On the variance of Shannon products of graphs

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Abstract

We study the combinatorial problem of finding an arrangement of distinct integers into the d-dimensional N-cube so that the maximal variance of the numbers on each ℓ-dimensional section is minimized. Our main tool is an inequality on the Laplacian of a Shannon product of graphs, which might be a subject of independent interest. We describe applications of the inequality to multiple description scalar quantizers (MDSQ), to getting bounds on the bandwidth of products of graphs, and to balanced edge-colorings of regular, d-uniform, d-partite hypergraphs.

1 Introduction

Applications of multiple description scalar quantizers (MDSQ) arise in speech and video coding over packet-switched networks, where packet losses can result in a degradation in signal quality, see [8, 10]. Here it is desired to send messages across multiple independent channels in such a way that certain guarantees on the reconstructed message fidelity apply if one or more channels are broken, and it is also desired not to decrease the total rate of communication too greatly, see [8, 10].

The model translates to the combinatorial problem of finding an arrangement of the integers into $\mathbb{Z}^d$ so that line of $\mathbb{Z}^d$ contains exactly N numbers, such that the variance of the numbers in each ℓ-dimensional section is minimized, where a *line* corresponding to fixed integers $a_1, \ldots, a_d$ and direction k is the set $\{\langle a_1, \ldots, a_{k-1}, i, a_{k+1}, \ldots, a_d\rangle \in \mathbb{Z}^d : \ i \in \mathbb{Z}\}$, and an *$\ell$-dimensional section* defined similarly, here all but $d-\ell$ coordinates are fixed. Then minimizing the distortion given the rate amounts to finding, for a given N, d and ℓ, an arrangement with the smallest possible *variance* $\mathrm{V}ar_\infty(N, d, \ell)$, which is defined formally as $\mathrm{V}ar_\infty(N, d, \ell) = \max\{(1/N)\sum(\bar{X} - X_i)^2\}$, where the maximum is over all ℓ-dimensional sections of $\mathbb{Z}^d$, and the summation is over all elements

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X_i of a section, and $\bar{X}$ is the mean of these numbers (in one section). This problem was considered in [10] and in [2]. In [2] it was proved that $(1/60)N^4 \leq \mathrm{V}ar_\infty(N,2,1) \leq (1/54)N^4 + O(N^3)$.

One of the key observations in [2] was (in the case $d = 2$ and $\ell = 1$) that to obtain good bounds for $\mathrm{V}ar_\infty(N,d,\ell)$, the following problem needed to be considered: Write the integers $1,\ldots,N^d$ into an N^d cuboid such that the maximal variance of the numbers appearing in an ℓ-dimensional section of the cuboid is as small as possible. We denote this minimum by $\mathrm{V}ar(N,d,\ell)$.

For this problem the bounds $(1+10^{-5})N^4/24 < \mathrm{V}ar(N,2,1) < (1/22)N^4$ were obtained in [2]. To achieve the $(1/24)N^4$ lower bound, an inequality was used, whose extension is the main aim of this note. In Section 2, as a warm-up, we prove it for two-dimensional rectangles, in Section 3 we shall state and prove it for any dimension. We also prove a generalization to the Laplacian of a Shannon product of graphs. In Section 4 we state some conclusions for bandwidth of products of cliques, and in Section 5 we give bounds on a problem concerning edge-colorings of hypergraphs. Note that isoperimetric problems on the products of regular graphs were studied in the literature, see for example [3].

2 A lower bound for the variance in a rectangle

For the sake of completeness we recall a lemma from [2], considering N by M matrices: As usual, let the *variance* of a list $(X_1, X_2, \ldots, X_n)$ of real numbers be denoted by

$$\mathrm{V}ar(X_1, X_2, \ldots, X_n) = \frac{1}{n}\sum_i (X_i - \overline{X})^2,$$

where $\overline{X} = \frac{1}{n}\sum_i X_i$ is the *mean* of $(X_1, X_2, \ldots, X_n)$. The following identity motivates many of our forthcoming definitions:

$$\mathrm{V}ar(X_1, X_2, \ldots, X_n) = \frac{1}{n^2}\sum_{i<j}(X_i - X_j)^2. \tag{1}$$

