Problem 1
In order not to make the statement vacuously true, when we say that a graph is connected, we mean that every two of its vertices are joined by a path. First we define a directed graph \( D \) on the same set of vertices as \( G \) by putting two directed edges \((v,w)\) and \((w,v)\) for each \( \{v,w\} \in E(G) \). It is obvious that any two vertices \( v,w \in V(G) \) are joined by a path in \( G \) if and only if there is a directed path from \( v \) to \( w \) (and also a path from \( w \) to \( v \)) in \( D \). Hence it suffices to prove that every two distinct vertices of \( D \) are joined by a directed path if and only if for every partition of \( V \) into two non-empty sets \( S \) and \( S' \), there is an edge in \( D \) from \( S \) to \( S' \).

The implication from left to right is trivial. Suppose there are non-empty \( S \) and \( S' \) with no edges from \( S \) to \( S' \). Take any \( s \in S \) and \( t \in S' \). By the assumption there is a directed \((s,t)\)-path in \( D \). This path has to cross from \( S \) to \( S' \), which contradicts the choice of \((S,S')\).

To prove the other implication, take two distinct vertices \( s \) and \( t \). We may view \( D \) as a network \( N \) by assigning unit capacities to all directed edges. Since by assumption there are no empty cuts in \( D \), the minimum capacity of a cut in \( N \) is at least 1. Hence by max-flow min-cut theorem a maximum \((s,t)\)-flow in \( N \) is non-zero. Therefore the all-zero flow \( f_0 \) is not a maximum flow, and by max-flow min-cut theorem the incremental network \( N(f_0) = N \) contains an augmenting path. In particular this augmenting path is an \((s,t)\)-path in \( D \).

Problem 2
Given the list of \( k \) departments \( D_1, \ldots, D_k \), construct a network \( N \) with the set of vertices \( V(N) = \{\text{Assist, Assoc, Full}, D_1, \ldots, D_k\} \) by adding for each professor one directed edge with unit capacity that connects professor’s academic position to the department they are in (multiple edges are allowed). Now, vertices Assist, Assoc, Full will become sources with supply \( k/3 \) each, and the remaining vertices \( D_1, \ldots, D_k \) will all be sinks with unit demand.

Suppose we have an integer solution to the above supply-demand problem (since all supplies, demands and capacities are integers, there will be an integer solution unless the problem is not feasible). Every edge of the network carries either unit or zero flow. It is easy to see that professors corresponding to saturated edges form a valid committee, since there are exactly \( k/3 \) saturated edges leaving each academic position vertex and exactly one saturated incoming edge at every departmental vertex.

Problem 3
(a) One consistent rounding is given below

\[
\begin{pmatrix}
0 & 0 & 1 \\
0 & 1 & 1 \\
0 & 0 & 1
\end{pmatrix}
\]

(b & c) We will prove the more general statement that every \( n \times n \) matrix has a consistent rounding, for any \( n \). Let \( A \) be an \( n \times n \) matrix. Without loss of generality, we may assume that \( A \) contains only nonnegative entries, since adding an integer to any entry of \( A \) does not affect whether or not it has a consistent rounding. Let \( r_i \) and \( c_j \) denote the sums of the \( i \)th row and \( j \)th column of \( A \), respectively. Construct a network \( N \) as in lecture, i.e. \( N \) has vertex set \( \{s,t,u,v\} \cup \{x_i\}_{i=1}^n \cup \{y_j\}_{j=1}^n \), edges

- \( sv \), with capacity \( \sum_j c_j \);
- \( su \), with capacity \( |r_i| - |r_j| \);
- \( sx_i \), with capacity \( |r_i| - \sum_j a_{ij} \);
- \( y_jv \), with capacity \( |c_j| - |c_j| \);
- \( ut \), with capacity \( \sum_i |r_i| \);
- \( xiy_j \), with capacity \( |a_{ij}| - |a_{ij}| \);
- \( yjt \), with capacity \( |c_j| - \sum_i |a_{ij}| \);
- \( vu \), with capacity \( \infty \).

