1 Lecture 8: Random Turán Theorem (2/4/19)

In this section, we shall deduce the sparse random analogue of a classical theorem by Turán [7]. We start by asking the following question.

Question 1.1. *For what values of p is it true that the maximum size of a K_r-free subgraph of $G(n,p) \approx \left(1 - \frac{1}{r-1} + o(1)\right) p \cdot \binom{n}{2}$?*

We first consider what happens for certain choices of p. When $p = 1$, then $G(n,1) = K_n$, so we can use Turán's theorem directly. What is the situation for other values of p?
If the expected number of K_r's is much less than the expected number of edges in the random graph, then we can remove one edge from each K_r and still get a large K_r-free subgraph with $\approx (1 - o(1))p \cdot \binom{n}{2}$ edges. For what value of p will this be the case?

$$\begin{aligned} \text{Expected number of } K_r\text{'s} &\ll \text{Expected } |E(G(n,p))| \\ \binom{n}{r} p^{\binom{r}{2}} &\ll pn^2 \\ n^r p^{(r^2-r)/2} &\ll pn^2 \\ p &\ll n^{-2/(r+1)}. \end{aligned}$$

This raises the following question. Is Question 1.1 true for $p \geqslant C \cdot n^{-2/(r+1)}$? This theorem was already proved by Conlon and Gowers [2] and Schacht [5].

Theorem 1.2 ([2])**.** *Given $\epsilon > 0$ and a strictly 2-balanced graph H, there exists a positive constant C such that*

$$\lim_{n\to\infty} \mathbb{P}\left(G(n,p) \text{ is } (H,\epsilon) - \text{Turán}\right) = 1, \text{ if } p > Cn^{-1/m_2(H)},$$

where $m_2(H) = (|E(H)| - 1)/(|V(H)| - 2)$.

Recall that a graph H is said to be *strictly 2-balanced* if, for every subgraph $K \subset H$,

$$\frac{|E(H)| - 1}{|V(H)| - 2} > \frac{|E(K)| - 1}{|V(K)| - 2},$$

so clearly K_r is strictly 2-balanced. In addition, we define a graph G to be (H,ϵ)-*Turán* if every subgraph of G with at least

$$\left(1 - \frac{1}{\chi(H) - 1} + \epsilon\right) |E(G)|$$

edges contains a copy of H. We prove the following version of the above theorem for $H := K_r$ using the Hypergraph Container Lemma.

Theorem 1.3 (**Random Turán Theorem** [1])**.** *Given a K_r-free subgraph $F \subseteq G(n,p)$, if $p \geqslant C \cdot n^{-2/(r+1)}$, then* $\max|F| \approx \left(1 - \frac{1}{r-1} + o(1)\right) p \cdot \binom{n}{2}$.

We recall the statement of the Hypergraph Container Lemma.

Theorem 1.4 (**Hypergraph Container Lemma** [1])**.** *For every $k \in \mathbb{N}$ and $c, \varepsilon > 0$, there exists a positive constant C for which the following holds. Let $\mathcal{H}$ be a k-uniform hypergraph and $\mathcal{F} \subseteq 2^{V(\mathcal{H})}$ be an upset such that $|A| \geqslant \varepsilon \cdot v(\mathcal{H})$ for all $A \in \mathcal{F}$. Suppose $\mathcal{H}$ is $(\mathcal{F}, \varepsilon)$-dense and $q \in (0,1)$ such that for every $\ell \in [k]$,*

$$\Delta_\ell(\mathcal{H}) \leqslant c \cdot q^{\ell-1} \frac{e(\mathcal{H})}{v(\mathcal{H})}.$$

Then there exists a family $\mathcal{S} \subseteq \binom{V(\mathcal{H})}{\leqslant Cq \cdot v(\mathcal{H})}$ and functions $f : \mathcal{S} \to \overline{\mathcal{F}}$ and $g : \mathcal{I}(\mathcal{H}) \to \mathcal{S}$ such that for every $I \in \mathcal{I}(\mathcal{H})$,

$$g(I) \subseteq I \text{ and } I \subseteq f(g(I)) \cup g(I).$$

Proof of Theorem 1.3. We build the hypergraph $\mathcal{H}_n$ starting from K_n and the copies of K_r in that complete graph. Define $V(\mathcal{H}_n) := E(K_n)$ and $E(\mathcal{H}_n) :=$ the edge sets of the copies of K_r. Note that $|V(\mathcal{H}_n)| = \binom{n}{2}$ and $|E(\mathcal{H}_n)| = \binom{n}{r}$. Finally, if we let $k = \binom{r}{2}$, we know that $\mathcal{H}_n$ is k-uniform.