Theorem 2.1. *Let $X_{i,j}$ $(1 \leq i \leq N, 1 \leq j \leq M)$ denote the elements of an N-by-M matrix. Then*

$$\mathrm{V}ar(X_{1,1}, X_{1,2}, \ldots, X_{N,M}) \leq \frac{1}{N}\sum_i \mathrm{V}ar(X_{i,1}, X_{i,2}, \ldots, X_{i,M}) + \frac{1}{M}\sum_j \mathrm{V}ar(X_{1,j}, X_{2,j}, \ldots, X_{N,j}). \tag{2}$$

Proof: Substitute the definitions of all the variances into (2), multiply by N^2M^2, and move all terms to the right side. It is easy to check that the coefficients of the monomials on the right hand side are as follows. The coefficients of the terms of form $X_{i,j}^2$ (where i and j need not be distinct) are $(N-1)(M-1)$, the coefficients of the terms of form $X_{i,j}X_{\ell,k}$ (with $i \neq \ell$ and $j \neq k$) are 2, and the coefficients of the remaining terms of form $X_{i,j}X_{i,k}$ are $2(1-N)$ and of $X_{i,j}X_{\ell,j}$ are $2(1-M)$. There are no other terms.

This transformation shows that our assertion is equivalent to stating that a certain quadratic form in the variables $X_{i,j}$ is positive semi-definite. Let G denote the NM-by-NM matrix whose rows (resp. columns) are labeled with the variables $X_{i,j}$ that represents the quadratic form in question. The entries of G can be read off from the calculation above. The diagonal entries are all $(N-1)(M-1)$. The off-diagonal entry corresponding to row $X_{i,j}$ and column $X_{\ell,k}$ is the half of the coefficients described above.

The matrix G is the Kronecker (tensor) product of an N-by-N matrix K_N and an M-by-M matrix K_M whose diagonal elements are equal to $N-1$, (resp. $M-1$) and whose other elements are -1. Since in general the matrix K_N has eigenvalues 0 (with multiplicity 1) and N (with multiplicity $N-1$ times), the matrix G has eigenvalues 0 ($N+M-1$ of them) and NM ($(N-1)(M-1)$ of them). This proves our result. □

Remark. An alternative proof is the following. Denote $A_{i,j} := X_{i,j} - (1/N)\sum_{i=1}^{N} X_{i,j}$. Let X be a randomly (uniformly) chosen element of the matrix, Y be a randomly (uniformly) chosen row, and Z be a column (to clarify notation; the first row consists of $X_{1,1}, X_{1,2}, \ldots, X_{1,M}$. With this notation the relation (2) is equivalent to the following:

$$\mathrm{V}ar(X) \leq \mathbb{E}[\mathrm{V}ar(X|Y)] + \mathbb{E}[\mathrm{V}ar(X|Z)]. \tag{3}$$

Using the identity $\mathrm{V}ar(X) = \mathbb{E}[\mathrm{V}ar(X|Y)] + \mathrm{V}ar[\mathbb{E}(X|Y)]$,

$$\mathrm{V}ar[\mathbb{E}(X|Y)] = \frac{1}{N}\sum_{i=1}^{N}\left(\frac{1}{M}\sum_{j=1}^{M} X_{i,j} - \mathbb{E}X\right)^2 = \frac{1}{N}\sum_{i=1}^{N}\left(\frac{1}{M}\sum_{j=1}^{M} A_{i,j}\right)^2 \leq \frac{1}{NM}\sum_{i,j} A_{i,j}^2 = \mathbb{E}[\mathrm{V}ar(X|Z)] \tag{4}$$

proves the claim.