Let \( M := \sum_i |r_i| + \sum_j |c_j| - \sum_{i,j} |a_{ij}| \). Then \( A \) has a consistent rounding if and only if \( N \) admits an \( s,t \)-flow of value \( M \). To show that \( N \) has such a flow, it suffices to show that any \( s,t \)-cut in \( N \) has value at least \( M \).
Fix $S \subset V(N)$ such that $s \in S$ and $t \in \overline{S}$. We have four cases to consider.

- $u \in \overline{S}, v \in S$: The size of the cut is at least $c(vu) = \infty > M$.
- $u \in S, v \in \overline{S}$: The size of the cut is at least $c(u) + c(sv) = \sum_i |r_i| + \sum_j |c_j| \geq M$.
- $u \in S, v \in S$: This case is more difficult. The size of the cut is now
  \[
  c(ut) + \sum_{x_i \in S} (c(wx_i) + c(sx_i)) + \sum_{y_j \in S} c(y_j t) + \sum_{x_i \in S, y_j \in S} c(x_i y_j)
  \]
  \[
  = \sum_i |r_i| + \sum_{x_i \in S} (|r_i| - \sum_{j} |a_{ij}|) + \sum_{y_j \in S} (|c_j| - \sum_i |a_{ij}|) + \sum_{x_i \in S, y_j \in S} (|a_{ij}| - |a_{ij}|)
  \]
  \[
  = \sum_i |r_i| + \sum_{x_i \in S, y_j \in S} |r_i| + \sum_{x_i \in S, y_j \in S} |a_{ij}| + \sum_{y_j \in S} |c_j| - \sum_i |a_{ij}|
  \]
  \[
  \geq \sum_i |r_i| + \sum_{i,j} a_{ij} + \sum_{i,j} a_{ij} + \sum_{i,j} |c_j| - \sum_i |a_{ij}|
  \]
  \[
  \geq \sum_i |r_i| + \sum_{i,j} |c_j| + \sum_{i,j} |c_j| - \sum_i |a_{ij}|
  \]
  \[
  = M.
  \]

- $u \in \overline{S}, v \in \overline{S}$: This is similar to the previous case. The size of the cut is
  \[
  c(sv) + \sum_{y_j \in S} (c(y_j v) + c(y_j t)) + \sum_{x_i \in S} c(sx_i) + \sum_{x_i \in S, y_j \in S} c(x_i y_j)
  \]
  \[
  = \sum_j |c_j| + \sum_{x_i \in S, y_j \in S} (|c_j| - \sum_{i} |a_{ij}|) + \sum_{x_i \in S} (|r_i| - \sum_j |a_{ij}|) + \sum_{x_i \in S, y_j \in S} (|a_{ij}| - |a_{ij}|)
  \]
  \[
  = \sum_j |c_j| + \sum_{x_i \in S, y_j \in S} |c_j| + \sum_{x_i \in S} |a_{ij}| + \sum_{x_i \in S} |r_i| - \sum_j |a_{ij}|
  \]
  \[
  \geq \sum_j |c_j| + \sum_{x_i \in S} a_{ij} + \sum_{x_i \in S} a_{ij} + \sum_{x_i \in S} |r_i| - \sum_j |a_{ij}|
  \]
  \[
  \geq \sum_j |c_j| + \sum_{x_i \in S} |r_i| + \sum_{x_i \in S} |r_i| - \sum_j |a_{ij}|
  \]
  \[
  = M.
  \]

This completes the proof.

**Problem 4**

Since the case $n = 1$ is trivial, let $n \geq 2$ and let $A$ be an $n$-by-$n$ matrix with every entry in the interval $(1/n, 1/(n - 1)) \subseteq (0, 1)$. Simple computation shows that the sum of every row and column is a number in the interval $(1, 2)$. Therefore every consistent rounding of $A$ is simply a 0, 1-matrix $A'$ with 1 or 2 ones in every row and column. We can view $A'$ as a bi-adjacency matrix of a bipartite graph $G$ with partite sets of size $n$. The condition on $A'$ means exactly that degree of every vertex in $G$ is either 1 or 2. This is equivalent to saying that $G$ is a disjoint union of (even) cycles and paths.
**Problem 5**

