Limits & partial derivatives:

\[ \lim_{(x,y) \to (a,b)} f(x,y) = L \]

means

For every \( \varepsilon > 0 \) there is a \( \delta > 0 \) such that

\[ 0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta \]

\( \iff \) distance from \( (x,y) \) to \( (a,b) \)

Then \( |f(x,y) - L| < \varepsilon \)

Notice: in order for the limit to exist, we need \( f \to L \) as \( (x,y) \to (a,b) \) along any path.

Limit along a path

Let \((x(t), y(t))\) be a parametric path such that \((x(0), y(0)) = (a, b)\).

Consider \( \lim_{t \to 0} f(x(t), y(t)) \)

\[ \lim_{(x,y) \to (a,b)} f(x,y) = L \] then this also \( \Rightarrow \) \[ \lim_{t \to 0} f(x(t), y(t)) = L. \]
Eq. \( f(x,y) = \frac{x^2 - y^2}{x^2 + y^2} \quad (a, b) = (0, 0) \)

Path 1: \( x(t) = t, \quad y(t) = 0 \quad \) x-axis
\[ f(x(t), y(t)) = \frac{t^2}{t^2} = 1 \quad \lim_{t \to 0} f(x(t), y(t)) = 1 \]

Path 2: \( x(t) = 0, \quad y(t) = t \quad \) y-axis
\[ f(x(t), y(t)) = \frac{-t^2}{t^2} = -1 \quad \lim_{t \to 0} f(x(t), y(t)) = -1 \]

\[ \min_{(x, y) \to (0, 0)} \frac{x^2 - y^2}{x^2 + y^2} \quad \text{does not exist.} \]

Eq. \( f(x,y) = \frac{xy}{x^2 + y^2} \)

Path 1: \( x(t) = t, \quad y(t) = 0 \)
\[ f(t, 0) = \frac{t \cdot 0}{t^2 + 0^2} = 0 \quad \lim_{t \to 0} f(t, 0) = 0 \]

Path 2: \( x(t) = 0, \quad y(t) = t \)
\[ f(0, t) = \frac{0 \cdot t}{0^2 + t^2} = 0 \quad \lim_{t \to 0} f(0, t) = 0 \]
Path 3: \( y = x, \quad x(t) = t, \quad y(t) = t \)

\[
f(t, t) = \frac{t \cdot t}{t^2 + t^2} = \frac{t^2}{2t^2} = \frac{1}{2}
\]

\[
\lim_{t \to 0} f(t, t) = \frac{1}{0}
\]

\( \lim \frac{xy}{x^2 + y^2} \) does not exist.

Path 4: \( y = mx, \quad x(t) = t, \quad y(t) = mt \)

\[
f(t, mt) = \frac{t \cdot mt}{t^2 + (mt)^2} = \frac{mt^2}{(1+m^2)t^2} = \frac{m}{1+m^2}
\]

Eq.: \( f(x, y) = \frac{xy^2}{x^2 + y^4} \)

Path \( y = mx, \quad x(t) = t, \quad y(t) = mt \)

\[
f(t, mt) = \frac{t \cdot (mt)^2}{t^2 + (mt)^4} = \frac{m^2 t^3}{t^2 + m^4 t^4}
\]

\[
\lim_{t \to 0} \left( \frac{m^2 t^3}{t^2 + m^4 t^4} \right) = \lim_{t \to 0} t \left( \frac{m^2}{1 + m^4 t^2} \right) = 0
\]

Path 2: \( y^2 = x, \quad x(t) = t^2, \quad y(t) = t \)
\[ f(t^2, t) = \frac{t^2 + t^2}{(t^2)^2 + t^4} = \frac{t^4}{t^4 + t^4} = \frac{1}{2} \]

So \( \lim_{(x,y) \to (a,b)} \frac{xy^2}{x^2 + y^4} \) does not exist.

Checking, cleverly chosen paths can show that a limit does not exist, but you can't prove that it does exist.

Positive results:

Limit properties hold:

\[ \lim f + g = \lim f + \lim g \]
\[ \lim f \cdot g = \lim f \cdot \lim g \]
\[ \lim cf = c \lim f \]
\[ \lim \frac{f}{g} = \frac{\lim f}{\lim g} \text{ if } \lim g \neq 0 \]

Squeeze: if \( f(x,y) \leq g(x,y) \leq h(x,y) \)
and \( \lim_{(x,y) \to (a,b)} f(x,y) = \lim_{(x,y) \to (a,b)} h(x,y) = L \)
then \( \lim_{(x,y) \to (a,b)} g(x,y) \) exists and equals \( L \).
Def \( f(x,y) \) is continuous at \((a,b)\)
if \( \lim_{(x,y) \to (a,b)} f(x,y) \) exists and equals \( f(a,b) \).