Remark. We can sharpen relation (2) to an equality. First, observe that w.l.o.g. $\sum_{i,j} X_{i,j} = 0$. Letting $H_{i,j} := X_{i,j} - (1/M)\sum_k X_{i,k} - (1/N)\sum_k X_{k,j}$ one can prove

$$\mathrm{V}ar(X_{1,1}, X_{1,2}, \ldots, X_{N,M}) + \frac{1}{NM}\sum_{i,j} H_{i,j}^2 = \frac{1}{N}\sum_i \mathrm{V}ar(X_{i,1}, X_{i,2}, \ldots, X_{i,M}) + \frac{1}{M}\sum_j \mathrm{V}ar(X_{1,j}, X_{2,j}, \ldots, X_{N,j}) \tag{5}$$

3 The lower bound for the variance

We will generalize Theorem 2.1 to tensor products of matrices, and in other directions.

Let G be a graph. For $g, g' \in V(G)$ we use the notation $g \sim_G g'$ for $\{g, g'\} \in E(G)$. We also use $g \sim g'$ if the graph in which g, g' lie is clear from the context. We define a number of matrices in $\mathbb{R}^{V(G)\times V(G)}$ associated with the graph G. The *adjacency matrix* of G, $A = A_G$, is the matrix with $A_{g,g'} = 1$ iff $g \sim g'$ and $A_{g,g'} = 0$ otherwise. The *degree matrix* of G, $D = D_G$, is the diagonal matrix with $D_{g,g} = \deg_G(g)$. The *Laplacian* of G is $L_G = D_G - A_G$. Other matrices that will prove useful are I_G, the *identity matrix*, and $D_G^+ := D_G + I_G$.

The *order* of a graph G, is the number of its vertices $|G| := |V(G)|$. For $x \in \mathbb{R}^{V(G)}$, motivated by (1), we define

$$\mathrm{V}ar_G(x) := \frac{1}{|G|^2}\, x^t L_G x = \frac{1}{|G|^2} \sum_{g \sim g'} (x_g - x_{g'})^2.$$

Note that $\mathrm{V}ar_{K_n}(x) = \mathrm{V}ar(x_1, \ldots, x_n)$.

Let $I_1, \ldots, I_n,\ J_1, \ldots, J_n$ be finite non-empty sets and $I = \prod I_k,\ J = \prod J_k$, be their Cartesian products. For $1 \leq k \leq n$, let $M_k \in \mathbb{R}^{I_k \times J_k}$. The *Kronecker* or *tensor* product of the matrices M_k is $M = \bigotimes_k M_k \in \mathbb{R}^{I \times J}$ where the entries of M are $M_{i,j} := \prod_k (M_k)_{i_k, j_k}$ for all $i \in I, j \in J$. Let $M'_k \in \mathbb{R}^{I_k \times J_k}$ for $k \in [n]$. We will encounter situations where we replace some M_i in M with M'_i, so we introduce the following notation:

Convention 3.1. *We write*

$$(\bigotimes_{k \in S} M_k) \otimes (\bigotimes_{k \notin S} M'_k)$$

as a shorthand for $\bigotimes_k A_k$ *where* $A_k = M_k$ *for* $k \in S$ *and* $A_k = M'_k$ *for* $k \notin S$.

We define the *Shannon* product of a sequence of graphs $G_1, G_2, \ldots, G_n$ to be the graph $G = \bigodot_k G_i$ with vertex set $V(G) = \prod_k G_k$ and the following adjacency relation: $g \sim_G g'$ if and only if $g \neq g'$ and for every k, $g_k = g'_k$ or $g_k \sim_{G_k} g'_k$. Note that $K_n \bigodot K_m = K_{nm}$. It is easy to observe:

Lemma 3.2. *Let* $G_1, \ldots, G_n$ *be a sequence of graphs and* $G = \bigodot_k G_k$ *be the Shannon product of the graphs* G_i. *Then the following hold:*

(i) $I_G + A_G = \bigotimes_k (I_{G_k} + A_{G_k})$,

(ii) $D_G^+ = \bigotimes_k D_{G_k}^+$,

(iii) $L_G = \bigotimes_k D_{G_k}^+ - \bigotimes_k (D_{G_k}^+ - L_{G_k})$.

Proof: (i) Let $g, g' \in V(G)$. By the definition of the Shannon product, we know that $g = g'$ or $g \sim_G g'$ iff for all k, $g_k = g'_k$ or $g_k \sim_{G_k} g'_k$.

