

THE STRUCTURE OF ALMOST ALL GRAPHS IN A HEREDITARY PROPERTY

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ABSTRACT. A hereditary property of graphs is a collection of graphs which is closed under taking induced subgraphs. The speed of \mathcal{P} is the function $n \mapsto |\mathcal{P}_n|$, where \mathcal{P}_n denotes the graphs of order n in \mathcal{P} . It was shown by Alekseev, and by Bollobás and Thomason, that if \mathcal{P} is a hereditary property of graphs then

$$|\mathcal{P}_n| = 2^{(1-1/r+o(1))n^2/2},$$

where $r = r(\mathcal{P}) \in \mathbb{N}$ is the so-called ‘colouring number’ of \mathcal{P} . However, their results tell us very little about the structure of a typical graph $G \in \mathcal{P}$.

In this paper we describe the structure of almost every graph in a hereditary property of graphs, \mathcal{P} . As a consequence, we derive essentially optimal bounds on the speed of \mathcal{P} , improving the Alekseev-Bollobás-Thomason Theorem, and also generalizing results of Balogh, Bollobás and Simonovits.

1. INTRODUCTION

In this paper we shall describe the structure of almost every graph in an arbitrary hereditary property of graphs, \mathcal{P} . As a corollary, we shall obtain bounds on the speed of \mathcal{P} which improve those proved by Alekseev [1] and Bollobás and Thomason [16, 17], and generalize a theorem of Balogh, Bollobás and Simonovits [7, 8] on monotone properties of graphs. We begin with some definitions.

A collection of labelled graphs, \mathcal{P} , is called a *hereditary property* if it is closed under re-labelling vertices, and under taking induced subgraphs. It is called *monotone* if it is moreover closed under taking arbitrary subgraphs. Note that a hereditary property may be characterized by a (possibly infinite) collection of forbidden induced subgraphs.

Given a property of graphs, \mathcal{P} , let $\mathcal{P}_n = \{G \in \mathcal{P} : |V(G)| = n\}$ denote the graphs in \mathcal{P} with vertex set $[n]$. The *speed* of \mathcal{P} , introduced in 1976 by Erdős, Kleitman and Rothschild [21], is the function

$$n \mapsto |\mathcal{P}_n|.$$

The speed is a natural measure of the ‘size’ of a property.

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The possible structures and speeds of a hereditary or monotone property of graphs have been extensively studied, originally in the special case where a single subgraph is forbidden, and more recently in general. For example, Erdős, Kleitman and Rothschild [21] and Kolaitis, Prömel and Rothschild [28] studied K_r -free graphs, Erdős, Frankl and Rödl [20] studied monotone properties when a single graph is forbidden, and Prömel and Steger [33, 34] obtained (amongst other things) very precise results on the structure of almost all (induced-) C_4 -free and C_5 -free graphs. They also were the first to define the following parameter of a property of graphs, known as the ‘colouring number’ of \mathcal{P} , which will be important in what follows.

First, for each $r \in \mathbb{N}$ and each vector $v \in \{0, 1\}^r$, define a collection $\mathcal{H}(r, v)$ of graphs as follows. Let $G \in \mathcal{H}(r, v)$ if $V(G)$ may be partitioned into r sets A_1, \dots, A_r such that $G[A_j]$ is the empty graph if $v_j = 0$, and is the complete graph if $v_j = 1$.

Definition. The colouring number $\chi_c(\mathcal{P})$ of a property of graphs, \mathcal{P} , is defined to be

$$\chi_c(\mathcal{P}) := \max \{r \in \mathbb{N} : \mathcal{H}(r, v) \subset \mathcal{P} \text{ for some } v \in \{0, 1\}^r\}.$$

The following result, proved by Alekseev [1] and Bollobás and Thomason [16, 17], generalizes the Erdős-Frankl-Rödl Theorem to a general hereditary property of graphs.

The Alekseev-Bollobás-Thomason Theorem. *Let \mathcal{P} be a hereditary property of graphs, and suppose $\chi_c(\mathcal{P}) = r$. Then*

$$|\mathcal{P}_n| = 2^{(1-1/r+o(1))n^2/2}.$$

The Alekseev-Bollobás-Thomason Theorem shows that the set of possible values for the ‘entropy’ of a hereditary property of graphs,

$$\text{ent}(\mathcal{P}) := \lim_{n \rightarrow \infty} \frac{1}{\binom{n}{2}} \log_2(|\mathcal{P}_n|)$$

is not continuous, but in fact undergoes a series of discrete ‘jumps’, from $1 - \frac{1}{r}$ to $1 - \frac{1}{r+1}$, where $r \in \mathbb{N}$. However, the proofs of Alekseev and of Bollobás and Thomason tell us very little about the structure of a *typical* graph $G \in \mathcal{P}$. Their theorem also gives rather weak bounds on the rate of convergence of the entropy as $n \rightarrow \infty$.

For monotone properties of graphs, these problems were addressed by Balogh, Bollobás and Simonovits [7, 8, 9] in a series of papers in which they proved *very* precise structural results, and obtaining correspondingly precise bounds on the rate of convergence of $\text{ent}(\mathcal{P})$.

The following theorem was the main result of [7]. Let $\text{ex}(n, \mathcal{M})$ denote the usual extremal number of a collection of graphs \mathcal{M} .

The Balogh-Bollobás-Simonovits Theorem. *Let \mathcal{P} be a monotone property of graphs, with colouring number $\chi_c(\mathcal{P}) = r$. Then there exists a family of graphs \mathcal{M} (containing a bipartite graph), and a constant $c = c(\mathcal{P})$, such that the following holds.*

For almost all graphs $G \in \mathcal{P}$, there exists a partition (A, S_1, \dots, S_r) of $V(G)$, such that

- (a) $|A| \leq c(\mathcal{P})$,
- (b) $G[S_j]$ is \mathcal{M} -free for every $j \in [r]$,

and moreover

$$2^{(1-1/r)\binom{n}{2}} n^{\text{ex}(n/r, \mathcal{M})} \leq |\mathcal{P}_n| \leq 2^{(1-1/r)\binom{n}{2}} n^{\text{ex}(n, \mathcal{M}) + cn}.$$

For even more precise results see [8] and [9]. Balogh, Bollobás and Simonovits also had the following (unpublished) conjecture regarding the speed of hereditary graph properties. Let $\mathcal{P}^i(n, \mathcal{M})$ denote the collection of induced- \mathcal{M} -free graphs on $[n]$.

Conjecture 1 (Balogh, Bollobás, Simonovits). *Let \mathcal{P} be a hereditary property of graphs, and suppose that $\chi_c(\mathcal{P}) = r$. Then there exists a family of graphs \mathcal{M} (with $\chi_c(\mathcal{M}) = 1$), and a constant $c = c(\mathcal{P})$, such that*

$$2^{(1-1/r)\binom{n}{2}} 2^{|\mathcal{P}^i(n/r, \mathcal{M})|} \leq |\mathcal{P}_n| \leq 2^{(1-1/r)\binom{n}{2}} n^{r|\mathcal{P}^i(n/r, \mathcal{M})| + cn}.$$

We remark that, by the results of Prömel and Steger [33, 34], Conjecture 1 holds for the properties $\mathcal{P} = \{G : C_4 \not\leq G\}$ and $\mathcal{P} = \{G : C_5 \not\leq G\}$. (Here, and throughout, we write $H \leq G$ to mean that H is an induced subgraph of G .) To be precise, they proved that the vertex set of almost all C_4 -free graphs can be partitioned into an independent set and a complete graph, and that for almost every C_5 -free graph G , either G or \overline{G} has the following structure: the vertex set may be partitioned into two classes, $V(G) = A \cup B$, so that A induces a clique, and B induces a vertex disjoint union of cliques.

Finally, we note that even more precise structural results have been obtained for hereditary properties of graphs with much lower speeds, by Balogh, Bollobás and Weinreich [10, 11, 12], and for hereditary properties of other combinatorial structures, see for example [2, 4, 5, 6, 23, 30]. Note in particular [2], where Sauer's Lemma (which will be a crucial tool in our proof) is used. There has also been some important recent progress on hereditary properties of hypergraphs, by Dotson and Nagle [18] and (independently) Ishigami [22], who (building on work of Nagle and Rödl [31] and Kohayakawa, Nagle and Rödl [27]) proved a version of the Alekseev-Bollobás-Thomason Theorem for k -uniform hypergraphs, and by Person and Schacht [32], who showed that almost every Fano-plane-free 3-uniform hypergraph is bipartite.

2. MAIN RESULTS

In this section we state our main results. We begin with a definition.

Definition. For each $k \in \mathbb{N}$, the *universal graph* $U(k)$ is the bipartite graph with parts $A \cong [2]^k$ and $B \cong [k]$, and edge set

$$E(U(k)) = \{ab : a \in A, b \in B \text{ and } b \in a\}.$$

A graph G is said to be *$U(k)$ -free* if there do not exist disjoint subsets $A, B \subset V(G)$ such that $G[A, B] = U(k)$. ($G[A, B]$ denotes the bipartite graph induced by the pair (A, B) .)

The following theorem is the main result of this paper.

Theorem 1. *Let \mathcal{P} be a hereditary property of graphs, with colouring number $\chi_c(\mathcal{P}) = r$. Then there exist constants $k = k(\mathcal{P}) \in \mathbb{N}$ and $\varepsilon = \varepsilon(\mathcal{P}) > 0$ such that the following holds. For almost all graphs $G \in \mathcal{P}$, there exists a partition (A, S_1, \dots, S_r) of $V(G)$, such that*

- (a) $|A| \leq n^{1-\varepsilon}$,
- (b) $G[S_j]$ is $U(k)$ -free for every $j \in [r]$.

Moreover

$$2^{(1-1/r)n^2/2} \leq |\mathcal{P}_n| \leq 2^{(1-1/r)n^2/2 + n^{2-\varepsilon}}$$

for every sufficiently large $n \in \mathbb{N}$.

We shall in fact prove, not just that this structural description holds for almost all graphs $G \in \mathcal{P}$, but that the number of graphs in \mathcal{P}_n for which it fails is at most

$$2^{-n^{2-\varepsilon}} |\mathcal{P}_n|$$

if n is sufficiently large. The final part of Theorem 1 is an immediate consequence of conditions (a) and (b), and the following theorem.