(a) We shall prove that a \((k + l)\)-regular graph \(G\) is \((k, l)\)-orientable if and only if there is a partition \(X, Y\) of \(V(G)\) such that for every \(S \subseteq V(G)\)

\[
(k - l)(|X \cap S| - |Y \cap S|) \leq |[S, \bar{S}]|,
\]

Suppose that the graph has a \((k, l)\)-orientation \(D\). Let \(X\) and \(Y\) denote the sets of vertices with outdegree \(k\) and \(l\) respectively. Obviously they form a partition of \(V(G)\). Let \(S\) be any subset of \(V(G)\). Simple edge counting yields

\[
\sum_{v \in S} d^+(v) + |[S, \bar{S}]_D| = \sum_{v \in S} d^-(v) + |[S, \bar{S}]_D|.
\]

We finish the proof by observing that

\[
|[S, \bar{S}]| \geq |[S, \bar{S}]_D| \geq \sum_{v \in S} (d^+(v) - d^-(v)) = \sum_{v \in X \cap S} (d^+(v) - d^-(v)) + \sum_{v \in Y \cap S} (d^+(v) - d^-(v)) = \sum_{v \in X \cap S} (k - l) + \sum_{v \in Y \cap S} (l - k) = (k - l)(|X \cap S| - |Y \cap S|).
\]

The proof of the other direction is a bit more intricate. We construct a network \(N\) on the vertex set \(V(N) = V(G) \cup E(G)\) with edge set \(E(N) = \{v, e : v \in e \in E(G)\}\), i.e. for each edge \(e\) in \(G\) there we add two arcs from both endpoints of \(e\) to \(e\) itself. All arcs in \(N\) are assigned unit capacities. Finally define a supply-demand problem in \(N\) by assigning unit demand \(\delta\) to all vertices in \(E(G)\) and supplies \(\sigma = k, l\) to all vertices in \(X, Y\) respectively.

Suppose there is an integer solution for the above supply-demand problem. Pick an edge \(e\) of \(G\). Since the demand of that edge is \(1\) and it has only two incoming arcs, both coming from endpoints of \(e\) in \(G\), exactly one of these arcs has flow value \(1\). Orient \(e\) in \(G\) such that it leaves the vertex from which it receives the unit flow. This procedure gives a unique orientation to all edges of \(G\). Since a vertex \(v \in V(G)\) sends a unit flow to exactly those edges that are coming out of it in the produced orientation, all vertices in \(X, Y\) have outdegrees \(k, l\) respectively. To finish the proof, it is enough to show that there exists an integer solution to the supply-demand problem. By integrality, it is enough to show that the problem is feasible.

Now we turn to Gale’s Theorem. Let \((A, \bar{A})\) be any partition of \(V(N)\). It uniquely corresponds to a partition \((S, \bar{S})\) of \(V(G)\) and a partition \((E, \bar{E})\) of \(E(G)\). To use Gale’s Theorem, we only have to show that

\[
\sigma(A \cap V(G)) + c(\bar{A}, A) \geq \delta(A \cap E(G)),
\]

or in other words

\[
\sigma(S) + c(\bar{S}, E) \geq \delta(E) = |E|.
\]

First note that \(\sigma(S) = k|X \cap S| + l|Y \cap S|\) and

\[
c(\bar{S}, E) = 2|E \cap \left(\begin{array}{c} \bar{S} \\ 2 \end{array}\right)| + |E \cap [S, \bar{S}]| \geq |E| - |E \cap \left(\begin{array}{c} S \\ 2 \end{array}\right)|.
\]