Part (ii) is the assertion that $\deg_G(g) + 1 = \prod_k (\deg_{G_k}(g_k) + 1)$ for all $g \in V(G)$. Fix a vertex $g \in V(G)$. There are $\deg_G(g) + 1$ vertices $h \in V(G)$ such that $h = g$ or $h \sim g$, these are precisely the vertices where for all k, $h_k = g_k$ or $h_k \sim_{G_k} g_k$. Thus there are $\prod_k (\deg_{G_k}(g_k) + 1)$ of them.

Now we prove (iii). For any graph H we have $D_H^+ = D_H + I_H$ and $L_H = D_H - A_H$ so $I_H + A_H = D_H^+ - L_H$. Thus $\bigotimes_k (D_{G_k}^+ - L_{G_k}) = \bigotimes_k (I_{G_k} + A_{G_k}) = I_G + A_G$ by part (i). Thus by part (ii),

$$\bigotimes_k D_{G_k}^+ - \bigotimes_k (D_{G_k}^+ - L_{G_k}) = D_G^+ - (I_G + A_G) = L_G.$$

□

Let $S \subset [n]$ and $\overline{S} = [n] \setminus S$. We define $G_S = \bigodot_{k \in S} G_k$, and for $h \in V(G_{\overline{S}})$, we let

$$\mathrm{V}ar_{G,h}(x) := \frac{1}{|G_S|^2} \sum_{\substack{g \sim_G g' \\ \forall k \in \overline{S} \;\; g_k = g'_k = h_k}} (x_g - x_{g'})^2,$$

and

$$f(S,x) := \mathbb{E}_{h \in V(G_{\overline{S}})} \mathrm{V}ar_{G,h}(x) = \frac{1}{|G_{\overline{S}}|} \sum_{h \in V(G_{\overline{S}})} \mathrm{V}ar_{G,h}(x),$$

where $\mathbb{E}$ refers to the uniform distribution. Thus

$$|G|^2 f(S,x) = |G_{\overline{S}}| x^t (L(G_S) \otimes I(G_{\overline{S}}))x. \tag{6}$$

Here $L(G_S) \otimes I(G_{\overline{S}})$ is interpreted according to Convention 3.1, and by Lemma 3.2 (iii)

$$L(G_S) \otimes I(G_{\overline{S}}) = \left(\bigotimes_{k \in S} D^+_{G_k} - \bigotimes_{k \in S} (D^+_{G_k} - L_{G_k}) \right) \otimes \bigotimes_{k \in \overline{S}} I_{G_k}.$$

Theorem 3.3. *Let G_k, $k \in [n]$, be graphs. Let $G = \bigodot_k G_k$. Let $S_1, \ldots, S_t$ be a partition of $[n]$. Then for all $x \in \mathbb{R}^{V(G)}$ we have*

$$\mathrm{V}ar_G(x) \le \sum_{i=1}^{t} \mathbb{E}_{h \in G_{\overline{S}_i}} \mathrm{V}ar_{G,h}(x).$$

Proof: The conclusion of the theorem is that $\sum_i f(S_i, x) \ge f(S_1 \cup \ldots \cup S_t, x)$ (since $f([n], x) = \mathrm{V}ar_G(x)$). We will show that if $S_1 \cap S_2 = \emptyset$ then $f(S_1, x) + f(S_2, x) \ge f(S_1 \cup S_2, x)$. Applying this repeatedly proves the theorem.

