0 y)\), \(g(x, y) = w(x, J_0 y)\)

\[J = J_0 (I + S)(I - S)^{-1}\]

So \(b(x, y) = w(x, J_0 (I + S)(I - S)^{-1} y) = g(x, (I + S)(I - S)^{-1} y)\)
Now \( b(x,y) \) is symmetric iff \( b((I-S)x, (I-S)y) \) is symmetric. The latter equals:

\[
\begin{align*}
g((I-S)x, (I+S)y) &= g(x-Sx, y+Sy) \\
&= g(x,y) + g(x, Sy) - g(Sx, y) + g(Sx, Sy)
\end{align*}
\]

This is symmetric iff \( g(x, Sy) - g(Sx, y) = 0 \) re. \( g(x, Sy) = g(Sx, y) = g(x, S^Ty) \)

This is equivalent to \( S = S^T \).

**Corollary** \( J_+(w) \) and \( J_-(w) \) are diffeomorphic to convex subsets of vector spaces. Hence they are contractible.

**Proof** For \( J_+(w) \) the vector space is \( \{ S \in \text{Mat}_{2n}(\mathbb{R}) \mid J_0 S + S J_0 = 0 \} \)

For \( J_-(w) \) it is \( \{ S \in \text{Mat}_{2n}(\mathbb{R}) \mid J_0 S + S J_0 = 0 \text{ and } S = S^T \} \)

The convex set is \( \| S \| < 1 \) in both cases.

Now we consider a manifold \( (M, w) \). There are fiber bundles of tame or compatible almost complex structures.

\[
\begin{array}{ccc}
J_+(wp) & \xrightarrow{J} & J_+(w) \\
\downarrow & & \downarrow \\
p \in M & & p \in M
\end{array}
\]

Both smooth \( \text{End}(TM) \)

To see they really are manifolds, pick Darboux coordinates, standard \( J_0 \) in those coordinates, and apply proposition parametrically.
So in each case we have a fibration of manifolds with contractible fibers.

Therefore both $\mathcal{J}_+^t$ and $\mathcal{J}_c$ have sections, and their spaces of sections are in fact contractible.

Denote their spaces of smooth sections by
\[ \mathcal{J}_+^t(M,\omega) = \Gamma(\mathcal{J}_+^t), \quad \mathcal{J}_c(M,\omega) = \Gamma(\mathcal{J}_c) \]

Lesson: because the space of tame/compatible ACS is contractible, from a topological viewpoint we can always assume that $(M,\omega)$ also is equipped with some ACS $\mathcal{J}$, and we don’t necessarily care that much how it is chosen. (Though we will care for the purposes of analysis).

**Local theory of $J$-holomorphic curves**

Let $(S,j)$ be a Riemann surface. Can just think of DCC the closed unit disk for now.

Let $(M,J)$ be a manifold with ACS $J$.

A function $f:S \to M$ is $J$-holomorphic if
\[ df \circ j = J \circ df \]

That is, if the differential $df$ is complex linear.

wrt. $j$ on source, $J$ on target.

We can reuse the trick to study $J$-holomorphic curves locally.

Let $(M,\omega,J)$ be tame. In a local Darboux coordinate patch $U$ in $M$, introduce the standard complex structure $J_0$
$J_0$ is integrable in $U$, and it makes $U$ look like an open set in $\mathbb{C}^n$ with coads $\{z_1, \ldots, z_n\}$. We consider maps $f : D \to U$ in $\mathbb{C} \to \mathbb{C}^n$.

$J_0$-holomorphic maps $f : D \to U$ are entirely classical. They are vector-valued holomorphic functions.

In coordinates $z = x + iy$ on $D$, introduce operators

$$\partial f = \frac{1}{2} \left( \frac{\partial f}{\partial x} - i \frac{\partial f}{\partial y} \right)$$

$$\overline{\partial} f = \frac{1}{2} \left( \frac{\partial f}{\partial x} + i \frac{\partial f}{\partial y} \right)$$

this $i$ is really $J_0$.

Observe that $f$ is $J_0$-holomorphic iff $\overline{\partial} f = 0$.

There is a classical parametrix for the $\overline{\partial}$ operator on $D$:

$$P g(z) = \frac{1}{2\pi i} \int_D \frac{g(\overline{\zeta})}{\overline{\zeta} - z} \, d\zeta \wedge d\overline{\zeta} \quad \text{for } g \in C_c^\infty(D)$$

$[P$ defines a map $L^2(D) \to L^1_c(D)]$

Then $\overline{\partial} \circ P = \text{Id}$ (and $P \circ \overline{\partial} = \text{Id}$ when restricted to functions with compact support in $\text{int}(D)$).

In our chart $U$, $J(p) + J_0(p)$ is invertible because both are tame so we may define $\sigma(p) = (J + J_0)^{-1}(J - J_0)$ at each point (cf. proposition).
The equation \( df \circ i = J \circ df \) boils down to
\[
\frac{\partial f}{\partial y} = J(f) \frac{\partial f}{\partial x}
\]

Now use
\[
\frac{\partial f}{\partial y} = \frac{i}{2} (\partial f - \overline{\partial f}) \quad \Rightarrow \quad \frac{J_0}{2} (\partial f - \overline{\partial f})
\]
\[
\frac{\partial f}{\partial x} = \frac{i}{2} (\partial f + \overline{\partial f})
\]

\((J_{hol}) \Leftrightarrow \)
\[
J_0(f)(\partial f - \overline{\partial f}) = J(f)(\partial f + \overline{\partial f})
\]
\[
\Leftrightarrow \quad J_0 \partial f - J_0 \overline{\partial f} = J_0 \partial f + J_0 \overline{\partial f}
\]
\[
\Leftrightarrow \quad (J_0 - J) \partial f = (J_0 + J) \overline{\partial f}
\]
\[
\Leftrightarrow \quad (J_0 + J)^{-1}(J_0 - J) \partial f = \overline{\partial f}
\]
\[
\Leftrightarrow \quad -\sigma(f) \partial f
\]
\[
\Leftrightarrow \quad \overline{\partial f} + \sigma(f) \partial f = 0
\]

(Somewhat similar to Beltrami equation \( \overline{\partial f} = \mu(z) \partial f \)).

Key observation: \( \overline{\partial f} + \sigma(f) \partial f = 0 \) is equivalent to \( \overline{\partial g} = 0 \)
where \( g = f + P \sigma(f) \partial f \) (since \( \overline{\partial P} = I_d \))

This means that in a local chart, \( J \)-holomorphic maps \( f : D \rightarrow U \) are equivalent to ordinary \( (J_0-\cdot) \)-holomorphic maps known from complex analysis.

The local theory of \( J \)-holomorphic curves tracks closely the theory for ordinary complex analytic curves.
Theorem: For $p \in M$ and $v \in T_p M$ sufficiently small, there exists a $J$-holomorphic curve $f : (D, 0) \to (M, p)$ such that $df_0(1) = v$.

Theorem: The critical points of a $J$-holomorphic map are isolated.

etc.