To apply the Hypergraph Container Lemma, we need to find the number q that bounds the codegrees of the hypergraphs with the following proposition.

Proposition 1.5. *Let n be an integer and let H be a 2-uniform hypergraph. Set $k = e(H)$ and let $\mathcal{H}$ be the k-uniform hypergraph of copies of H in K_n. There exists a positive constant c such that, letting $q = n^{-1/m_2(H)}$,*

$$\Delta_\ell(\mathcal{H}) \leqslant c \cdot q^{\ell-1} \frac{e(\mathcal{H})}{v(\mathcal{H})}$$

for every $\ell \in [k]$.

Proof of Proposition 1.5. Note that $v(\mathcal{H}) = \binom{n}{2} = \Theta(n^2)$ and that $e(\mathcal{H}) = \frac{v(H)!}{|\mathrm{Aut}(H)|} \cdot \binom{n}{v(H)} = \Theta(n^{v(H)})$. By the definition of q and $m_2(H)$, we have

$$q^{e(H')-1} n^{v(H')-2} \geqslant 1$$

for every $H' \subseteq H$. Now, for each $\ell \in [k]$,

$$\Delta_\ell(\mathcal{H}) \leqslant c' \cdot \max\left\{n^{v(H)-v(H')} : H' \subseteq H \text{ with } e(H') = \ell\right\}$$

for some positive constant c'. Since $e(\mathcal{H})/v(\mathcal{H}) \geqslant c'' \cdot n^{v(H)-2}$ for some constant c'', it follows that

$$\begin{aligned}\Delta_\ell(\mathcal{H}) \cdot \left(q^{\ell-1}\frac{e(\mathcal{H})}{v(\mathcal{H})}\right)^{-1} &\leqslant c' \cdot \frac{v(\mathcal{H})}{e(\mathcal{H})} \cdot \max_{H' \subseteq H \,:\, e(H')=2} \left(\frac{n^{v(H)}}{q^{e(H')-1}n^{v(H')}}\right) \\ &\leqslant \frac{c'}{c''} \cdot \max_{H' \subseteq H \,:\, e(H')=2} \left(\frac{1}{q^{e(H')-1}} n^{v(H')-2}\right) \leqslant \frac{c'}{c''},\end{aligned}$$

where the last inequality follows since $q^{e(H')-1}n^{v(H')-2} \geqslant 1$. □

So we let $q = n^{-1/m_2(K_r)} = n^{-2/r+1}$. We now compute the codegrees of the hypergraphs. That is, we find how many copies of K_r will contain any fixed ℓ vertices. We have that

$$\Delta_\ell(\mathcal{H}_n) \leqslant \binom{n-t}{r-t},$$

where we choose t to be the minimum such that $\ell \leqslant \binom{t}{2}$. Note that it is always the case that $1 \leqslant \ell \leqslant \binom{r}{2}$. So, for $\ell = 1$, we need $t = 2$, which implies $\Delta_1(\mathcal{H}_n) \leqslant \binom{n-2}{r-2} \approx n^{r-2}$. Is there a constant C_r such that $\Delta_1(\mathcal{H}_n) \leqslant n^{r-2} \leqslant C_r\frac{|E(\mathcal{H}_n)|}{|V(\mathcal{H}_n)|} = C_r \cdot \binom{n}{r}/\binom{n}{2}$? In this case, the answer is yes. Therefore, there is no restriction on q.