Let $S_3 = [n] \setminus (S_1 \cup S_2)$. Let $H_i = G_{S_i}$ for $i = 1, 2, 3$. Multiplying $f(S_1, x) + f(S_2, x) - f(S_1 \cup S_2, x) \ge 0$ by $|G|^2$, and using (6), we obtain $x^t A x \ge 0$, where

$$A = A' \otimes (|H_3| I(H_3))$$

and

$$A' = |H_2| L(H_1) \otimes I(H_2) + |H_1| I(H_1) \otimes L(H_2) - L(H_1 \odot H_2).$$

Thus it remains to show A' (and hence A) is positive semidefinite. Using the identities $|V(H_i)| I(H_i) = D^+(H_i) + D(\overline{H_i})$ and this corollary of Lemma 3.2 (iii)

$$L(H_1 \odot H_2) = D^+(H_1) \otimes D^+(H_2) - (D^+(H_1) - L(H_1)) \otimes (D^+(H_2) - L(H_2)),$$

it is not hard to see that

$$A' = L(H_1) \otimes L(H_2) + D(\overline{H_1}) \otimes L(H_2) + L(H_1) \otimes D(\overline{H_2}).$$

Since $L(H_i)$, $D(\overline{H_i})$ are all positive semidefinite so are their tensor products and thus so is A'.

□

Consider the following inequality:

$$f(S_1, x) + f(S_2, x) \geq f(S_1 \cup S_2, x) + f(S_1 \cap S_2, x). \tag{7}$$

Note that $f(\emptyset, x) = 0$. So in the proof of the previous theorem we proved (7) for $S_1 \cap S_2 = \emptyset$. If (7) were true in general then we would have a proof of the following:

Statement 3.4. *Let G_k, $k \in [n]$, be graphs. Let $G = \bigodot_k G_k$. Let $S_1, \ldots, S_t$ be the edges of an r-regular hypergraph on $[n]$. Then for all $x \in \mathbb{R}^{V(G)}$ we have*

$$r\mathrm{V}ar_G(x) \leq \sum_{i=1}^{t} \mathbb{E}_{h \in G_{\overline{S}_i}} \mathrm{V}ar_{G,h}(x).$$

This statement would be very reminiscent of Shearer's Lemma on entropy, and inequality (7) would be reminiscent of the method of proof of Shearer's Lemma [4]. Unfortunately (7) is false in general. A counterexample is found when $n = 3$, $G_1 = P_2, G_2 = P_3, G_3 = P_2$ and $S_1 = \{1, 2\}$, $S_2 = \{2, 3\}$, and the value of a vertex (a, b, c) is b is modulo 2 (and P_k is the path with k vertices). Even Statement 3.4 is false, for a counterexample, see $S_1 = \{1, 2\}, S_2 = \{2, 3\}, S_3 = \{1, 3\}$ and $G_1 = G_2 = G_3 = K_{6,6}$ (the complete bipartite graph).

However (7) holds if all the G_k are complete graphs:

Theorem 3.5. *Let G_k, $k \in [n]$, be cliques. Let $G = \bigodot_k G_k$. Let $S_1, \ldots, S_t$ be the edges of an r-regular hypergraph on $[n]$. Then for all $x \in \mathbb{R}^{V(G)}$ we have*

$$r\mathrm{V}ar_G(x) \leq \sum_{i=1}^{t} \mathbb{E}_{h \in G_{\overline{S}_i}} \mathrm{V}ar_{G,h}(x).$$

Proof: Clearly, it is sufficient to prove for every S_1, S_2 inequality (7). For fixed S_1, S_2, let $T_1 = S_1 \setminus S_2$, $T_2 = S_1 \cap S_2$, $T_3 = S_2 \setminus S_1$, $T_4 = [n] \setminus (S_1 \cup S_2)$. Let $H_i = G_{T_i}$, $i = 1, 2, 3, 4$. Multiplying (7) by $|G|^2$, and using (6), we see that the matrix $A = A' \otimes (|H_4| I(H_4))$ must be positive semidefinite, where

$$A' = |H_3| L(H_1 \odot H_2) \otimes I(H_3) + |H_1| I(H_1) \otimes L(H_2 \odot H_3) - L(H_1 \odot H_2 \odot H_3) - |H_1||H_3| I(H_1) \otimes L(H_2) \otimes I(H_3).$$

Since all the G_k's are complete so are the H_i's, and thus $|H_i| I(H_i) = D^+(H_i)$. Using this and the identities for $L(H_i \odot H_j)$ and $L(H_1 \odot H_2 \odot H_3)$, from part (iii) of Lemma 3.2, we find that A' simplifies to

$$A' = L(H_1) \otimes (D^+(H_2) - L(H_2)) \otimes L(H_3).$$

H_2 is a complete graph, hence $D^+(H_2) - L(H_2)$ is a matrix of all ones, $J(H_2)$. Thus A' is positive semidefinite as desired.