Theorem 2. *For each $k \in \mathbb{N}$ there exists $\varepsilon = \varepsilon(k) > 0$ such that the following holds. There are at most $2^{n^{2-\varepsilon}}$ distinct $U(k)$ -free graphs on $[n]$.*

The structure of the remainder of the paper is as follows. First, in Section 3, we shall state the main tools we shall use in the paper: these include the Szemerédi Regularity Lemma, the Erdős-Simonovits Stability Theorem, and Sauer’s Lemma. In Section 4 we give a sketch of the proof of our main result in the case $\chi_c(\mathcal{P}) = 2$, and in Section 5 we prove Theorem 2, and deduce our bounds on the speed of \mathcal{P} . In Section 6 we prove various lemmas on $U(k)$ -free graphs, and in Section 7 we prove Theorem 1. In Section 8 we shall show how to prove even sharper results in the bipartite case, and in Section 9, we finish by stating a couple of questions and open problems.

3. TOOLS

In this section we shall recall some of the important tools we shall use in order to prove Theorems 1 and 2. In particular, we shall recall the Szemerédi Regularity Lemma [38], the Erdős-Simonovits Stability Theorem [19, 37], and Sauer’s Lemma [35].

Let G be a graph, let $A, B \subset V(G)$ with $A \cap B = \emptyset$, and let $\varepsilon > 0$. We write $d(A, B)$ for the density of the bipartite graph $G[A, B]$. We say that the pair (A, B) is ε -regular if

$$|d(A, B) - d(X, Y)| < \varepsilon$$

for every $X \subset A$ and $Y \subset B$ with $|X| \geq \varepsilon|A|$ and $|Y| \geq \varepsilon|B|$.

Definition. A partition $A_1 \cup \dots \cup A_k$ of $V(G)$ is said to be a *Szemerédi partition* of G for ε if $|A_1| \leq \dots \leq |A_k| \leq |A_1| + 1$, and all but εk^2 of the pairs (A_i, A_j) are ε -regular.

Szemerédi’s Regularity Lemma (Szemerédi, 1976). *Let $\varepsilon > 0$ and $m \in \mathbb{N}$. There exists an $M = M(m, \varepsilon) \in \mathbb{N}$ such that, given any graph G , there exists a Szemerédi partition of G for ε into k parts, for some $m \leq k \leq M$.*

We shall also need the following ‘refinement’ version of Szemerédi’s Lemma. Let G be a graph and $P = (U_1, \dots, U_t)$ be a partition of $V(G)$. A *Szemerédi refinement* of the partition P for ε is a refinement of the partition P which is also a Szemerédi partition of G for ε .

Szemerédi's Lemma: refinement version. *Let $\varepsilon > 0$ and $m \in \mathbb{N}$. There exists an $M = M(m, \varepsilon) \in \mathbb{N}$ such that, given any graph G , and any partition P of $V(G)$ into at most m parts, there exists a Szemerédi refinement of P for ε into k parts, for some $m \leq k \leq M$.*

Let $T_r(n)$ denote the Turán graph, and $t_r(n) = e(T_r(n))$ the Turán number, as usual.

Erdős-Simonovits Stability Theorem (Erdős, Simonovits, 1968). *For each $r \in \mathbb{N}$ and $\varepsilon > 0$, there exists a $\delta > 0$ such that the following holds. Let G be a graph, and suppose that $K_{r+1} \not\subset G$, but*

$$e(G) \geq t_r(n) - \delta n^2.$$

Then we can change G into $T_r(n)$ by switching at most εn^2 edges.

We say a set $X \subset [n]$ is shattered by a family of subsets $\mathcal{A} \subset \mathcal{P}(n)$ if for every set $B \subset X$, there exists an $A \in \mathcal{A}$ such that $B \cap X = A$.

Sauer's Lemma (Sauer, 1972). *Let $\mathcal{A} \subset \mathcal{P}(n)$, and suppose that*

$$|\mathcal{A}| > \sum_{i=0}^{k-1} \binom{n}{i}.$$

Then there exists a k -set $X \subset [n]$ which is shattered by \mathcal{A} .

Given $A, B \subset V(G)$, we shall also write $A \rightarrow B$ to mean A 'shatters' B , i.e., that $G[A', B] = U(k)$ for some $A' \subset A$, where $k = |B|$. Note that $A \rightarrow B$ if and only if the family of sets $\mathcal{A} = \{\Gamma(v) \cap B : v \in A\}$ shatters B .

Given $\varepsilon, \delta > 0$, a pair (A, B) of subsets $A, B \subset V(G)$ is said to be (ε, δ) -grey if the graph $G[A, B]$ is ε -regular and has density $\delta \leq d(A, B) \leq 1 - \delta$.

The following Embedding Lemma is well-known (see [29], Theorem 2.1).

Embedding Lemma. *Let $\delta > 0$, and let H be a graph on r vertices. There exist $\varepsilon > 0$ and $n_0 = n_0(\varepsilon, \delta, r) \in \mathbb{N}$ such that the following holds.*

Let G be a graph on vertex set $A_1 \cup \dots \cup A_r$, where $|A_j| \geq n_0$ for each $j \in [r]$. Then there exist distinct vertices $v_k \in A_k$ for each $k \in [r]$ such that, whenever the pair (A_i, A_j) is (ε, δ) -grey, we have

$$v_i v_j \in E(G) \Leftrightarrow ij \in E(H).$$

We shall also use the following simple result, known as the Slicing Lemma (see [29], Fact 1.5).

The Slicing Lemma. *If (A, B) is (ε, δ) -grey and $X \subset A$, $Y \subset B$ with $|X| \geq \alpha|A|$ and $|Y| \geq \alpha|B|$, then the pair (X, Y) is (ε', δ') -grey, where $\varepsilon' = 2\varepsilon/\alpha$ and $\delta' = \delta/\alpha - \varepsilon$.*

Finally, we make a trivial observation.

Observation 3. *Let \mathcal{P} be a hereditary property of graphs, and suppose $\chi_c(\mathcal{P}) = r$. Then*

$$|\mathcal{P}_n| \geq 2^{(1-1/r+o(1))n^2/2}.$$

Proof. By definition: there are this many graphs in $\mathcal{H}(r, v)$. □

As noted in the introduction, Alekseev [1] and Bollobás and Thomason [16, 17] independently proved the corresponding upper bound.

4. SKETCH OF THE PROOF

Before proving Theorem 1, let us give a brief (and imprecise) sketch of the proof. For simplicity we shall only consider the case $\chi_c(\mathcal{P}) = 2$.

Let $\varepsilon, \delta, \gamma, \alpha > 0$ be sufficiently small, with $\varepsilon \ll \delta \ll \gamma \ll \alpha$, and let $G \in \mathcal{P}_n$, where n is sufficiently large. We shall say that the bipartite subgraph of G induced by a pair of sets is *grey* if it is (ε, δ) -grey, i.e., it is ε -regular and of density between δ and $1 - \delta$. We shall also use the following definition of a *generalized universal graph*:

Let $U(3, k)$ denote the 3-partite graph on vertex set $A \cup B \cup C$, where

$$|A| = 2^{|B|+|C|}, |B| = 2^k \text{ and } |C| = k, \text{ such that } B \rightarrow C \text{ and } A \rightarrow B \cup C.$$

The first step in the proof is to show that, for almost all $G \in \mathcal{P}$, there is a partition (S_1, S_2) of $V(G)$ such that,

- (a) Each part is a union of (an almost equal number of) Szemerédi sets.
- (b) Each part contains at most γm^2 grey pairs (where m is the total number of Szemerédi sets).

The proof of this (see Lemma 16) follows as in [7], by applying the Szemerédi Regularity Lemma and Erdős-Simonovits Stability Theorem. The key observation is that the ‘cluster graph’ contains no grey triangles (see Lemma 9). We call such a partition (S_1, S_2) a *BBS-partition* of G .

Next we consider a maximal set $B \subset V(G)$ of vertices such that, for each $j = 1, 2$, and each pair $b, b' \in B$,

$$|(\Gamma(b) \cap S_j) \Delta (\Gamma(b') \cap S_j)| \geq \alpha n.$$

We shall sometimes refer to a set with this property as a set of ‘bad’ vertices. The main step in the proof is to show that, for almost every $G \in \mathcal{P}$, $|B|$ is bounded.

Indeed, we show (see Lemmas 10, 11 and 12) that, for any $t \in \mathbb{N}$, if $|B|$ is sufficiently large then there exist a set $B' \subset B$ of size 2^{2t} , and sets $T_1^{(1)}, \dots, T_t^{(1)} \subset S_1$ and $T_1^{(2)}, \dots, T_t^{(2)} \subset S_2$, with $|T_j^{(i)}| \geq \delta n$, such that the following holds:

- (a) All vertices of $T_j^{(i)}$ have the same neighbourhood in B' .
- (b) If $b_1, b_2 \in B'$ with $b_1 \neq b_2$, then $\Gamma(b_1) \cap \bigcup T_j^{(i)} \neq \Gamma(b_2) \cap \bigcup T_j^{(i)}$. (Since there are 2^{2t} vertices in B' , this means they shatter any set of representatives of the sets $T_j^{(i)}$.)

The proof of Lemma 10 uses the so-called ‘sparsening method’, together with a repeated application of the ‘reverse’ Sauer’s Lemma; that is, Sauer’s Lemma combined with Lemma 4, the observation that if $U \rightarrow V$, then $V \rightarrow U'$ for some (large) $U' \subset U$.

Now, suppose such a set B' exists in G . We show (see Lemma 17) that in almost every such graph we can find subsets $W_j^{(i)} \subset T_j^{(i)}$ such that, for each $p, q \in [t]$, the pair $(W_p^{(1)}, W_q^{(2)})$ is grey. Hence, by the Embedding Lemma, we can find a copy of $U(3, k)$ in

G , for arbitrarily large k (see Lemma 18). But this is a contradiction, since $\chi_c(\mathcal{P}) = 2$ (see Lemma 7).

We have shown that $|B|$ is bounded for almost every $G \in \mathcal{P}$. Since B is maximal, it follows that each vertex $v \in V(G)$ is a ‘clone’ of some vertex $b \in B$ with respect to one of the sets S_j , i.e.,

$$|(\Gamma(v) \cap S_j) \Delta (\Gamma(b) \cap S_j)| \leq \alpha n.$$

Since we expect to have few choices inside the sets S_j , it would be natural to expect that $v \in S_j$. Although this is not necessarily true for every vertex $v \in V(G)$, it turns out that, for almost every graph $G \in \mathcal{P}$, we can make it true by ‘adjusting’ the partition (S_1, S_2) (see Lemmas 20 and 21). We obtain a new partition, (S'_1, S'_2) , which is ‘close’ to the original partition (in the sense that $|S_j \Delta S'_j| \leq \alpha n$), such that for each $j = 1, 2$, and every $v \in S'_j$, v is a clone of some $b \in B$ with respect to S'_j .

