Lecture 6: Level sets in 3rd (14.1),
quadric surfaces (12.6), review of limits (14.2)

Last time: \( f: \mathbb{R}^2 \to \mathbb{R} \quad f(x, y) = x^2 - y^2 \)

Graph

For \( f: \mathbb{R}^3 \to \mathbb{R} \) can't draw the graph
(in \( \mathbb{R}^4 \)) but can still
look at level sets.

[Did \( f(x, y, z) = x^2 + y^2 + z^2 \) last time]

Ex: \( f(x, y, z) = x^2 + y^2 - z^2 \)

First, look at \( xz \)-plane

\( f(x, 0, z) = x^2 - z^2 \)

so the level sets
in this plane look like what we had on Wed. Also, as \(x^2 + y^2 = r^2\)

we have \(f(x, y, z) = r^2 - z^2\) and so each level set is symmetric about \(z\)-axis:

[These level sets are all examples of quadric surfaces.]
Conic Sections: Solutions in $\mathbb{R}^2$

of $A x^2 + B x y + C y^2 + D x + E y + F = 0$

Circle  Ellipse  Parabola  Hyperbola

Quadric Surfaces in $\mathbb{R}^3$ (Section 12.6)

$A x^2 + B y^2 + C z^2 + D x y + E x z + F y z + G x + H y + I z + J = 0$

Ex: Ellipsoid

$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

Elliptic paraboloid:

$\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$

Hyperbolic paraboloid:

$\frac{z}{c} = \frac{x^2}{a^2} - \frac{y^2}{b^2}$
The other quadric surfaces are the double cone and the hyperboloids of 1 and 2 sheets that we saw at the start.

\[ f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \]

Limits (14.2) [To talk about derivatives, first need to understand limits of fns of several variables.]

Can take different perspectives on limits — I’ll focus on them as a way of estimating error.

Suppose we want to fabricate a square with area 1 m\(^2\) and it comes back with sides of length 1 + h

\[ \text{Error} = \text{actual area} - 1 = (1+h)^2 - 1 \]
\[ = h^2 + 2h \]

Q: If we want error < \(\frac{1}{10}\), to what tolerance do we need to make the square?
Consider $E : \mathbb{R} \to \mathbb{R}$ (an "error function")

We say

$$\lim_{h \to 0} E(h) = 0 \quad \text{if given } \varepsilon > 0 \text{ we can always find } \delta > 0 \text{ so that whenever } 0 < |h| < \delta$$

we have $|E(h)| < \varepsilon$

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$E(h) = h^2$

View as challenge-response process.

**Ex:** $E(h) = h^2 \quad \varepsilon = \frac{1}{10}$.

Take $\delta = \frac{1}{4}$. If $|h| < \delta = \frac{1}{4}$

then $|E(h)| = |h^2| = |h|^2 < \frac{1}{16} < \frac{1}{10}$.

**Ex:** $\varepsilon = \frac{1}{100} \quad \delta = \text{Audience Response}$

$\varepsilon = \frac{1}{1000} \quad \delta = \underline{\text{Audience Response}}$
Claim: \( \lim_{h \to 0} h^2 = 0 \)

Reason: if you give me \( \varepsilon > 0 \), we'll take \( S = \sqrt{\varepsilon} \). Then if \( |h| < S \) we have

\[
|h^2| = |h|^2 < S^2 = \varepsilon.
\]

Ex: \( E(h) = 2h + h^2 \)  

Know \( \lim_{h \to 0} 2h + h^2 = 0 \)

Given \( \varepsilon = \frac{1}{10} \) take \( S = \frac{1}{100} \).

If \( |h| < S \), then \( |2h + h^2| \leq 2|h| + |h|^2 \) 

\[
< 2 \cdot \frac{1}{100} + \frac{1}{10000} < \frac{3}{100} < \frac{1}{10} = \varepsilon.
\]

In general, say

\[
\lim_{x \to a} f(x) = C
\]

if \( f(a+h) = C + E(h) \)

where \( \lim_{h \to 0} E(h) = 0 \).

Differentiability:

\[
f(x+h) = f(x) + f'(x)h + E(h)
\]

where \( E(h) \) is really small, i.e.

\[
\lim_{h \to 0} \frac{E(h)}{h} = 0.
\]