□

4 Bandwidth of products of complete graphs

The d-wise product, K_n^d, of complete graphs K_n has n^d vertices, and two vertices (d-vectors) are joined with an edge iff they differ in only one coordinate. (Or with a little bit different terminology, two vertices are connected with an edge iff their Hamming distance is 1, for more details see [7].) A numbering of a graph G is a (bijective) labeling of the vertices with $1, \ldots, |V(G)|$. The *bandwidth* of a labeling of G is the maximum difference appearing between numbers of the endpoints of the edges of G, and the *bandwidth* of a graph G is $BW(G)$, which is the minimum of that maximum difference considering all the vertex labelings of G. It is easy to see that

$$BW(G) \geq \max_k \min_{S \subset V(G), |S|=k} |\Phi(S)|, \tag{8}$$

where $\Phi(S)$ is the neighborhood of the vertex set S outside of S. Harper [6] determined asymptotically (n tends to infinity)

$$\min_{S \subset V(G), |S|=k} |\Phi(S)|$$

for any k, when $G = K_n^d$. However the bound given by (8) does not seem to be sharp for $d \geq 3$. For $d = 3$ it is known that $0.4437n^3 + O(n^2) \leq BW(K_n^3) \leq 0.4498n^3 + O(n^2)$ (see [1]).

Harper's [7] proved that

$$\binom{d}{d/2} \leq BW(K_n^d) \leq \sum_{k=0}^{d-1} \binom{k}{\lceil k/2 \rceil},$$

where the lower bound valid for $n \to \infty$ and d even, and the upper bound valid for $n \to \infty$ and any d. His conclusion was that

$$BW(K_n^d) \sim \sqrt{(2/\pi d)} n^d,$$

when $n \to \infty$ and (every) $d \to \infty$ but $d = o(n)$. We can conclude from our Theorem 3.5 a weaker bound, which is valid for all n and d. Certainly, our bounds cannot be sharp (the variance is more robust, i.e., for example (9) is usually not sharp), but it is still an improvement on the constant of n^d compared to the constant of the trivial bound, $1/d$. To apply Theorem 3.3, note that $\mathrm{V}ar(1, \ldots, n^d) = (n^{2d} - 1)/12$, and we use $S_i = \{i\}$ for $1 \leq i \leq d$. Furthermore, observe that for $x_1 \leq x_2 \leq \ldots \leq x_t$

$$2\sqrt{\mathrm{V}ar(x_1, \ldots, x_t)} \leq x_t - x_1. \tag{9}$$

Thus if x is a labeling of the vertices of K_n^d by $1, \ldots, n^d$ achieving bandwidth $BW(K_n^d)$, then using Theorem 3.3 and (9) we obtain

$$\frac{n^{2d} - 1}{12} = \mathrm{V}ar_{K_n^d}(x) \leq \sum_{i=1}^{d} \mathbb{E}_{h \in V(K^d_{n\overline{S_i}})} \mathrm{V}ar_{K_n^d, h}(x) \leq \frac{d \cdot BW(K_n^d)^2}{4}.$$

Here, we used that for $h \in V(K^d_{n\bar{S}_i})$, $\mathrm{V}ar_{K^d_n,h}(x)$ is the variance of the numbers of the line containing h in direction i.