Finally, let $U_j \subset S'_j$ be the vertex set of a maximal collection of disjoint copies of $U(k)$ in S'_j . (When $\chi_c(\mathcal{P}) \geq 3$ this step is more complicated, see the algorithm before Lemma 23). We claim that $|U_j| = O(n^{1-\varepsilon})$ for almost every $G \in \mathcal{P}$; to prove this, we simply count (see Lemma 23). First, note that there are at most n^n choices for the partition of $V(G)$, and for the edges incident with B , and at most

$$2^{\alpha|U_j|n+n^{2-\varepsilon}}$$

choices for the edges inside S'_j . (This follows because each vertex of U_j is a clone of a vertex in B with respect to S'_j , and by Theorem 2, using the fact that $S'_j \setminus U_j$ is $U(k)$ -free.) We will show further (see Lemma 15) that we have at most $2^{(1/2-2\alpha)|U_j|n}$ choices for the edges between U_j and $V(G) \setminus S'_j$. Thus the total number of choices, $|\mathcal{P}_n|$, satisfies

$$\begin{aligned} \log_2(|\mathcal{P}_n|) &\leq (|S'_1| - |U_1|)(|S'_2| - |U_2|) + \left(\frac{1}{2} - \alpha\right) (|U_1| + |U_2|)n + O(n^{2-\varepsilon}) \\ &\leq \frac{n^2}{4} - \alpha(|U_1| + |U_2|)n + O(n^{2-\varepsilon}), \end{aligned}$$

which implies that $|U_1| + |U_2| = O(n^{1-\varepsilon})$, as required. (We have assumed for simplicity that $|U_j| = o(n)$; the calculation in the other case is essentially the same.) Letting $A = U_1 \cup U_2 \cup B$, we obtain Theorem 1.

5. PROOF OF THEOREM 2

In this section we give a short proof of Theorem 2. Our main tool is Sauer’s Lemma.

Proof of Theorem 2. Let $k \in \mathbb{N}$, and let G be a $U(k)$ -free graph on n vertices. We first claim that, given any bipartition (A, B) of the vertex set $V(G)$, there are at most $2^{n^{2-\varepsilon}}$ choices for the cross-edges.

Indeed, let $0 < \varepsilon < 1/(k+1)$, and partition B into sets B_1, \dots, B_t of size about n^ε . By Sauer’s Lemma, for each $j \in [t]$ we have

$$|\{S \subset B_j : \exists a \in A \text{ such that } \Gamma(a) \cap B_j = S\}| \leq k \binom{|B_j|}{k-1} < n^{\varepsilon k},$$

since $G[A, B_j]$ is $U(k)$ -free. Thus the number of choices for $G[A, B]$ is at most

$$\prod_{j=1}^t \binom{2^{|B_j|}}{n^{\varepsilon k}} (n^{\varepsilon k})^n \leq 2^{\sum_j |B_j| n^{\varepsilon k}} (2^{\varepsilon k n \log n})^{n^{1-\varepsilon}} \leq 2^{n^{2-\varepsilon} \log n},$$

since $\sum_j |B_j| = n$ and $1 + \varepsilon k < 2 - \varepsilon$.

Finally, we may cover $E(G)$ with $\log n$ bipartite graphs, and so the number of choices for G is at most $2^{n^{2-\varepsilon} (\log n)^2} \ll 2^{n^{2-\varepsilon'}}$ for any $\varepsilon' < \varepsilon$, as required. \square

The reader will have noticed that the value of ε obtained above is not best possible; a more precise calculation is undertaken in Section 8.

6. SOME LEMMAS ON UNIVERSAL GRAPHS

In this section we state some of the lemmas we'll use to prove Theorem 1. We begin with a simple but key observation. Recall that we write $A \rightarrow B$ to mean that A shatters B , i.e., $G[A', B] = U(k)$ for some $A' \subset A$, where $|B| = k$.

Lemma 4. *Let G be a graph and let $t \in \mathbb{N}$. Suppose $A, B \subset V(G)$, with $A \rightarrow B$ and $|B| \geq 2^t$. Then there exist subsets $A' \subset A$ and $B' \subset B$ such that $B' \rightarrow A'$, and $|A'| = t$.*

Proof. Let $A \rightarrow B$ be as described, and assume (taking subsets if necessary) that $|B| = 2^t$ and $G[A, B] = U(2^t)$. Label the elements of B with the vertices of the hypercube $[2]^t$ arbitrarily, and consider the faces of this cube (i.e., the subcubes of dimension $t-1$) which contain the origin. Denote by F_1, \dots, F_t the corresponding subsets of B .

Now, there is an obvious bijection ϕ between vertices of A and subsets of B (a vertex is mapped to its neighbourhood), and so each set $F_j \subset B$ corresponds to a vertex of A . Let $A' = \{\phi(F_1), \dots, \phi(F_t)\}$.

We claim that for each pair of vertices $b, b' \in B$, we have $\Gamma(b) \cap A' \neq \Gamma(b') \cap A'$. Indeed, if $b \neq b'$ then their labels must differ in some direction on the cube, so $b \in F_j, b' \notin F_j$, say. But then $\phi(F_j) \in \Gamma(b) \setminus \Gamma(b')$, as claimed. Thus A' and B' are sets as required by the lemma. \square

We shall also use the following slight extension of Lemma 4.

Lemma 5. *Let G be a graph and let $r, t \in \mathbb{N}$. Let $A_1, \dots, A_r, B \subset V(G)$ be disjoint sets, with $|B| \geq 2^{rt}$, and $A_j \rightarrow B$ for each $j \in [r]$. Then there exist subsets $A'_j \subset A_j$ and $B' \subset B$ such that $B' \rightarrow \bigcup_j A'_j$, and $|A'_j| = t$ for each $j \in [r]$.*

Proof. Assume (by taking a subset if necessary) that $|B| = 2^{rt}$. By Lemma 4, there exist subsets $A_j^* \subset A_j$ such that $|A_j^*| = rt$ and $B \rightarrow A_j^*$ for each $j \in [r]$. Moreover, we can choose the sets A_j^* so that the following holds:

- Let $(v_1^{(1)}, \dots, v_1^{(rt)})$ be an arbitrary order for the elements of A_1^* . Then, for each $i \in [rt]$ and $j \in [r]$, there exists a vertex $v_j^{(i)} \in A_j^*$ with the same neighbourhood in B as $v_1^{(i)}$.

Indeed, to do this we simply use the same $[2]^t$ -labelling of B (see the proof of Lemma 4 above) for each set A_j .

Now we simply have to choose r disjoint t -subsets $A'_j \subset A_j^*$ for $j \in [r]$. To spell it out, let $A'_j = \{v_j^{(i)} : (j-1)t + 1 \leq i \leq jt\}$ for each $j \in [r]$. It is clear that $B \rightarrow \bigcup_j A'_j$, as required. \square

In Section 4 we used the 3-partite graph $U(3, k)$. We now make the natural generalization to r -partite graphs, which we shall denote $U(r, k)$.

Definition. For each $k, r \in \mathbb{N}$, define the *generalized universal graph*, $U(r, k)$, to be the r -partite graph on vertex set $A_1 \cup \dots \cup A_r$, where $|A_1| = k$ and $|A_{j+1}| = 2^{\sum_1^j |A_i|}$ for each $1 \leq j \leq r-1$, such that

$$A_{j+1} \rightarrow A_1 \cup \dots \cup A_j$$

for each $1 \leq j \leq r-1$.

Moreover, for each $v \in \{0, 1\}^r$, define $U_v^*(r, k)$ to be the graph on vertex set $A_1 \cup \dots \cup A_r$ such that the induced r -partite graph is $U(r, k)$, and $G[A_j]$ is either complete or empty for each $j \in [r]$, with $G[A_j]$ complete if and only if $v_j = 1$.

We next apply Lemma 4 to prove a Ramsey-type theorem for the graphs $U(r, k)$.

Lemma 6. *For each $r, k \in \mathbb{N}$, there exists $K = K(r, k) \in \mathbb{N}$ such that the following holds. Let G be a graph on vertex set $A_1 \cup \dots \cup A_r$, and suppose $G[A_1, \dots, A_r] = U(r, K)$. Then $U_v^*(r, k) \leq G$ for some $v \in \{0, 1\}^r$.*

Proof. The proof is by induction on r . For $r = 1$ the lemma is just Ramsey's Theorem. For $r = 2$ we first apply Ramsey to A_1 , to obtain a subset $B_1 \subset A_1$ such that $G[B_1]$ is either complete or empty. Note that $A_2 \rightarrow B_1$, so by Lemma 4, there exists a set $B'_2 \subset A_2$ such that $B_1 \rightarrow B'_2$. Finally, applying Ramsey to B'_2 , we obtain a subset $B_2 \subset B'_2$ such that $G[B_2]$ is complete or empty, and $B_1 \rightarrow B_2$.

So let $r \geq 3$, and assume the result holds for smaller values of r . Let $t = |U_{v'}^*(r-1, k)|$, $T = 2^t$ and $m = \binom{R(T)}{T}$, where $R(T)$ is the Ramsey number. Let $K' = mk$, and let $K = K(r-1, K')$. We claim that the lemma holds for K .

First, by the induction hypothesis, there exists a copy H of $U_{v'}^*(r-1, K')$ in $G[A_1 \cup \dots \cup A_{r-1}]$, for some $v' \in \{0, 1\}^{r-1}$. Note that $A_r \rightarrow V(H)$, and let $V(H) = B_1 \cup \dots \cup B_{r-1}$, where $B_{j+1} \rightarrow B_1 \cup \dots \cup B_j$ for each $1 \leq j \leq r-2$.

Since $A_r \rightarrow V(H)$, the bipartite graph $G[A_r, V(H)]$ contains every (small) bipartite graph as an induced subgraph. We shall define a specific such bipartite graph, F , and show that it contains $U_v^*(r, k)$ for some $v \in \{0, 1\}^r$.

Indeed, since $K' = mk$, H contains m disjoint copies of $U_{v'}^*(r-1, k)$. To see this, simply partition B_1 into m equal-size parts, $B_1^{(1)}, \dots, B_1^{(m)}$, and successively choose disjoint sets $B_j^{(1)}, \dots, B_j^{(m)} \subset B_j$ such that $B_j^{(i)} \rightarrow B_1^{(i)} \cup \dots \cup B_{j-1}^{(i)}$. Let $H^{(i)}$ denote the graph induced by $B_1^{(i)} \cup \dots \cup B_{r-1}^{(i)}$.

Now, define the bipartite graph F as follows. Let $\mathcal{I}(T)$ denote the set of subsets of $\{1, \dots, R(T)\}$ of size T .

