Conditional Probability

Office hours M 1-2, 3-4, W 9:30-10:30

\[ P(E|F) = \text{probability that } E \text{ occurs given that } F \text{ occurs} \]

Condition

Importance
(i) compute probability w/partial information
(ii) break up problems into conditional ones which may be easier.
(iii) Reasoning about hypotheses/evidence (Bayes’s Formula)
(iv) Can define “independent events”

Suppose we are dealt 2 cards from a 52 card deck:

\[
\begin{bmatrix}
\text{Cards} & 13 \text{ values } 2, 3, 4, \ldots, 10, J, Q, K, A \\
\text{4 suits} & \text{Spades, hearts, diamonds, clubs}
\end{bmatrix}
\]

eg. Aspadee
Suppose dealt 2 cards \( P(2 \text{ aces}) \)?

\[
\binom{52}{2} = \text{ total \# 2 card hands}
\]

= \# points in \( S \)

\[
\binom{4}{2} = \text{ pairs of aces}
\]

\[
P(2 \text{ aces}) = \frac{\binom{4}{2}}{\binom{52}{2}} = \frac{\frac{4 \cdot 3}{2}}{\frac{52 \cdot 51}{2}} = \frac{4 \cdot 3}{52 \cdot 51}
\]

Suppose get 1 card at a time

\[P(2 \text{ aces} \mid \text{ first card is an Ace})\]

Prob of getting 2 aces given first card is an ace.

Thinking of drawing second card as a new experiment.

3 aces out of 51 cards left

\[
P(2 \text{ aces} \mid \text{ first is Ace}) = \frac{3}{51}
\]

\[
P(\text{1st card is an Ace}) = \frac{4}{52}
\]
See \( P(2\text{ aces}) = P(1\text{st card is Ace}) \cdot P(2\text{ aces | 1st card is A}) \)

\[
\frac{4 \cdot 3}{52 \cdot 51} = \frac{4}{52} \cdot \frac{3}{51}
\]

\[P(2\text{ aces | 1st and is A}) = \frac{P(2\text{ aces})}{P(1\text{st and is Ace})}\]

Formulate
\[
F = 1\text{st and is A}
\]
\[
E = 2\text{nd and is A}
\]

"2 aces" = EF

\( \Box \)

\[
P(E | F) = \frac{P(EF)}{P(F)}
\]

We promote \( \Box \) to a definition

If \( E \) and \( F \) are events, we define \( P(E | F) \) by \( \Box \)

\[P(E | F) = \text{"Prob E given F"}, \]

"Prob E conditional on F",

Suppose \( x \) is an outcome. If \( F \) occurs then \( x \) is in \( F \).

If we want \( E \) to also occur, we need \( x \) in \( E \) also.

So ultimately \( x \) is in \( EF \).

That's why we take \( P(EF) \)
\[ 1 = P(F \mid F) = \frac{P(FF)}{C} = \frac{P(F)}{C} \quad \text{so} \quad c = P(F) \]

So \( P(F) \) is a normalization.

Another interpretation: Given that \( F \) is known to occur: then we can replace the sample space \( S \) with the subset \( F \).

(reduced sample space)

Example: urn with \( r \) red and \( b \) blue balls

\( n \) balls chosen in order \( w/o \) replacement

\( n \leq r+b \)

Suppose \( k \) of \( n \) chosen are blue

what is \( P(1\text{st ball chosen is blue}) \)

We work in reduced sample space

\( B_k = \) event that \( k \) blue balls are chosen.

Each of the outcomes in \( B_k \) is equally likely.

(need to think)

Among the \( n \) balls chosen, the first is equally likely to be any of these \( n \), and there are \( k \) chances for it to be blue.
so \( P(\text{4th is bhw} \mid B_k) = \frac{k}{n} \)

Working w/ full sample space

\( B = \text{first ball chosen is bhe} \)

\( B_k = k \text{ bhw balls are chosen} \)

\[
P(B \mid B_k) = \frac{P(BB_k)}{P(B_k)}
\]

\[
P(BB_k) = P(B) \ P(B_k \mid B)
\]

\[
P(B_k \mid B) = \frac{P(Bk \mid B) \ P(B)}{P(B_k)}
\]

\[
P(B) = \frac{b}{r+b} \quad P(B_k) = \frac{(b^r)(n-k)}{(r+b)^n}
\]

\[
P(B_k \mid B) = \frac{(b-1)^r (n-k)}{(r+b-1)^{n-1}}
\]
\[ P(B | B_K) = \frac{P(B_K | B) \cdot P(B)}{P(B_K)} = \frac{k}{n} \]