Consider another case. When $\ell = \binom{r}{2}$, then $t = r$. That is, $\Delta_\ell(\mathcal{H}_n) = 1$, and we want $1 \leqslant C_r \cdot q^{\ell-1}n^{r-2}$. Therefore, we choose $q = n^{-2/(r+1)}$ by Proposition 1.5.

We may now apply the Hypergraph Container Lemma. This implies that there exists some $S \subseteq V(\mathcal{H}_n)$ such that $|S| \leqslant C \cdot n^{2-2/(r+1)}$. For such S, there exists an associated graph $f(S)$ with at most $\epsilon\binom{n}{r}$ copies of K_r. We apply the following supersaturation argument (with stability) to $f(S)$.

For small $\gamma > 0$ and $\beta > 0$, then $|f(S)| \leqslant (1 - 1/(r-1) + \beta)\binom{n}{2}$ by supersaturation (i.e. if the number of K_r's in a graph is at most $q\binom{n}{r}$, then the graph can only be a small, constant proportion bigger than Turán's bound). By stability, either $|f(S)| \leqslant (1 - 1/(r-1) - \beta)\binom{n}{2}$ or, by removing at most γn^2 edges from $f(S)$, the graph can be made $(r-1)$-partite.

Consider the counting statement which follows from the Hypergraph Container Lemma: the number of K_r-free graphs is at most (the number of choices of S)$\cdot 2^{\max|f(S)|}$. We obtain the following stability version of the same counting statement: almost all K_r-free graphs are 'almost' $(r-1)$-partite. So we need to prove that the number of K_r-free graphs that are not $(r-1)$-partite $\ll$ (# of choices of S) $\cdot\, 2^{(1-1/(r-1))\binom{n}{2}}$. This is, in fact true, proven earlier using the Regularity Lemma.

Now let $F \subseteq G(n,p)$ with $|E(F)| > (1+\gamma)(1-\frac{1}{r-1}) \cdot p \cdot \binom{n}{2}$, where F is K_r-free. What we hope is for the number of such subgraphs F to be small. First note that all such subsets form an independent set in the hypergraph $\mathcal{H}_n$. Therefore, there exists $S \subseteq F$ and $f(S)$ such that $F \subseteq S \cup f(S)$.

We know that $|f(S)| < (1-\frac{1}{r-1}+o(1))\binom{n}{2}$ because it is a container. Therefore

$$\mathbb{E}(|G(n,p)\cap f(S)|) < p \cdot \left((1-\frac{1}{r-1}+o(1))\binom{n}{2}\right).$$

By Chernoff bound, we obtain the following.

$$\mathbb{P}\left(|G(n,p)\cap f(S)| < (1+\gamma/2)\left(1-\frac{1}{r-1}\right)\cdot p \cdot \binom{n}{2}\right) < \exp\left(-\frac{\gamma^2}{8}\cdot p \cdot \frac{n^2}{2}\right).$$

Since $F \subseteq S \cup f(S)$, $F \subseteq G(n,p)$, and F has more edges than the expected size of $G(n,p)\cap f(S)$, we get $|F \cap f(S) \cap G(n,p)| > (1+\gamma/2)\cdot p \cdot (1-1/(r-1))\binom{n}{2}$.

By the first approach, we get that

$$\text{number of choices for } S = \binom{n^2/2}{C \cdot n^{2-\frac{2}{r+1}}} \cdot e^{C' n^{2-\frac{2}{r+1}}} \approx 2^{\log n \cdot n^{2-\frac{2}{r+1}}}.$$

We want this number to be $o(1)$. We apply the counting lemma to show that this is true for the desired p.

!!!!!MISSING COMPUTATION!!!!! TO BE DONE AT ONE POINT!!!!!!

For a second approach, we consider $S \subseteq G(n,p)$ and count the number of choices for S by summing over the possible sizes of S as follows.

$$\sum_{|S| \leqslant C \cdot n^{2-\frac{2}{r+1}}} p^{|S|}\binom{n^2/2}{|S|} \cdot e^{C' n^{2-\frac{2}{r+1}}},$$

where again, the desired p will make this sum equal to $o(1)$.

$\square$

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