- (a) $V(F) = X \cup Y$, with $X = [R(T)]$ and $Y = \bigcup_{Z \in \mathcal{I}(T)} Y_Z$, where the sets Y_Z are disjoint, and $|Y_Z| = t$ for each $Z \in \mathcal{I}(T)$.
- (b) For each subset $W \subset X$ of size T , we have $W \rightarrow Y_W$.
- (c) The other edges may be chosen arbitrarily.

Note that no two vertices of X have the same neighbourhood in Y , since each such pair is contained in some T -set $W \subset X$, so differs on Y_W . Therefore, since $A_r \rightarrow V(H)$, it follows that there exists a set $A'_r \subset A_r$ such that

$$G[A'_r, V(H^{(1)}) \cup \dots \cup V(H^{(m)})] = F,$$

with $Y_Z = V(H^{(\phi(Z))})$ for each $Z \in \mathcal{I}(T)$, for some bijection $\phi : \mathcal{I}(T) \rightarrow [m]$.

Finally we apply Ramsey's Theorem to A'_r , to obtain a set B_r of size T such that $G[B_r]$ is complete or empty. By the definition of F we have $B_r \rightarrow Y_{B_r}$, so $B_r \rightarrow V(H^{(i)})$ for some $i \in [m]$. It follows that the set $B_r \cup Y_{B_r}$ induces a copy of $U_v^*(r, k)$ for some $v \in \{0, 1\}^r$, as required. \square

The following immediate consequence of Lemma 6 says that if $\chi_c(\mathcal{P}) = r$, and there is an arbitrarily large copy of $U(r+1, k)$ in G , then G contains a forbidden graph of \mathcal{P} .

Lemma 7. *Let $r, m \in \mathbb{N}$, and for each $v \in \{0, 1\}^r$ choose a graph $H_v \in \mathcal{H}(r, v)$ with $|V(H_v)| \leq m$. Let $k = k(r, m) \in \mathbb{N}$ be sufficiently large, and let G be any graph.*

Then either G is $U(r, k)$ -free, or $H_v \leq G$ for some $v \in \{0, 1\}^r$.

Proof. Suppose that G contains a copy of $U(r, k)$. Then, by Lemma 6, G also contains an induced copy of $U_v^*(r, m)$ for some $v \in \{0, 1\}^r$. But $H_v \leq U_v^*(r, k)$ for every $H_v \in \mathcal{H}(r, v)$ with $|H_v| \leq m$, and so we are done. \square

Before continuing, we note that we have already proved Theorem 1 in the case $r = 1$.

Corollary 8. *Let \mathcal{P} be a hereditary property of graphs with $\chi_c(\mathcal{P}) = 1$. Then there exists $k = k(\mathcal{P}) \in \mathbb{N}$ such that G is $U(k)$ -free for every $G \in \mathcal{P}$, and hence there exists $\varepsilon > 0$ such that*

$$|\mathcal{P}_n| \leq 2^{n^{2-\varepsilon}}$$

for every sufficiently large $n \in \mathbb{N}$.

Proof. Since $\chi_c(\mathcal{P}) = 1$, there exists, for each $v \in \{0, 1\}^2$, a graph $H_v \in \mathcal{H}(2, v)$ such that $H_v \notin \mathcal{P}$. By Lemma 7, it follows that there exists $k \in \mathbb{N}$ such that G is $U(k)$ -free for every $G \in \mathcal{P}$. By Theorem 2, it follows that

$$|\mathcal{P}_n| \leq 2^{n^{2-\varepsilon}}$$

for every sufficiently large $n \in \mathbb{N}$, as required. \square

The following 'induced' embedding lemma is a simple consequence of the Embedding Lemma, the Slicing Lemma and (the proof of) Lemma 7.

Lemma 9. *Given $\delta > 0$ and $m, r \in \mathbb{N}$, there exist $\varepsilon > 0$ and $n_0 = n_0(\varepsilon, \delta, m, r) \in \mathbb{N}$ such that the following holds. Let G be a graph on $A_1 \cup \dots \cup A_r$, where $|A_j| \geq n_0$ for each $j \in [r]$, and suppose each pair (A_i, A_j) is (ε, δ) -grey.*

Then for some $v \in \{0, 1\}^r$, $H \leq G$ for every $H \in \mathcal{H}(r, v)$ with $|H| \leq m$.

Proof. We claim that G is not $U(r, k)$ -free, where $k = k(r, m)$ is the constant in Lemma 7, if n_0 is sufficiently large. Indeed, by the Slicing Lemma, if we partition each set A_j into $t = |U(r, k)|$ almost equal sets $(A_j^{(1)}, \dots, A_j^{(t)})$, then each pair $(A_j^{(i)}, A_{j'}^{(i')})$ (with $j \neq j'$) is (ε', δ') -grey, where $\varepsilon' = 2\varepsilon/t$ and $\delta' = \delta/t - \varepsilon$. Therefore, by the Embedding Lemma, there exists a copy of $U(r, k)$ in G , as required.

But now, by the proof of Lemma 7, G also contains an induced copy of $U_v^*(r, m)$ for some $v \in \{0, 1\}^r$, and so $H_v \leq U_v^*(r, k)$ for every $H_v \in \mathcal{H}(r, v)$ with $|H_v| \leq m$, as required. \square

We next prove our key lemma whose proof uses the so-called ‘sparsening’ method.

Lemma 10. *For each $\alpha > 0$ and $t \in \mathbb{N}$, there exist $c = c_1(\alpha, t) \in \mathbb{N}$ and $\delta = \delta_1(c, \alpha, t) > 0$ such that the following holds. Let G be a bipartite graph with parts U and V , satisfying $|U| \geq c$, $|V| = n \in \mathbb{N}$ and*

$$|\Gamma(u) \Delta \Gamma(u')| \geq \alpha n$$

for each $u, u' \in U$ with $u \neq u'$.

Then there exists a subset $U' \subset U$, with $|U'| = t$, and sets $T_1, \dots, T_{2^t} \subset V$, with $|T_j| \geq \delta n$ for each $j \in [2^t]$, such that the following holds:

- (a) *If $u, v \in T_j$ then $\Gamma(u) \cap U' = \Gamma(v) \cap U'$.*
- (b) *If $W = \{w_1, \dots, w_{2^t}\}$, where $w_j \in T_j$ for each $j \in [2^t]$, then $W \rightarrow U'$.*

Proof. Assume that $|U| = c \geq \left(\frac{5 \log c}{\alpha}\right)^{2^t}$, and with foresight, let $p = \frac{5 \log c}{\alpha n}$. We claim that there exists a subset $X \subset V$, with $|X| = pn$, such that

$$\Gamma(x) \cap X \neq \Gamma(y) \cap X$$

for each $x, y \in U$ with $x \neq y$. Indeed, if we choose a random subset $X \subset V$ of size pn , then

$$\mathbb{P}\left(\Gamma(x) \cap X = \Gamma(y) \cap X\right) \leq \binom{n - \alpha n}{pn} / \binom{n}{pn} \leq (1 - \alpha)^{pn}$$

for each such pair $\{x, y\}$, and so

$$\mathbb{E}\left|\left\{\{x, y\} \in \binom{U}{2} : \Gamma(x) \cap X = \Gamma(y) \cap X\right\}\right| \leq \binom{|U|}{2} (1 - \alpha)^{pn} \leq c^2 e^{-p\alpha n} < 1,$$

by our choice of p .

Thus such a set X must exist, as claimed. Now, since c is sufficiently large so that

$$c \geq |X|^{2^t} = (pn)^{2^t} = \left(\frac{5 \log c}{\alpha}\right)^{2^t}$$

then, by Sauer's Lemma, there exist sets $U^* \subset U$ and $X^* \subset X$, with $|X^*| \geq 2^t$, such that $U^* \rightarrow X^*$. Thus, by Lemma 4, there exist sets $U_1 \subset U^*$ and $X_1 \subset X^*$, with $|U_1| = t$, such that $X_1 \rightarrow U_1$.

Now, let us remove X_1 from V , and repeat the process, obtaining disjoint sets X_2, \dots, X_ℓ . Since $|X_j| = 2^t$, we can do this so long as $\ell \leq \frac{\alpha n}{2^{t+1}}$. (It is easy to see that nothing goes wrong in the calculation above when we replace α by $\alpha/2$.) By the pigeonhole principle, there is a set $U' \subset U$ which occurs (as the set U_j) at least

$$\ell \binom{|U|}{t}^{-1} \geq \frac{\alpha n}{2^{t+1} c^t} \geq \delta n$$

times. Let $\mathcal{J} = \{j : X_j \rightarrow U'\}$, and note that $|\mathcal{J}| \geq \delta n$. Write $X_j = \{x_j(1), \dots, x_j(2^t)\}$, where $\Gamma(x_j(k)) \cap U' = \Gamma(x_{j'}(k)) \cap U'$ for every $j, j' \in \mathcal{J}$, and let $T_k = \{x_j(k) : j \in \mathcal{J}\}$, for each $k \in [2^t]$. The sets T_1, \dots, T_{2^t} satisfy conditions (a) and (b), as required. \square

The corresponding result for r -partite graphs follows as an easy corollary.

Lemma 11. *For each $\alpha > 0$ and $r, t \in \mathbb{N}$, there exist $c = c_2(\alpha, r, t) \in \mathbb{N}$ and $\delta = \delta_2(c, \alpha, r, t) > 0$ such that the following holds. Let G be a graph on $n \in \mathbb{N}$ vertices, let (S_1, \dots, S_r) be a partition of $V(G)$, and let $B \subset V(G)$ satisfy $|B| \geq c$ and*

$$|(\Gamma(b) \cap S_j) \Delta (\Gamma(b') \cap S_j)| \geq \alpha n$$

for every $j \in [r]$, and each $b, b' \in B$ with $b \neq b'$.

Then there exists a subset $B' \subset B$, with $|B'| = t$, and sets $T_1^{(i)}, \dots, T_{2^t}^{(i)} \subset S_i$ for each $i \in [r]$, with $|T_j^{(i)}| \geq \delta n$, such that the following holds:

- (a) If $u, v \in T_j^{(i)}$ then $\Gamma(u) \cap B' = \Gamma(v) \cap B'$.
- (b) If $W = \{w_1, \dots, w_{2^t}\}$, where $w_j \in T_j^{(1)} \cup \dots \cup T_j^{(m)}$ for each $j \in [2^t]$, then $W \rightarrow B'$.

