Lagrange Multipliers: Constrained Optimization

**Problem:** Maximize (or minimize) \( f(x, y, z) \)
subject to constraint \( g(x, y, z) = C \)
\( (C = \text{constant}) \)

\[ P = 6 L^\alpha K^{1-\alpha} \quad \text{Cobb-Douglas production function} \]
\[ 0 < \alpha < 1 \]

\( L = \text{labour} \)
\( K = \text{capital} \)
\( P = \text{production} \)
\( C = \text{cost} \)
\( C = mL + nK \)

**Problem:** Maximize \( P \) for a fixed cost \( C \)
Minimize \( C \) for a fixed production \( P \).

**Rectangular:**
\[ \begin{array}{c}
Z \\
\hline
x \\
\hline
y \\
\end{array} \]

Maximize volume \( = xyz \)
for a fixed surface area \( A = 2xy + 2xz + 2yz \)

One way to solve: solve constraint equation and substitute into \( f \)

Eg. minimize \( f(x, y) = x^2 + y^2 \) subject to \( x + y = 1 \)

constraint equation \( x + y = 1 \) \( \Rightarrow \) \( y = 1 - x \)

\[ f(x, y) = f(x, 1-x) = x^2 + (1-x)^2 = x^2 + 1 - 2x + x^2 \]
\[ f = 2x^2 - 2x + 1 \]
\[ 0 = \frac{d}{dx} (2x^2 - 2x + 1) = 4x - 2 \]
\[ x = \frac{1}{2} \]
\[ y = 1 - x = \frac{1}{2} \]
\[ f \left( \frac{1}{2}, \frac{1}{2} \right) = \left( \frac{1}{2} \right)^2 + \left( \frac{1}{2} \right)^2 = \frac{1}{4} + \frac{1}{4} = \frac{1}{2} \]
\[ f = \left( \text{dist to } 0 \right)^2 \]

Observe: constraint curve is tangent to the level curve of \( f \) at the minimum.

Not always possible to solve constraint equation. Lagrange multipliers let us optimize without solving constraint.

2D Lagrange multipliers: Optimize \( f(x,y) \) subject to \( g(x,y) = c \)

Solve:
\[
\begin{cases}
    \nabla f (x,y) = \lambda \nabla g (x,y) \\
    g (x,y) = c
\end{cases}
\]

\( \lambda \) is a new variable called the Lagrange multiplier.
3 equations \[ \begin{cases} f_x = \lambda g_x \\ f_y = \lambda g_y \\ g(x, y) = c \end{cases} \] 3 variables \( x, y, \lambda \)

Ex. \( f(x, y) = x^2 + y^2 \) subject to \( g(x, y) = x + y = 1 \)

\[ \nabla f = \lambda \nabla g \quad \nabla f = \langle 2x, 2y \rangle \]

\[ \nabla g = \langle 1, 1 \rangle \]

\[ \langle 2x, 2y \rangle = \lambda \langle 1, 1 \rangle = \langle \lambda, \lambda \rangle \]

\[ \begin{cases} 2x = \lambda \\ 2y = \lambda \\ x + y = 1 \end{cases} \quad \frac{\lambda}{2} + \frac{\lambda}{2} = 1 \quad \lambda = 1 \quad x = \frac{1}{2}, \ y = \frac{1}{2} \]

Minimum occurs at \( \left( \frac{1}{2}, \frac{1}{2} \right) \)

Minimum value \( f \left( \frac{1}{2}, \frac{1}{2} \right) = \frac{1}{2} \)

Multiplier equation says \( \nabla f \) is proportional to \( \nabla g \)

\[ \nabla f = \lambda \nabla g \]

\[ g(x, y) = c \]

Constraint

Level curves for \( f \)

Directional derivative \( > 0 \)

So it's possible to increase \( f \) while staying in \( g(x, y) = c \)
To summarize: at a constrained max or min

* The level curve for $f$ is tangent to the curve $g(x,y) = C$.

$\nabla f$ is parallel to the level curve of $f$

$\iff \nabla f$ is parallel to $g(x,y) = C$ (because level curves of $f$ are tangent to each other)

OTOH $\nabla g$ is parallel to $g(x,y) = C$

This is a level curve of $g$.

$\nabla f$ and $\nabla g$ are perpendicular to each other, so

the are parallel: $\nabla f = \lambda \nabla g$

3D Lagrange multipliers: Optimize $f(x,y,z)$
subject to $g(x,y,z) = C$ (constraint surface)

Solve

\[
\begin{cases}
\nabla f(x,y,z) = \lambda \nabla g(x,y,z) \\
g(x,y,z) = C
\end{cases}
\]
Maxima & minima occur when level surface of $f$ is tangent to constraint surface $g = c$

4 equations in 4 unknowns $x, y, z, \lambda$

\[ f(x, y, z) = x^2 + y^2 + z^2 \]
Constraint: $g(x, y, z) = x^4 + y^4 + z^4 = 1$

\[ \nabla f = \langle 2x, 2y, 2z \rangle \]
\[ \nabla g = \langle 4x^3, 4y^3, 4z^3 \rangle \]

\[ \nabla f = \lambda \nabla g; \quad 2x = \lambda 4x^3 \]
\[ 2y = \lambda 4y^3 \]
\[ 2z = \lambda 4z^3 \]

\[ x^4 + y^4 + z^4 = 1 \]

\[ 2x = \lambda 4x^3 \]
\[ x = \lambda 2x^3 \rightarrow x = 0 \quad \text{or} \quad 1 = \lambda \cdot 2x^2 \]
\[ y = \lambda 2y^3 \rightarrow y = 0 \quad \text{or} \quad 1 = \lambda \cdot 2y^2 \]
\[ z = \lambda 2z^3 \rightarrow z = 0 \quad \text{or} \quad 1 = \lambda \cdot 2z^2 \]

