Students in the three credit hour course must solve five of the six problems. Students in the four credit hour course must solve all six problems.

1. For \( k \geq 2 \), prove that every \( k \)-regular bipartite graph has no cut-edge and construct a bipartite graph with all vertex degrees in \( \{k, k + 1\} \) that has a cut-edge.

2. Given an integer \( k \geq 2 \), let \( G(k) \) be the subgraph of the cube \( Q_{2k} \) induced by the vertices in which the number of ones is either \( k \) or \( k + 1 \). Compute the number of vertices, the number of edges, and the girth (the length of a shortest cycle) of \( G(k) \).

3. Given a nonincreasing list \( d = (d_1, \ldots, d_n) \) of nonnegative integers and \( 1 \leq k \leq n \), let \( d(k) \) be obtained from \( d \) by deleting \( d_k \) and subtracting 1 from the \( d_k \) largest elements remaining in the list. Prove that \( d \) is graphic if and only if \( d(k) \) is graphic. (Hint: Mimic the proof of Havel–Hakimi Theorem.)

4. For every odd \( n \), construct an \( n \)-vertex tournament in which every vertex is a king. Does there exist such a tournament with 4 vertices?

5. Let \( n \geq 3 \) and \( G \) be an \( n \)-vertex graph. Prove that the following are equivalent.
   (A) \( G \) is connected and has exactly one cycle.
   (B) \( G \) is connected and has \( n \) edges.
   (C) \( G \) has exactly one cycle and has \( n \) edges.

6. For \( n \geq 3 \), prove that if an \( n \)-vertex graph \( G \) has three vertices \( v_1, v_2, v_3 \) such that the subgraphs \( G - v_1, G - v_2, G - v_3 \) are acyclic, then \( G \) has at most one cycle.

Problems below review basic concepts and their ideas could be used in the tests.

WARMUP PROBLEMS: Section 1.3: # 1, 4, 5, 9, 12. Section 1.4: # 3, 4, 7. Section 2.1: # 1, 2, 4, 6, 13. Do not write these up!

OTHER INTERESTING PROBLEMS: Section 1.3: # 24, 40, 41, 57, 63. Section 1.4: # 9, 10, 21, 26, 28, 29, 36, 37. Section 2.1: # 15, 19, 27, 29, 31, 44, 52, 53.

Do not write these up!