Proof. Let $t_1 > t_2 > \dots > t_{r+1} = t$ be a sequence satisfying $t_j \geq c_1(\alpha, t_{j+1})$ for each $j \in [r]$ (where c_1 is the function in Lemma 10), and assume that $|B| = c \geq t_1$. Applying Lemma 10 (with $t = t_1$) to the pair $(B, S_1) = (U, V)$, we obtain a set $B_1 \subset B$ and a collection of sets $T_1^{(1)}, \dots, T_{2^{t_1}}^{(1)} \subset S_1$ given by that lemma. In particular, we have $|B_1| \geq t_2$ and $|T_j^{(i)}| \geq \delta n$, where $\delta = \delta_1(t_1, \alpha, t_2)$.

Similarly, for each $q \in [2, r]$ we may apply Lemma 10 (with $c = t_q$ and $t = t_{q+1}$) to the pair (B_{q-1}, S_q) , to obtain sets $B_q \subset B_{q-1}$ and $T_1^{(q)}, \dots, T_{2^{t_q}}^{(q)} \subset S_q$, with $|B_q| = t_{q+1}$ and $|T_j^{(q)}| \geq \delta n$, where $\delta = \delta_1(t_q, \alpha, t_{q+1})$. Let $\delta_2(c, \alpha, t) = \min_q \{\delta_1(t_q, \alpha, t_{q+1})\}$.

Finally, for each $q \in [r]$, re-number so that the sets $T_1^{(q)}, \dots, T_{2^t}^{(q)}$ shatter $B_r \subset B_q$. It follows that the sets $B' = B_r$ and $\{T_j^{(i)} : i \in [r], j \in [2^t]\}$ are those required by the lemma, and so we are done. \square

We shall in fact use the following immediate corollary of Lemmas 5 and 11.

Lemma 12. *For each $\alpha > 0$ and $r, t \in \mathbb{N}$, there exist $c = c_3(\alpha, r, t) \in \mathbb{N}$ and $\delta = \delta_3(c, \alpha, r, t) > 0$ such that the following holds. Let G be a graph on $n \in \mathbb{N}$ vertices, let (S_1, \dots, S_r) be a partition of $V(G)$, and let $B \subset V(G)$ satisfy $|B| \geq c$ and*

$$|(\Gamma(b) \cap S_j) \Delta (\Gamma(b') \cap S_j)| \geq \alpha n$$

for every $j \in [r]$, and each $b, b' \in B$ with $b \neq b'$.

Then there exists a subset $B' \subset B$, with $|B'| = 2^{rt}$, and sets $T_1^{(i)}, \dots, T_t^{(i)} \subset S_i$ for each $i \in [r]$, with $|T_j^{(i)}| \geq \delta n$, such that the following holds:

- (a) If $u, v \in T_j^{(i)}$ then $\Gamma(u) \cap B' = \Gamma(v) \cap B'$.
- (b) If $W = \{w_{11}, \dots, w_{rt}\}$, where $w_{ij} \in T_j^{(i)}$ for each $i \in [r], j \in [t]$, then $B' \rightarrow W$.

Proof. First we apply Lemma 11 to get sets $U_j^{(i)}$ (for each $i \in [r]$ and $j \in [2^{|B'|}]$) which shatter B' . Applying Lemma 5 to these sets (or, if the reader prefers, to an arbitrarily chosen element from each set) gives the required sets. \square

The following two observations will be useful in Section 7.

Observation 13. *Let $\delta > 0$ be sufficiently small, and let $|A| = |B| = n$. There are at most $2^{\delta n^2}$ bipartite graphs on $A \cup B$ of density at most δ^2 .*

Proof. If G is such a graph then $e(G) \leq m = \delta^2 n^2$, so the number of choices is at most

$$\sum_{j=0}^m \binom{n^2}{j} \leq 2 \binom{n^2}{m} \leq 3^m \left(\frac{1}{\delta^2}\right)^{\delta^2 n^2} < 2^{\delta n^2},$$

as required, since $(1/x^2)^x \rightarrow 1$ as $x \rightarrow 0$. \square

Recall that $K_r(t)$ denotes the Turán graph on rt vertices, i.e., the complete r -partite graph with t vertices in each part.

Observation 14. *For each $r, t \in \mathbb{N}$, there exist $\varepsilon > 0$ and $n_0 = n_0(r, t) \in \mathbb{N}$ such that the following holds. Let G be an r -partite graph on vertex set $A_1 \cup \dots \cup A_r$ and, for each $j \in [r]$, let $B_j(1) \cup \dots \cup B_j(t)$ be an equipartition of A_j . Suppose $n \geq n_0$, $|A_1| = \dots = |A_r| = n$, and $e(G) \geq (1 - \varepsilon) \binom{r}{2} n^2$.*

Then there exists a copy H of $K_r(t)$ in G with $|H \cap B_j(k)| \leq 1$ for each j, k .

Proof. Since ε may be chosen so that $\varepsilon r^3 t^3 \ll 1$, the result is trivial by the greedy algorithm. To spell it out, for each i, j there exists a vertex $v \in B_j^{(i)}$ such that $\Gamma(v)$ avoids at most $|B_{j'}^{(i)}|/rt$ vertices of each other set $B_{j'}^{(i)}$. \square

Finally, we prove the following easy lemma, which bounds the number of ways in which a copy of $U(r, k)$ in G can be attached to the rest of the graph without creating a copy of $U(r + 1, k)$.

Lemma 15. *For each $r, k \in \mathbb{N}$, there exists $K = K(r, k) \in \mathbb{N}$ and $\delta = \delta(r, k) > 0$ such that the following holds. Given a vertex set $A \cup B$, with $|A| = |U(r, k)|$ and $|B| = n$, let*

$$\mathcal{G}(r, k, n) := \left\{ G[A, B] : \exists \text{ a } U(r+1, k)\text{-free graph } G \text{ on } A \cup B \text{ with } G[A] = U(r, k) \right\},$$

the set of bipartite graphs on $A \cup B$ which do not create a copy of $U(r+1, k)$. Then

$$|\mathcal{G}(r, k, n)| \leq 2^{|U(r, k)|n - \delta n}.$$

Proof. Since G is $U(r+1, k)$ -free, no set $X \subset B$ shatters A . This means that there exists a subset $Y \subset A$ such that $\Gamma(u) \cap A \neq Y$ for every $u \in B$. We therefore have at most

$$(2^{|A|} - 1)^n \leq 2^{|U(r, k)|n - \delta n}$$

choices for the edges of $G[A, B]$, where $\delta = 2^{-|U(r, k)|}$, as required. \square

7. PROOF OF THEOREM 1

In this section we shall describe several ‘bad’ properties of a graph, and prove that the number of graphs in a hereditary property of graphs \mathcal{P} with one of these properties is $o(|\mathcal{P}_n|)$. We then deduce Theorem 1 by observing that all remaining graphs have the required structure.

We begin with an important definition, motivated by [7].

Definition. Let $\varepsilon, \delta, \gamma > 0$, let $r \in \mathbb{N}$, let G be a graph, and let $V(G) = S_1 \cup \dots \cup S_r$ be a partition of $V(G)$. We say that $P = (S_1, \dots, S_r)$ is a *BBS-partition* of G (for $(\varepsilon, \delta, \gamma)$) if there exists a Szemerédi partition of G (for ε) into m parts (for some $1/\varepsilon < m \in \mathbb{N}$) such that:

- (a) Each part S_j is a union of (an almost equal number of) Szemerédi sets.
- (b) Each part S_j contains at most γm^2 pairs which are (ε, δ) -grey.

Note that if (S_1, \dots, S_r) is a BBS-partition of G for $(\varepsilon, \delta, \gamma)$, then it follows that

$$\left(\frac{1}{r} - \varepsilon \right) n \leq |S_j| \leq \left(\frac{1}{r} + \varepsilon \right) n$$

for each $j \in [r]$, by condition (a).

The following lemma was proved (in the monotone case) by Balogh, Bollobás and Simonovits [7] using the Szemerédi Regularity Lemma, the Erdős-Simonovits Stability Theorem, and the Embedding Lemma. The proof is essentially the same in our case, but for the sake of completeness we shall give a fairly complete sketch.

Lemma 16 (Balogh, Bollobás and Simonovits [7]). *Let $r \in \mathbb{N}$, let $\gamma > 0$, and let $\delta = \delta(\gamma, r) > 0$ and $\varepsilon = \varepsilon(\delta, \gamma, r) > 0$ be sufficiently small. Let \mathcal{P} be a hereditary property of graphs with $\chi_c(\mathcal{P}) = r$.*

For almost every graph $G \in \mathcal{P}$, there exists a BBS-partition of G for $(\varepsilon, \delta, \gamma)$ into r parts.

Proof. First, since $\chi_c(\mathcal{P}) = r$, it follows that for each $v \in \{0, 1\}^{r+1}$ there exists a ‘forbidden’ graph $H_v \in \mathcal{H}(r+1, v)$, such that $H_v \notin \mathcal{P}$. Choose such a graph for each $v \in \{0, 1\}^{r+1}$, and let $t = \max\{|V(H_v)| : v \in \{0, 1\}^{r+1}\}$.

Let $G \in \mathcal{P}_n$, with $n \in \mathbb{N}$ sufficiently large, and apply the Szemerédi Regularity Lemma (for ε and $m = 1/\varepsilon$) to the graph G . We obtain a collection of k parts, B_1, \dots, B_k ; define a graph H on k vertices by letting $ij \in E(H)$ if and only if the pair (B_i, B_j) is (ε, δ^2) -grey. (This is called the cluster graph of G .)

Suppose first that $K_{r+1} \subset H$. Then, applying Lemma 9, we deduce that $H \leq G$ for every $H \in \mathcal{H}(r+1, v)$ with $|H| \leq t$, for some $v \in \{0, 1\}^{r+1}$. Therefore $H_v \in \mathcal{P}$, which is a contradiction.

Suppose next that the number of edges in the cluster graph H satisfies

$$e(H) \leq \left(1 - \frac{1}{r} - 2\delta\right) \binom{k}{2}.$$

It is easy to bound the number of graphs G with at most this many edges. Indeed, there are at most n^n ways of choosing the Szemerédi partition, and, by Observation 13 and our choice of m , at most $2^{(\varepsilon+\delta)n^2}$ ways of choosing the edges inside the parts, and between non-grey pairs. Moreover, there are at most $2^{(1-1/r-2\delta)n^2/2}$ ways of choosing the edges between grey pairs. But

$$|\mathcal{P}_n| \geq 2^{(1-1/r+o(1))n^2/2},$$

by Observation 3, so the number of such graphs G on n vertices is $o(|\mathcal{P}_n|)$.

Hence we may assume that H is K_{r+1} -free, and has at least $(1 - 1/r - 2\delta)\binom{k}{2}$ edges. By the Erdős-Simonovits Stability Theorem, it follows that we can change H into the Turán graph $T_r(k)$ by changing at most γk^2 edges. But this is exactly the definition of a BBS-partition, and so we are done. \square

Next, we need to count those graphs which have large ‘irregularities’ between pairs (S_i, S_j) of their BBS-partition. The following definition is designed to allow us to take advantage of the δn -sets given by Lemma 12.