Suppose $x, y, z$ are all not zero.
\[ x^2 = \frac{1}{2\lambda}, \quad y^2 = \frac{1}{2\lambda}, \quad z^2 = \frac{1}{2\lambda} \]
\[ 1 - x^4 + y^4 + z^4 = \left(\frac{1}{2\lambda}\right)^2 + \left(\frac{1}{2\lambda}\right)^2 + \left(\frac{1}{2\lambda}\right)^2 \]

\[ 1 = \frac{3}{4\lambda^2} \quad \frac{4\lambda^2}{3} = 1 \quad \lambda^2 = \frac{3}{4} \]

\[ \lambda = \pm \sqrt{\frac{3}{4}} = \pm \frac{\sqrt{3}}{2} \]

\[ x^2 = \frac{1}{2\lambda} = \pm \frac{1}{\sqrt{3}} \quad \text{minima is impossible} \]

\[ x^2 = \frac{1}{\sqrt{3}} \quad x = \pm \frac{1}{\sqrt{3}} \]

So get solutions \( \lambda = \frac{1}{\sqrt{3}} \), \((x,y,z) = \left( \pm \frac{1}{\sqrt{3}}, \pm \frac{1}{\sqrt{3}}, \pm \frac{1}{\sqrt{3}} \right) \)

8 solutions already!

Also need to consider what happens when \( x=0 \), \( y, z \neq 0 \)

\( x=0, y=0, z \neq 0 \) etc.

**Example** Global max/min for

\[ f(x,y) = x^2 + y^2 + 4x - 4y \quad \text{on disk} \quad x^2 + y^2 \leq 9 \]

(i) critical points in interior
(ii) max/min on boundary \( x^2 + y^2 = 9 \)

Can think of (ii) as a constrained optimization
(i) \( \nabla f = \langle 2x+4, 2y-4 \rangle = 0 \)

\[
\begin{align*}
2x+4 &= 0 & x &= -2 & \text{(-2, 2)} \\
2y-4 &= 0 & y &= 2 \\
\end{align*}
\]

\[
x^2+y^2 = (-2)^2 + (2)^2 = 4 + 4 = 8 \leq 9
\]

So the critical point lies inside disk \( x^2 + y^2 \leq 9 \)

\[
f(-2, 2) = (-2)^2 + (2)^2 + 4(-2) - 4(2) = -8
\]

(ii) Max/min of \( f \) on curve \( g(x, y) = x^2 + y^2 = 9 \)

\[
\nabla f = \lambda \nabla g \quad \langle 2x+4, 2y-4 \rangle = \lambda \langle 2x, 2y \rangle
\]

\[
\begin{align*}
2x+4 &= \lambda \cdot 2x & \Rightarrow (2-2\lambda)x + 4 &= 0 \\
2y-4 &= \lambda \cdot 2y & x &= \frac{-4}{2-2\lambda} = \frac{2}{\lambda-1} \\
(2-2\lambda)y &= 4 & y &= \frac{4}{2-2\lambda} = \frac{2}{1-\lambda}
\end{align*}
\]

\[
\left( \frac{2}{\lambda-1} \right)^2 + \left( \frac{2}{1-\lambda} \right)^2 = 9
\]

\[
\begin{align*}
\frac{4}{(\lambda-1)^2} + \frac{4}{(\lambda-1)^2} &= 9 \\
\frac{8}{(\lambda-1)^2} &= 9 \\
(\lambda-1)^2 &= \frac{8}{9} \\
(\lambda-1) &= \pm \frac{2\sqrt{2}}{3} \\
\lambda &= 1 \pm \frac{2\sqrt{2}}{3}
\end{align*}
\]
\[ x = \frac{2}{\lambda - 1} = \frac{2}{\pm \frac{3\sqrt{3}}{2}} = \pm \frac{3}{\sqrt{2}} \]

\[ y = \frac{-2}{\lambda - 1} = -x \]

\[ (x = \frac{3}{\sqrt{2}}, \ y = -\frac{3}{\sqrt{2}}), \quad f = (\frac{3}{\sqrt{2}})^2 + (\frac{3}{\sqrt{2}})^2 + 4(\frac{3}{\sqrt{2}}) - 4(\frac{3}{\sqrt{2}}) \]

\[ (x = -\frac{3}{\sqrt{2}}, \ y = \frac{3}{\sqrt{2}}), \quad f = (\frac{3}{\sqrt{2}})^2 + (\frac{3}{\sqrt{2}})^2 + 4(\frac{3}{\sqrt{2}}) - 4(\frac{3}{\sqrt{2}}) \]

\[ = \frac{9}{2} + \frac{9}{2} + \frac{8.3}{\sqrt{2}} \]

\[ = \frac{9}{2} + \frac{9}{2} - \frac{8.3}{\sqrt{2}} \]

**Multiple constraints**: Optimize \( f(x,y,z) \) subject to two constraints \( g(x,y,z) = C \)

\[ h(x,y,z) = k \]

Two Lagrange multipliers \( \lambda, \mu \)

\[ \nabla f = \lambda \nabla g + \mu \nabla h \]

\[ g(x,y,z) = C, \quad h(x,y,z) = k \]

5 equations in 5 variables \( x, y, z, \lambda, \mu \).