Corollary 4.1. *For every positive integers n and d,*

$$\sqrt{\frac{1-n^{-2d}}{3d}}n^d \leq BW(K^d_n). \tag{10}$$

The following version was also considered in [7]. Let $(K^d_n)^{(h)}$ be the graph with the same vertex set as K^d_n, such that two vertices are connected with an edge if their distance in K^d_n is at most h. In other words, two vertices are neighbors in $(K^d_n)^{(h)}$, if there is an h-dimensional face which contains both of them. Harper [7] concluded that for $n \to \infty$ and $d \to \infty$ and $d = o(n)$, and additionally $h = o(\sqrt{d})$

$$BW((K^d_n)^{(h)}) \sim h\sqrt{(2/\pi d)}n^d.$$

From Theorem 3.5, where the sets S_is form a complete r-regular hypergraph on $[d]$, we can conclude a weaker lower bound, but one valid for every n, d, h:

$$\sqrt{\frac{(1-n^{-2d})h}{3d}}n^d \leq BW((K^d_n)^{(h)}). \tag{11}$$

5 Edge-colorings of hypergraphs

Let $\mathcal{H}$ be a d-uniform, d-equipartite, r-regular hypergraph on nd vertices (i.e. $V(\mathcal{H})$ is partitioned into d classes of sizes n each, $V_1 \cup \dots \cup V_d$, such that for every $F \in E(\mathcal{H})$ and every $1 \leq i \leq d$, $|F \cap V_i| = 1$). Consider a 2-edge-coloring of $\mathcal{H}$ (say with -1 and 1) such that the number of edges labeled with 1 is $\alpha|E(\mathcal{H})|$ for some constant α. To be more formal, a coloring

$$\xi : E(\mathcal{H}) \to \{-1, 1\}$$

is α-*dense* if

$$|\xi^{-1}(1)| = \alpha|E(\mathcal{H})|.$$

The *unbalancedness* of a coloring of $\mathcal{H}$ could be measured as follows:

$$UB(\mathcal{H}, \xi) := \min_{x \in V(\mathcal{H})} \left| \sum_{x \in E} \xi(E) \right|.$$

We are interested in determining the largest unbalancedness of any α-dense coloring of $\mathcal{H}$, i.e.:

$$UB(\mathcal{H}, \alpha) := \max\{|UB(\mathcal{H}, \xi)| : \xi \text{ is } \alpha\text{-dense}\},$$

or more generally in determining

$$UB(n, d, r, \alpha) := \max\{|UB(\mathcal{H}, \alpha)| : |V(\mathcal{H})| = nd,\ \mathcal{H} \text{ is } d\text{-uniform } d\text{-equipartite } r\text{-regular}\}.$$

The simplest case is when $d = 2, r = n$ and $\alpha = 1/2$. Note that under these restrictions $\mathcal{H} = K_{n,n}$ is unique. It can be proven elementarily, that $UB(K_{n,n}, 1/2) \leq n/2$. For even n, there is a coloring achieving equality: let $a_1, \dots, a_n$ and $b_1, \dots, b_n$ denote the vertices in the two classes, and the edge (a_i, b_j) should get color 1 iff $i, j \leq n/2$ or ($i+j$ is even and ($i \leq n/2 < j$ or $j \leq n/2 < i$)).

To give an upper bound on $UB(n, d, r, \alpha)$ we shall apply Theorem 3.5. For a coloring ξ of $\mathcal{H}$, where $\mathcal{H}$ is a d-equipartite, d-uniform hypergraph, with class sizes n, we can assign a d-dimensional n-cube C, where each entry represents a possible edge. A 0 is placed into an entry, if the edge represented by it does not exist in $\mathcal{H}$, and, otherwise, the color of the edge is written there.

The variance of all the entries of C can be obtained easily; the hypergraph has rn edges, so C has αrn entries containing 1, $n^d - rn$ containing 0 and $(1-\alpha)rn$ containing -1. To ease the notation, let $R = r/n^{d-1}$. Then the variance of the entries in C is

$$(1-(2\alpha-1)R)^2\alpha R + ((2\alpha-1)R)^2(1-R) + (1+(2\alpha-1)R)^2(1-\alpha)R. \tag{12}$$