Let $\varepsilon, \delta, \gamma, \alpha > 0$, let $n \in \mathbb{N}$, and let \mathcal{P} be a hereditary property of graphs. To simplify the notation in what follows, we shall suppress dependence on ε, δ and γ . Define a set $\mathcal{A}(\mathcal{P}_n, \alpha) \subset \mathcal{P}_n$ as follows:

$$\mathcal{A}(\mathcal{P}_n, \alpha) := \left\{ G \in \mathcal{P}_n : \exists \text{ a BBS-partition } (S_1, \dots, S_r) \text{ of } G \text{ for } (\varepsilon, \delta, \gamma), \text{ and sets } X \subset S_i \text{ and } Y \subset S_j, \text{ for some } i \neq j, \text{ with } |X|, |Y| \geq \alpha n, \text{ such that } d(X, Y) \notin (\delta, 1 - \delta) \right\}.$$

The following lemma says that the collection $\mathcal{A}(\mathcal{P}_n, \alpha)$ is small.

Lemma 17. *Let $\alpha > 0$, let $2 \leq r \in \mathbb{N}$, and let $\varepsilon > 0, \delta > 0$ and $\gamma > 0$ be sufficiently small. Let \mathcal{P} be a hereditary property of graphs with $\chi_c(\mathcal{P}) = r$, and let $n \in \mathbb{N}$. Then*

$$|\mathcal{A}(\mathcal{P}_n, \alpha)| \leq 2^{(1-1/r)n^2/2 - \alpha^2 n^2/3} = o(|\mathcal{P}_n|).$$

Proof. For the first inequality we simply count. We have at most n^n choices for the BBS-partition $S_1 \cup \dots \cup S_r$, and the sets $X \subset S_i$ and $Y \subset S_j$. By Observation 13, and the

definition of a BBS-partition, we have at most

$$2^{(\varepsilon + \sqrt{\delta} + \gamma)n^2}$$

choices for the edges inside the set S_k , for each $k \in [r]$.

Next, recall that $(1/r - \varepsilon)n \leq |S_\ell| \leq (1/r + \varepsilon)n$ for each $\ell \in [r]$, by the definition of a BBS-partition, and so we have at most

$$2^{(1/r + \varepsilon)^2 n^2}$$

choices for the edges between S_p and S_q , for each $p \neq q$. Moreover, we have at most

$$2^{(1/r + \varepsilon)^2 n^2 - (1 - \sqrt{\delta})\alpha^2 n^2 + 1}$$

choices for the edges between S_i and S_j . To see this, assume for simplicity that $|X| = |Y| = \alpha n$, and observe that we have at most $2^{(1/r + \varepsilon - \alpha)^2 n^2}$ choices for the edges between $S_i \setminus X$ and $S_j \setminus Y$, at most $2^{(1/r + \varepsilon - \alpha)\alpha n^2}$ choices for the edges between X and $S_j \setminus Y$ (and similarly for those between Y and $S_i \setminus X$), and, by Observation 13, at most $2^{\sqrt{\delta}\alpha^2 n^2 + 1}$ choices for the edges between X and Y .

Putting these bounds together, we obtain

$$\log_2(|\mathcal{A}(\mathcal{P}_n, \alpha)|) \leq \binom{r}{2} (1/r + \varepsilon)^2 n^2 - \frac{\alpha^2 n^2}{2} + O((\varepsilon + \sqrt{\delta} + \gamma)n^2).$$

The first inequality now follows if ε , δ and γ are sufficiently small. The final inequality follows by Observation 3. \square

Next, given a graph G , and a partition $P = (S_1, \dots, S_r)$ of G , we say that a set of vertices $B \subset V(G)$ is α -bad for (G, P) if

$$|(\Gamma(u) \cap S_j) \Delta (\Gamma(v) \cap S_j)| \geq \alpha n$$

for each $u, v \in B$ with $u \neq v$, and each $j \in [r]$. Let

$$B(G, P, \alpha) := \max\{|B| : B \subset V(G) \text{ is } \alpha\text{-bad for } (G, P)\}.$$

Now, given $\varepsilon, \delta, \gamma, \alpha > 0$, $c, n \in \mathbb{N}$, and a hereditary property of graphs \mathcal{P} , we define a set $\mathcal{B}(\mathcal{P}_n, \alpha, c) \subset \mathcal{P}_n$ as follows:

$$\mathcal{B}(\mathcal{P}_n, \alpha, c) := \{G \in \mathcal{P}_n : \exists \text{ a BBS-partition } P \text{ of } G \text{ for } (\varepsilon, \delta, \gamma) \text{ with } B(G, P, \alpha) \geq c\}.$$

We next show that, if $c = c(\mathcal{P})$ is sufficiently large then the collection $\mathcal{B}(\mathcal{P}_n, \alpha, c)$ is small.

Lemma 18. *Let $\alpha > 0$ and $r \in \mathbb{N}$, and let \mathcal{P} be a hereditary property of graphs with $\chi_c(\mathcal{P}) = r$. There exist constants $c = c(\mathcal{P}, \alpha) \in \mathbb{N}$ and $\alpha' = \alpha'(\mathcal{P}, \alpha) > 0$ such that the following holds. Let $\varepsilon > 0$, $\delta > 0$ and $\gamma > 0$ be sufficiently small, and let $n \in \mathbb{N}$ be sufficiently large. Then*

$$\mathcal{B}(\mathcal{P}_n, \alpha, c) \subset \mathcal{A}(\mathcal{P}_n, \alpha').$$

Proof. First, choose a graph $H_v \in \mathcal{H}(r+1, v) \setminus \mathcal{P}$ for each $v \in \{0, 1\}^{r+1}$. Such graphs must exist because $\chi_c(\mathcal{P}) = r$. Let $m = \max\{|H_v| : v \in \{0, 1\}^{r+1}\}$, and let $k = k(r, m) \in \mathbb{N}$ be the constant in Lemma 7.

Now, let $t = t(\alpha, r, k) \in \mathbb{N}$ be sufficiently large, let $c = c_3(\alpha, r, t)$ and $\tilde{\alpha} = \delta_3(c, \alpha, r, t)$ be the constants in Lemma 12, and let $\alpha' = \tilde{\alpha}/M(n_0, \varepsilon)$, where $M(\cdot, \cdot)$ is the constant in the refinement version of Szemerédi's Lemma, and $n_0 = n_0(r, t)$ is the constant in Observation 14.

Let $G \in \mathcal{P}_n \setminus \mathcal{A}(\mathcal{P}_n, \alpha')$, let ε, δ and γ be sufficiently small, and suppose that there exists a BBS-partition P of G for $(\varepsilon, \delta, \gamma)$ such that $B(G, P, \alpha) \geq c$, i.e., there exists a set $B \subset V(G)$, with $|B| \geq c$, which is α -bad for (G, P) .

Claim: G is not $U(r+1, k)$ -free.

Proof of claim. By Lemma 12, there exists a set $B' \subset B$, with $|B'| = 2^{rt}$, and disjoint sets $T_j^{(i)} \subset S_i$, with $|T_j^{(i)}| = \tilde{\alpha}n$ for each $i \in [r]$ and $j \in [t]$, such that

- (a) If $u, v \in T_j^{(i)}$ then $\Gamma(u) \cap B' = \Gamma(v) \cap B'$.
- (b) If $W = \{w_{11}, \dots, w_{rt}\}$, where $w_{ij} \in T_j^{(i)}$ for each $i \in [r], j \in [t]$, then $B' \rightarrow W$.

Let $T = \bigcup_{i,j} T_j^{(i)}$, and apply the refinement version of Szemerédi's Lemma (for ε) to the (equi-)partition

$$\bigcup_{i=1}^r \bigcup_{j=1}^t T_j^{(i)}$$

of T . We obtain, for each $i \in [r]$ and $j \in [t]$, a partition $(U_j^{(i)}(1), \dots, U_j^{(i)}(m))$ of $T_j^{(i)}$, such that the resulting partition of T is a Szemerédi partition. Moreover, by our choices of constants above, we have $m \geq n_0(r, t)$ and $|U_j^{(i)}| \geq \alpha'n$ for every $i \in [r]$ and $j \in [t]$.

Suppose first that there exists a pair $(U_j^{(i)}(\ell), U_{j'}^{(i')}(\ell'))$, where $i \neq i'$, which is ε -regular but not (ε, δ) -grey. Then the graph $G[U_j^{(i)}(\ell), U_{j'}^{(i')}(\ell')]$ has density in $[0, \delta) \cup (1 - \delta, 1]$, and so $G \in \mathcal{A}(\mathcal{P}_n, \alpha')$, a contradiction.

Thus every ε -regular pair $(U_j^{(i)}(\ell), U_{j'}^{(i')}(\ell'))$ is also (ε, δ) -grey. By the definition of a Szemerédi partition, at most $\varepsilon(mrt)^2$ pairs are irregular, and so the number of pairs $(U_j^{(i)}(\ell), U_{j'}^{(i')}(\ell'))$ with $i \neq i'$ which are not (ε, δ) -grey is at most $(1 - 2\varepsilon) \binom{r}{2} (mt)^2$.

We apply Observation 14 to the r -partite graph F where $V(F) = \{U_j^{(i)}(\ell) : i \in [r], j \in [t], \ell \in [m]\}$, and a pair of vertices $\{U_j^{(i)}(\ell), U_{j'}^{(i')}(\ell')\}$ (with $i \neq i'$) is an edge of F if and only if they form an (ε, δ) -grey pair. Since we chose $\varepsilon > 0$ sufficiently small, and $m \geq n_0$, it follows that there exist representatives $\{W_j^{(i)} : i \in [r], j \in [t]\}$, where $W_j^{(i)} \in \{U_j^{(i)}(1), \dots, U_j^{(i)}(m)\}$, such that every pair $(W_j^{(i)}, W_{j'}^{(i')})$ with $i \neq i'$ is (ε, δ) -grey.