Ex: A polynomial is continuous
monomial \( x, y, x^2, y^2, xy, x^2y, x^5y^0, x^m y^n \)
Polynomial sum of monomials
ex \( 2x^2 + y^2 + 5x^3y^4 + 2y x \)
To take limit of continuous func., just plug in
Rational funcn \( \frac{f(x,y)}{g(x,y)} \) where \( f \) and \( g \) are polynomials
A rational function is continuous everywhere the denominator does not vanish
\[
\lim_{(x,y) \to (1,0)} \frac{x^2 - y^2}{x^2 + y^2} = \frac{1^2 - 0^2}{1^2 + 0^2} = 1
\]
\[ f(x, y) = \frac{3x^2y}{x^2 + y^2} \]

Limit exists and \( = 0 \) as \((x, y) \to (0, 0)\)

\[
|f(x, y)| = \left| \frac{3x^2y}{x^2 + y^2} \right| = 3|y| \left| \frac{x^2}{x^2 + y^2} \right|
\]

Notice \( \left| \frac{x^2}{x^2 + y^2} \right| \leq 1 \)

\[ 0 \leq |f(x, y)| \leq 3|y| \]

\[ -3y \leq f(x, y) \leq 3y \]

\[ \Rightarrow \quad 0 \quad \Rightarrow \quad 0 \quad \text{as} \quad (x, y) \to 0 \]

\[ \Rightarrow \quad 0 \quad \text{by squeeze theorem.} \]

**Partial Derivative:** Just as we can take limits in different directions, we can take derivatives in different directions.

Take derivative in x-direction, while holding y constant

\[ \frac{\partial f}{\partial x} = f_x \quad \text{partial derivative with respect to } x \]
Take deriv. in y-direction, holding $x$ constant
\[ \frac{\partial f}{\partial y} = f_y \]
Partial derivative with respect to $y$.

There's no reason for \( \frac{\partial f}{\partial x} \) and \( \frac{\partial f}{\partial y} \) to be equal and they are almost never equal.

\[ \frac{\partial f}{\partial x} = f_x \]
hold $y$ constant, and vary $x$.

To compute \( f_x(a, b) \), Fix $y = b$ constant.

Then we get \( g(x) = f(x, b) \)

Then take \( g'(x) \) deriv w.r.t. $x$

Plug in a \( g'(a) \)

\[ f_x(a, b) = g'(a) = \lim_{h \to 0} \frac{g(a+h) - g(a)}{h} \]

\[ = \lim_{h \to 0} \frac{f(a+h, b) - f(a, b)}{h} \]

\[ \frac{\partial f}{\partial y} = f_y \]

\[ f_y(a, b) = \lim_{h \to 0} \frac{f(a, b+h) - f(a, b)}{h} \]
Symbolically, just treat one variable as constant.

\( f(x, y) = x^2 - y^2 \), find \( \frac{\partial f}{\partial x} \frac{\partial f}{\partial y} \) at \( (x, y) = (1, 1) \)

\[
\frac{\partial f}{\partial x} = 2x - 0
\]

\( \overline{\text{Note: } y^2 \text{ is constant w.r.t. } x. \quad (\text{constant}) = 0} \)

\[
\frac{\partial f}{\partial y} = 0 - 2y
\]

\[
\left. \frac{\partial f}{\partial x} \right|_{(1, 1)} = 2 \quad \left. \frac{\partial f}{\partial y} \right|_{(1, 1)} = -2
\]

\[
f(x, y) = xy \quad \frac{\partial f}{\partial x} = y \quad \frac{\partial f}{\partial y} = x
\]

\[
f(x, y) = e^{-y} \cos \pi x
\]

\[
\frac{\partial f}{\partial x} = \frac{\partial}{\partial x} (e^{-y} \cos \pi x) = e^{-y} \frac{\partial}{\partial x} (\cos \pi x)
\]

\[
= e^{-y} (-\pi \sin \pi x)
\]

\[
\frac{\partial f}{\partial y} = \frac{\partial}{\partial y} (e^{-y} \cos \pi x) = \frac{\partial}{\partial y} (e^{-y}) \cos \pi x
\]

\[
= -e^{-y} \cos \pi x
\]
\[ w = \ln(u + 2v) \] find \( \frac{\partial w}{\partial u} \) and \( \frac{\partial w}{\partial v} \)

\[
\frac{\partial w}{\partial u} = \frac{\partial}{\partial u} \ln(u + 2v) = \frac{1}{u + 2v} \frac{\partial}{\partial u} (u + 2v) = \frac{1}{u + 2v}
\]

\[
\frac{\partial w}{\partial v} = \frac{\partial}{\partial v} \ln(u + 2v) = \frac{1}{u + 2v} \frac{\partial}{\partial v} (u + 2v) = \frac{2}{u + 2v}
\]

Implicit differentiation works.

\[ x^2 + y^2 + z^2 = 1 \quad \frac{\partial z}{\partial x} \] holding \( y \) constant

\[
\frac{\partial}{\partial x} (x^2 + y^2 + z^2 = 1) \Rightarrow 2x + 0 + 2z \frac{\partial z}{\partial x} = 0
\]

\[ \frac{\partial z}{\partial x} = \frac{-2x}{2z} = \frac{-x}{z} \]

Higher derivatives \((f_x)_x = f_{xx} = \frac{\partial}{\partial x} (\frac{\partial f}{\partial x}) = \frac{\partial^2 f}{\partial x^2}\)

\((f_y)_y = f_{yy} = \frac{\partial}{\partial y} (\frac{\partial f}{\partial y}) = \frac{\partial^2 f}{\partial y^2}\)

\((f_x)_y = f_{xy} = \frac{\partial}{\partial y} (\frac{\partial f}{\partial x}) = \frac{\partial^2 f}{\partial y \partial x}\)

\((f_y)_x = f_{yx} = \frac{\partial}{\partial x} (\frac{\partial f}{\partial y}) = \frac{\partial^2 f}{\partial x \partial y}\)
Usually,

It doesn't matter what order you take partial derivatives in. \( f_{xy} = f_{yx} \)

(as long as both continuous)