Each $(d-1)$-dimensional section of the cube represents one vertex of the hypergraph. By the regularity of $\mathcal{H}$, each such section has $(1-R)n^{d-1}$ entries containing 0. Our aim is to prove that the coloring cannot be very unbalanced. Let us assume that for some fixed $t > R/2$, each vertex belongs to at least tn^{d-1} edges of color 1 or of color -1. The variance of the entries in each section is at most

$$t(1-2t+R)^2 + (1-R)(2t-R)^2 + (R-t)(1+2t-R)^2.$$

Applying Theorem 3.5, with $S_i = [d] - \{i\}$, we have the following inequality:

$$\begin{aligned}(d-1)\big[(1-(2\alpha-1)R)^2\alpha R + ((2\alpha-1)R)^2(1-R) + (1+(2\alpha-1)R)^2(1-\alpha)R\big] \\ \leq d\big[t(1-2t+R)^2 + (1-R)(2t-R)^2 + (R-t)(1+2t-R)^2\big].\end{aligned} \tag{13}$$

The inequality (13) is quadratic in t and under the condition of $t > R/2$ we obtain that

$$\frac{R}{2} < t \leq s := \frac{R}{2} + \frac{\sqrt{Rd} \cdot \sqrt{1+(1-2\alpha)^2(d-1)R}}{2d}. \tag{14}$$

This yields

$$UB(n, d, r, \alpha) \leq (2s - R)n^{d-1} = \frac{\sqrt{Rd} \cdot \sqrt{1+(1-2\alpha)^2(d-1)R}}{d} n^{d-1}$$

This is most likely not the best possible upper bound. We shall list several special cases to see what benchmark has been set:

1. $d = 2,\ R = 1,\ \alpha = 1/2:$

$$\frac{UB}{n} \leq \frac{\sqrt{2}}{2}.$$

(Note that in this case we know that 1/2 is the correct value.)

2. $R = 1, \ \alpha = 1/2$:

$$\frac{UB}{n^{d-1}} \leq \frac{1}{\sqrt{d}}.$$

(Note that this bound is sharp to a constant factor, independent of d and n.)

3. $d = 2, \ \alpha = 1/2$:

$$\frac{UB}{n} \leq \frac{\sqrt{R}}{2}.$$

4. $d = 2, \ R = 1$:

$$\frac{UB}{n} \leq \frac{\sqrt{2}\sqrt{1 + (2\alpha - 1)^2}}{2}.$$

6 Concluding Remarks

1. Let us apply Theorem 3.5 to our original problem, with $r = \binom{d}{\ell}$ and $S_1, \ldots, S_r$ enumerating all the ℓ-element subsets of $[d]$. The sets S_i cover each point of $[d]$ $\binom{d-1}{\ell-1}$ times, hence we obtain

$$\mathrm{V}ar(N, d, \ell) \geq \frac{\binom{d-1}{\ell-1}}{\binom{d}{\ell}} \mathrm{V}ar(1, \ldots, N^d) = \frac{\ell}{d} \frac{N^{2d} - 1}{12}.$$

For $d = 2$ in [2] it was proved that although this lower bound is not sharp (and an analogue of the rather technical argument would work for $d > 2$ also), it is not very far from the truth.

2. One could consider higher moments (for example fourth moments, instead of the variance). We could not prove anything, but we were able to formulate an inequality, which might be true. First we define

$$\mathrm{V}ar^{(4)}(X_1, X_2, \ldots, X_n) = \frac{1}{n} \sum_i (X_i - \overline{X})^4.$$

Does the following inequality hold for all values of $X_{1,1}, X_{1,2}, \ldots, X_{N,N}$?

$$\frac{1}{4}\mathrm{V}ar^{(4)}(X_{1,1}, X_{1,2}, \ldots, X_{N,N}) \leq \frac{1}{N} \sum_i \mathrm{V}ar^{(4)}(X_{i,1}, \ldots, X_{i,N}) + \frac{1}{N} \sum_j \mathrm{V}ar^{(4)}(X_{1,j}, \ldots, X_{N,j}). \tag{15}$$

This is true for $N = 2$. If it is true in general, then it is best possible for even N, as the following example shows: Let $X_{i,j}$ be -1 if $i, j \leq N/2$, 0 if $i \leq N/2 < j$ or $j \leq N/2 < i$, and 1 if $N/2 < i, j$.

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