It follows, by the Embedding Lemma, that there exists a copy of $U(r, k) \subset G$ with exactly one vertex in each set $W_j^{(i)}$. But B' shatters this copy of $U(r, k)$, by condition (b) above, and so G is not $U(r+1, k)$ -free, as claimed. \square

By Lemma 7 and our choice of k , it follows that $H_v \leq G \in \mathcal{P}$ for some forbidden graph H_v , which is a contradiction. Thus $\mathcal{B}(\mathcal{P}_n, \alpha, c) \subset \mathcal{A}(\mathcal{P}_n, \alpha')$, as required. \square

Given $\alpha > 0$, a graph G on n vertices, a subset $A \subset V(G)$ and two vertices $u, v \in V(G)$, we say that u is an α -clone of v with respect to A if

$$|(\Gamma(u) \cap A) \Delta (\Gamma(v) \cap A)| \leq \alpha n.$$

Let $P = (S_1, \dots, S_r)$ be a partition of G , and let B be an α -bad set for (G, P) . If B is chosen to be maximal, then for every vertex $v \in V(G)$, there exists a vertex $b \in B$, and an index $j \in [r]$ such that v is an α -clone of b with respect to S_j . Define

$$j(v) := \min \{j \in [r] : \exists b \in B \text{ such that } v \text{ is an } \alpha\text{-clone of } b \text{ with respect to } S_j\}.$$

Note that the function $j(\cdot)$ in fact depends on the triple (P, B, α) . It will usually be obvious which partition P , set B and constant $\alpha > 0$ we are using, so we suppress this dependence. When it is not obvious from the context, we shall clarify.

The following observation is an immediate consequence of the definition of an α -clone, together with Observation 13.

Observation 19. *Let $\alpha > 0$ be sufficiently small, let G be a graph on n vertices, let $A, B \subset V(G)$, and suppose $v \in V(G)$ is an α -clone of some vertex in B with respect to A . Then, given the edges of $G[A, B]$, we have at most*

$$|B|2^{\sqrt{\alpha}n}$$

choices for the edges between v and A .

We would like to have $v \in S_{j(v)}$ for every $v \in V(G)$. In fact we shall prove that, for almost every graph $G \in \mathcal{P}$, we can *adjust* any given BBS-partition of G to guarantee that this holds.

First, we shall show that almost all graphs $G \in \mathcal{P}_n$ have at most $\alpha n/2$ vertices with $v \notin S_{j(v)}$. Indeed, given $\alpha > 0$, a graph G , a BBS-partition $P = (S_1, \dots, S_r)$ of G , and a maximal α -bad set B for (G, P) , let

$$J(G, P, B, \alpha) := \{v \in V(G) : v \notin S_{j(v)}\}.$$

Now, given $n \in \mathbb{N}$, $\alpha, \varepsilon, \delta, \gamma > 0$ and a hereditary property of graphs \mathcal{P} , let

$$\mathcal{C}(\mathcal{P}_n, \alpha) := \left\{ G \in \mathcal{P}_n : \exists \text{ a BBS-partition } P \text{ of } G \text{ for } (\varepsilon, \delta, \gamma) \text{ and a maximal } (2\alpha)\text{-bad set } B \text{ for } (G, P) \text{ such that } |J(G, P, B, 2\alpha)| \geq \alpha n \right\}.$$

The next lemma says that the set $\mathcal{C}(\mathcal{P}_n, \alpha)$ is small.

Lemma 20. *Let $r \in \mathbb{N}$, and let \mathcal{P} be a hereditary property of graphs with $\chi_c(\mathcal{P}) = r$. Let $\alpha > 0$ be small, and let $\varepsilon, \delta, \gamma > 0$ be sufficiently small, and $n \in \mathbb{N}$ be sufficiently large. Then,*

$$|\mathcal{C}(\mathcal{P}_n, \alpha)| \leq 2^{(1-1/r)n^2/2 - \alpha n^2/3r^3} = o(|\mathcal{P}_n|).$$

Proof. The proof is almost the same as that of Lemma 17. Indeed, let $G \in \mathcal{C}(\mathcal{P}_n, \alpha)$, and let $P = (S_1, \dots, S_r)$ be a BBS-partition of G for $(\varepsilon, \delta, \gamma)$, and $B \subset V(G)$ be a maximal (2α) -bad set for (G, P) , such that $|J(G, P, B, 2\alpha)| \geq \alpha n$. By the pigeonhole principle, there exists $i, j \in [r]$ (with $i \neq j$) such that

$$|C| := |\{v \in S_i : j(v) = j\}| \geq \alpha' n,$$

where $\alpha' = \alpha/r^2$.

Now we simply count the graphs in $\mathcal{C}(\mathcal{P}_n, \alpha)$. We have at most n^n choices for the partition P , the set B , the index j and the set C . By Observation 13, and the definition of a BBS-partition, we have at most

$$2^{(\varepsilon + \sqrt{\delta} + \gamma)n^2}$$

choices for the edges inside the set S_k , for each $k \in [r]$.

Next, recall that $(1/r - \varepsilon)n \leq |S_k| \leq (1/r + \varepsilon)n$ for each $k \in [r]$, by the definition of a BBS-partition, and so we have at most

$$2^{(1/r + \varepsilon)^2 n^2}$$

choices for the edges between S_p and S_q , for each $p \neq q$. Moreover, we have at most

$$2^{(1/r + \varepsilon)^2 n^2 - (\alpha'/2r)n^2}$$

choices for the edges between S_i and S_j . Indeed, by Observation 19 we have at most $n2^{\sqrt{2\alpha'}|C|n}$ choices for the edges between C and S_j , and we have at most $2^{(1/r + \varepsilon)^2 n^2 - |C|n/r}$ choices for the edges between $S_i \setminus C$ and S_j .

Putting these bounds together, we obtain

$$\log_2 (|\mathcal{A}(\mathcal{P}_n, \alpha)|) \leq \binom{r}{2} (1/r + \varepsilon)^2 n^2 - \frac{\alpha n^2}{2r^3} + O\left((\varepsilon + \sqrt{\delta} + \gamma)n^2\right).$$

The first inequality now follows if ε, δ and γ are sufficiently small. The final inequality follows by Observation 3. \square

Now let G be a graph, let $r \in \mathbb{N}$, and let $\alpha, \alpha', \varepsilon, \delta, \gamma > 0$. Given a BBS-partition $P = (S_1, \dots, S_r)$ of G for $(\varepsilon, \delta, \gamma)$, and a maximal (2α) -bad set for (G, P) , we make the following definition.

Definition. An α -adjustment of (G, P) with respect to B is a partition $P' = (S'_1, \dots, S'_r)$ of $V(G)$ such that, for each $j \in [r]$, the following holds. $|S_j \Delta S'_j| \leq \alpha n$, and for every $v \in S'_j$, there exists $b \in B$ such that v is a (3α) -clone of b with respect to S'_j .

Given $n \in \mathbb{N}$, constants $\varepsilon, \delta, \gamma, \alpha > 0$, and a hereditary property of graphs \mathcal{P} , define

$$\mathcal{D}(\mathcal{P}_n, \alpha) := \left\{ G \in \mathcal{P}_n : \exists \text{ a BBS-partition } P \text{ of } G \text{ for } (\varepsilon, \delta, \gamma) \text{ and a maximal } (2\alpha)\text{-bad set } B \text{ for } (G, P) \text{ such that } \nexists \text{ an } \alpha\text{-adjustment of } (G, P) \text{ with respect to } B \right\}.$$

The next lemma follows easily from the definitions.

Lemma 21. *Let $r \in \mathbb{N}$, and let \mathcal{P} be a hereditary property of graphs with $\chi_c(\mathcal{P}) = r$. Let $\alpha > 0$, and let $\varepsilon, \delta, \gamma > 0$ be sufficiently small. Then,*

$$\mathcal{D}(\mathcal{P}_n, \alpha) \subset \mathcal{C}(\mathcal{P}_n, \alpha).$$

Proof. Let $G \in \mathcal{P}_n$, and suppose $G \in \mathcal{D}(\mathcal{P}_n, \alpha) \setminus \mathcal{C}(\mathcal{P}_n, \alpha)$. Let $P = (S_1, \dots, S_r)$ be an arbitrary BBS-partition of G for $(\varepsilon, \delta, \gamma)$, and let B be a maximal (2α) -bad set B for (G, P) . Note that, since $G \notin \mathcal{C}(\mathcal{P}_n, \alpha)$, we have

$$|J(G, P, B, 2\alpha)| = |\{v \in V(G) : v \notin S_{j(v)}\}| \leq \alpha n.$$

For each $j \in [r]$, let $S'_j = \{v \in V(G) : j(v) = j\}$. We claim that $P' = (S'_1, \dots, S'_r)$ is an α -adjustment of (G, P) with respect to B . Indeed, since $|J(G, P, B, 2\alpha)| \leq \alpha n$, it follows immediately that $|S_j \Delta S'_j| < \alpha n$ for every $j \in [r]$. Moreover, for each $v \in S'_j$ we have $j(v) = j$, and so there exists $b \in B$ such that v is a (2α) -clone of b with respect to S_j . But $|S_j \Delta S'_j| < \alpha n$, so v is a (3α) -clone of b with respect to S_j .

Thus P' is an α -adjustment of (G, P) with respect to B , as claimed. But P and B were chosen arbitrarily, so this contradicts the fact that $G \in \mathcal{D}(\mathcal{P}_n, \alpha)$. Thus $\mathcal{D}(\mathcal{P}_n, \alpha) \subset \mathcal{C}(\mathcal{P}_n, \alpha)$, as required. \square

Finally, for each graph G on vertex set $S_1 \cup \dots \cup S_r$, and each integer $k \in \mathbb{N}$, we choose a collection of vertex-disjoint copies of $U(t, k)$ for each $2 \leq t \leq r + 1$, using the following algorithm.

Algorithm. Set $\ell := 1$, $t := r + 1$ and $X = \emptyset$. Repeat the following steps until $t = 1$.

1. Suppose there exists a copy H of $U(t, k)$ in $G - X$, and a function $i : [t] \rightarrow [r]$ such that:
 - (a) $V(H) = A_1 \cup \dots \cup A_t$,
 - (b) $A_{j+1} \rightarrow A_1 \cup \dots \cup A_j$ for each $j \in [t - 1]$,
 - (c) $A_j \subset S_{i(j)} \setminus X$ for each $j \in [t]$, and
 - (d) $i(1) = i(2)$, and $i(j) = i(j') \Leftrightarrow j = j'$ for $j, j' \geq 2$.

Then set $U_\ell := V(H)$, $X := X \cup U_\ell$ and $\ell := \ell + 1$, and repeat Step 1.

2. Otherwise, set $t := t - 1$, and go to Step 1.

In other words, we first find a maximal collection of vertex-disjoint copies of $U(r + 1, k)$, such that for each copy, the smallest two classes are in the same set ($S_{i(1)} = S_{i(2)}$) as each other, and the other classes in different sets ($S_{i(3)}, \dots, S_{i(r+1)}$). We then find a maximal collection of vertex-disjoint copies of $U(r, k)$, which are also disjoint from each of the copies of $U(r + 1, k)$. We repeat this for each $2 \leq t \leq r + 1$, in decreasing order.