Hence it is enough to show that

\[
\frac{|S|(k + l) - |[S, \bar{S}]|}{2} = |E \cap \left(\begin{array}{c} S \\ 2 \end{array}\right)| \leq k|X \cap S| + l|Y \cap S| = (k + l)|S| - k|X \cap S| - l|Y \cap S|,
\]

which is equivalent to

\[
2k|X \cap S| + 2l|Y \cap S| \leq |[S, \bar{S}]| + (k + l)|S| = |[S, \bar{S}]| + (k + l)(|S \cap X| + |S \cap Y|),
\]

which is easily seen to be equivalent to the assumed inequality.
(b) Suppose first that \( k > l + 1 \) and \( G \) is \((k,l)\)-orientable. By part (a) there is a partition of \( V(G) \) to a pair of sets \((X,Y)\) such that the inequality from (a) holds for all \( S \subseteq V(G) \). It suffices to show that for that particular choice of the partition, we also have

\[
(k - l - 2)(|X \cap S| - |Y \cap S|) = (k - 1) - (l + 1))(|X \cap S| - |Y \cap S|) \leq |S, S^c|.
\]

This is immediate for \( S \) such that \( \Delta(S) = (|X \cap S| - |Y \cap S|) \geq 0 \), since then

\[
(k - l - 2)\Delta(S) \leq (k - l)\Delta(S) \leq |S, S^c|.
\]

Thus we may restrict our attention to the case when \( \Delta(S) < 0 \). But note that by our initial assumption on \( k \) and \( l \), we have \( k - l - 2 \geq 0 \), so we are also done, since the left-hand side of the inequality is non-positive. We are left with the case \( k = l + 1 \). But then \((k-1, l+1) = (l,k)\) and again we are done, because obviously being \((k,l)\)-orientable or \((l,k)\)-orientable is equivalent.

Problem 6
Let \( A \) be the clique-vertex incidence matrix of \( G \). We will first prove that if \( Ax \leq 1 \) is TDI, then \( G \) is perfect. Let \( H \) be an induced subgraph of \( G \) and let \( c \) be the characteristic vector of \( H \), i.e. \( c(v) = 1 \) for \( v \in H \) and \( c(v) = 0 \) otherwise. Let \((P)\) be the linear program \( \max cx, Ax \leq 1, x \) unrestricted. If \( x \) is feasible for \((P)\), then \( x_i \leq 1 \) for all \( i \) since the vertex \( x_i \) by itself is a clique.

Claim. Some optimal solution of \((P)\), satisfies \( x_i \geq 0 \) for all \( i \).

Proof. Let \( x \) be an optimal solution with minimum number of negative coordinates. Suppose for contradiction that there is an \( i \) such that \( x_i < 0 \). Let \( \tilde{x}_i = 0 \) and \( \tilde{x}_j = x_j \) for \( j \neq i \). \( c \tilde{x} \geq cx \), so by the choice of \( x, \tilde{x} \) is not feasible. In other words, there is a \( k \) such that \( a_k \tilde{x} > 1 \) (where \( a_k \) is the \( k \)-th row of \( A \)). Let \( Q \) be the clique corresponding to the \( k \)-th row of \( A \) and \( v_i \) the vertex corresponding to \( x_i \). Then \( Q \setminus \{v_i\} \) is a clique (nonempty since \( a_k \tilde{x} > 1 \)), and if \( a_s \) is the corresponding row of \( A \), then \( a_s x = a_k \tilde{x} > 1 \). This is a contradiction with the feasibility of \( x \).

Since the considered program is TDI, all basic feasible solutions of \((P)\) are integral, hence we have an optimal \([0,1]\)-solution. It corresponds to an independent set in \( H \) of size \( \alpha(H) \). In particular we have proved that for that particular choice of the vector \( c \), the dual program is feasible and bounded (since \((P)\) is feasible and bounded), hence by the assumption that \((P)\) is TDI, \((D)\) has an integral optimal solution.

Let \((D)\) be the dual of \((P)\). This is how it looks like:

\[
\begin{align*}
\min \ \sum_i \pi_i \\
A^T \pi = c \\
\pi \geq 0.
\end{align*}
\]

Note that if \( \pi \) is any feasible solution of \((D)\), then for each clique \( Q \) in the graph \( G \), \( \sum_{v \in Q} \pi(Q) \) is either 0 or 1, and so \( 0 \leq \pi \leq 1 \). Hence the integer optimal solution of \((D)\) guaranteed by TDI property is a \([0,1]\)-vector. It corresponds to a (minimum) set of disjoint cliques covering the vertex set of \( H \). The cost of the optimal solution is therefore equal to \( \chi(H) \).

By the duality theorem the optimal costs of \((P)\) and \((D)\) are the same, so \( \alpha(H) = \omega(H) = \chi(H) \). Since \( H \) was an arbitrary induced subgraph of \( G \), \( G \) is perfect, and hence \( G \) is perfect as well.

Now suppose that \( G \) is perfect. We want to show the existence of an integer optimal solution for every integer vector \( c \) such that \((D)\) is feasible and bounded for this \( c \).

Claim. \((D)\) is feasible and bounded if and only if \( c \geq 0 \).

Proof. First notice that if there is an \( i \) such that \( c_i < 0 \), then letting \( x_i = -M, x_j = 0 \) for \( j \neq i \) gives a feasible solution. If we let \( M \to \infty \), we get a feasible solution of arbitrarily large cost, which makes \((P)\) unbounded and in turn \((D)\) infeasible. Otherwise \( \pi(\{v_i\}) = c_i \geq 0 \) and \( \pi(Q) = 0 \) for \(|Q| > 1 \) is a feasible solution of \((D)\), so \((D)\) is feasible if and only if \( c \geq 0 \).

In either case \( x = 0 \) is a feasible solution for \((P)\), with cost 0, hence 0 is a lower bound for the cost of \((D)\). Therefore \((D)\) is bounded and feasible exactly when \( c \geq 0 \).
Now suppose $c$ is an integer vector. We will form a new graph $G'$ from $G$ by replacing vertex $v$ with an independent set $I_v$ of size $c(v)$, e.g., $I_v = \emptyset$ whenever $c(v) = 0$. For $x \in I_v$ and $y \in I_u$, $xy \in E(G')$ if and only if $vu \in E(G)$. By Lovász’s Replacement Lemma $G'$ is perfect.

Let $A'$ be the clique-vertex incidence matrix of the new graph $G'$ and consider a new primal problem ($P'$):

$$\max c'x' = \sum x'_i$$
$$A'x' \leq 1$$
$$x' << 0.$$

Since now all coefficients of the cost vector are equal to one, characteristic vector of any independent set is a feasible solution, so the optimal cost of ($P'$) is at least $\alpha(G')$. We will show that it is also true for the optimal cost of ($P$) by proving the following stronger statement.

**Claim.** For every feasible solution $x'$ of ($P'$) there is a feasible solution $x$ of ($P$) with $cx \geq c'x'$.

**Proof.** Note that if $x'$ is a feasible solution of ($P'$), then $x(v) = \max \{x'(v') : v' \in I_v\}$ is a feasible solution of ($P$). To check it, take any clique $Q = \{v_1, \ldots, v_k\}$ in $G$ and note that $Q' = \{v'_1, \ldots, v'_k\}$ is a clique in $G'$ whenever $v'_i \in I_{v_i}$ for all $i$. Take $v'_i$’s used in the max in the definition of $x(v)$ and note that

$$\sum_{i=1}^k x(v_i) = \sum_{i=1}^k x'(v'_i) \leq 1,$$

since $x'$ was feasible. It is straightforward that $cx \geq c'x'$.

By the duality theorem $\alpha(G')$ is also a lower bound for the optimal cost of ($D$). Note that we can easily produce an integer solution of ($D$) with cost $\chi(\bar{G}')$, simply start with a partition of $G'$ into $\chi(\bar{G}')$ cliques (color classes in an optimal coloring of $\bar{G}'$) and for each clique $Q$ of $G$ set

$$\pi(Q) = |\{Q' : Q' is a 'copy' of Q that occurs in the partition\}| \in \mathbb{Z}.$$

Hence we have

$$\alpha(G') \leq \text{opt}(P) = \text{opt}(D) \leq \chi(\bar{G'}).$$

But since $G'$ is perfect, $\alpha(G') = \chi(\bar{G'})$, and this proves that the integer vector we constructed above is in fact an optimal solution for ($D$), and therefore ($P$) is TDI.