We obtain from the algorithm a collection $\{U_1, \dots, U_L\}$, where $G[U_\ell] = U(t, k)$ for some $2 \leq t \leq r + 1$, and the sets U_ℓ are pairwise disjoint. The following observation describes the key property of these sets.

Observation 22. Let U_1, \dots, U_L be the sets obtained from the algorithm applied (for k) to the graph G and partition (S_1, \dots, S_r) of $V(G)$. Then, for each $j \in [r]$ and $\ell \in [L]$, if

$U_\ell \cap S_j = \emptyset$ then the set $S_j \setminus \bigcup_{j=1}^{\ell} U_j$ does not shatter U_ℓ .

Proof. Suppose $G[U_\ell] = U(t, k)$, and $A_{t+1} \rightarrow U_\ell$ for some $A_{t+1} \subset S_j \setminus \bigcup_{j=1}^{\ell} U_j$. Then

$U_\ell \cup A_{t+1}$ induces a copy of $U(t+1, k)$, and so this set would have been chosen at an earlier step of the algorithm. \square

Now, given $k \in \mathbb{N}$, a graph G , a BBS-partition P of G for $(\varepsilon, \delta, \gamma)$, a maximal (2α) -bad set B for (G, P) , and an α -adjustment $P' = (S'_1, \dots, S'_r)$ of (G, P) with respect to B , let

$$U(G, P', k) := \bigcup_{\ell=1}^L U_\ell,$$

where $\{U_1, \dots, U_L\}$ are the sets given by the algorithm, applied to the graph G and the partition P' .

Given $n, k \in \mathbb{N}$, constants $\varepsilon, \delta, \gamma, \alpha > 0$, and a hereditary property of graphs \mathcal{P} , let

$$\mathcal{U}(\mathcal{P}_n, \alpha, k) := \left\{ G \in \mathcal{P}_n : \exists \text{ a BBS-partition } P \text{ of } G \text{ for } (\varepsilon, \delta, \gamma), \text{ a maximal } (2\alpha)\text{-bad set } B \text{ for } (G, P), \text{ and an } \alpha\text{-adjustment } P' = (S'_1, \dots, S'_r) \text{ of } (G, P) \text{ with respect to } B \text{ with } |U(G, P', k)| \geq n^{1-\alpha} \right\}.$$

Theorem 1 is an easy corollary of the following lemma, together with Lemmas 16, 20 and 21. The proof of the lemma uses Theorem 2, and Lemmas 7, 15, 17, 18, 20 and 21.

Lemma 23. Let $r \in \mathbb{N}$, and let \mathcal{P} be a hereditary property of graphs with $\chi_c(\mathcal{P}) = r$. There exist $k = k(\mathcal{P}) \in \mathbb{N}$ and $\alpha = \alpha(k, \mathcal{P}) > 0$ such that the following holds. Let $\varepsilon, \delta, \gamma > 0$ be sufficiently small and $n \in \mathbb{N}$ be sufficiently large. Then

$$|\mathcal{U}(\mathcal{P}_n, \alpha, k)| \leq 2^{(1-1/r)n^2/2 - \alpha^2 n^2 - \alpha} = o(|\mathcal{P}_n|).$$

Proof. Let k be sufficiently large so that G is $U(r+1, k)$ -free for every $G \in \mathcal{P}$. Such a $k = k(\mathcal{P})$ exists by Lemma 7.

We simply count the graphs in

$$\mathcal{U}_n := \mathcal{U}(\mathcal{P}_n, \alpha, k) \setminus \left(\mathcal{B}(\mathcal{P}_n, \alpha, n^{1-2\alpha}) \cup \mathcal{D}(\mathcal{P}_n, \alpha) \right).$$

By Lemmas 17 and 18 we have

$$|\mathcal{B}(\mathcal{P}_n, \alpha, c)| \leq 2^{(1-1/r)n^2/2 - \alpha' n^2}$$

for some $\alpha' = \alpha'(\mathcal{P}, \alpha)$, if $c = c(\mathcal{P}, \alpha)$ is sufficiently large. (Recall that $|\mathcal{B}(\mathcal{P}_n, \alpha, c)|$ is monotone decreasing in c .) Also, by Lemmas 20 and 21 we have

$$|\mathcal{D}(\mathcal{P}_n, \alpha)| \leq 2^{(1-1/r)n^2/2 - \alpha'' n^2},$$

where $\alpha'' = \alpha/3r^3$. Thus it suffices to prove the claimed bound for the set \mathcal{U}_n .

So let $G \in \mathcal{U}_n$, and note that G has

- (a) a BBS-partition P for $(\varepsilon, \delta, \gamma)$,
- (b) a maximal set $B \subset V(G)$, which is (2α) -bad for (G, P) , with $|B| \leq n^{1-2\alpha}$,
- (c) an α -adjustment $P' = (S'_1, \dots, S'_r)$ of (G, P) with respect to B , such that

$$|U(G, P', k)| \geq n^{1-\alpha}.$$

Let U_1, \dots, U_L denote the sets given by the algorithm, applied (for k) to the partition P' of G . By definition,

$$U(G, P', k) = \bigcup_j U_j.$$

We have at most n^n choices for the partition P' , and the sets B and U_1, \dots, U_L . Now, given an edge $e = ab$, define the index $i(e)$ as follows:

- (a) If e has an endpoint in B then $i(e) = 0$.
- (b) If e has an endpoint in U_ℓ and the other endpoint is in

$$V(G) \setminus \left(B \cup \bigcup_{j=1}^{\ell-1} U_j \right)$$

then $i(e) = \ell$.

- (c) If e has both endpoints in $V(G) \setminus \left(B \cup U(G, P', k) \right)$ then $i(e) = \infty$.

We choose the edges of G in increasing order of index.

First, since $|B| \leq n^{1-2\alpha}$, we have at most $2^{|B|n} \leq 2^{n^{2-2\alpha}}$ choices for the edges incident with B . So let $1 \leq \ell \leq L$, and suppose that $G[U_\ell]$ is a copy of $U(t, k)$, where $2 \leq t \leq r$. (Note that $t \neq r+1$, since G is $U(r+1, k)$ -free by our choice of k .)

Claim: There is a constant $\lambda > 0$, depending only on k and r , such that we have at most

$$2^{(1-1/r-\lambda)|U_\ell|n}$$

choices for the edges with index ℓ .

Proof. Without loss of generality, let $U_\ell = A_1 \cup \dots \cup A_t$, where $A_1, A_2 \subset S'_2$, $A_j \subset S'_j$ for each $3 \leq j \leq t$, and

$$A_{j+1} \rightarrow A_1 \cup \dots \cup A_j$$

for each $1 \leq j \leq t-1$. Recall that, by the definition of an α -adjustment, each vertex $u \in S'_j$ is a (3α) -clone of b with respect to S'_j , for some $b \in B$. Note also that $(1/r - 2\alpha)n \leq |S'_j| \leq (1/r + 2\alpha)n$ for each $j \in [r]$.

Thus, for each $u \in U_\ell$, we have at most

$$2^{(1-2/r+2\sqrt{\alpha})n}$$

choices for the edges between u and $V(G) \setminus S'_1$, by Observation 19, since $\alpha > 0$ is sufficiently small. But S'_1 does not shatter U_ℓ , by Observation 22, and so, by Lemma 15, we have at most

$$2^{|U_\ell|n/r - \lambda_1 n}$$

choices for the edges between U_ℓ and S'_1 , where $\lambda_1 = \lambda_1(k, r) > 0$ is the constant in Lemma 15. Choosing $\alpha = \alpha(\mathcal{P}, k)$ sufficiently small, the result follows. \square

Now, let $T_j = S'_j \setminus U(G, P', k)$ for each $j \in [r]$, and note that, since the algorithm stopped, T_j is $U(k)$ -free. Thus, by Theorem 2, we have at most $2^{n^{2-2\alpha}}$ choices for the edges inside these sets. Also, trivially, we have at most

$$2^{(1-1/r)(n-U(G, P', k))^2/2}$$

choices for the edges between the sets T_j .

Multiplying the number of choices, we get

$$\begin{aligned} \log_2(|\mathcal{U}_n|) &\leq \left(1 - \frac{1}{r}\right) \frac{(n - |U(G, P', k)|)^2}{2} + \left(1 - \frac{1}{r} - \lambda\right) |U(G, P', k)|n + O(n^{2-2\alpha}) \\ &\leq \left(1 - \frac{1}{r}\right) \frac{n^2}{2} - \lambda |U(G, P', k)|n + \frac{|U(G, P', k)|^2}{2} + O(n^{2-2\alpha}). \end{aligned}$$

Since $|U(G, P', k)| \geq n^{1-\alpha}$, the result follows if $|U(G, P', k)| \leq \lambda n$.

Finally, suppose that $|U(G, P', k)| \geq \lambda n$. Then, by the pigeonhole principle and without loss of generality, there exists $2 \leq t \leq r$ and a subset $X \subset [L]$ such that, for each $j \in X$, $U_j = A_1 \cup \dots \cup A_t$, where $A_1, A_2 \subset S'_2$, $A_j \subset S'_j$ for each $3 \leq j \leq t$, and

$$A_{j+1} \rightarrow A_1 \cup \dots \cup A_j$$

for each $1 \leq j \leq t-1$, and $|X| \geq \lambda_2 n$, where λ_2 depends only on k and r .

We have at most n^n choices for the partitions (S_1, \dots, S_r) and (S'_1, \dots, S'_r) , and at most $2^{(\varepsilon + \sqrt{\delta} + \gamma)n^2}$ choices for the edges inside the sets S_j . We have at most $2^{\alpha n^2}$ choices for the edges incident with vertices in $\bigcup_j S_j \triangle S'_j$, and at most $2^{\binom{r-1}{2}n^2/r^2}$ choices for the edges between S'_i and S'_j for $i, j \neq 1$. Finally, we have at most $2^{n/r(n-n/r) - \lambda_3 n^2}$ choices for the edges incident with S'_1 , by Lemma 15.

Thus, choosing α sufficiently small, we obtain

$$\log_2(|\mathcal{U}_n|) \leq \left(\frac{(r-1)(r-2)}{2r^2} + \frac{r-1}{r^2}\right) n^2 - \lambda_4 n^2 = \left(1 - \frac{1}{r} - \lambda_4\right) n^2,$$

for some $\lambda_4 > 0$, as required. \square

Remark 1. Note that we in fact only needed $B(G, P, \alpha) \leq n^{1-2\alpha}$ for almost every graph $G \in \mathcal{P}$.

The proof of Theorem 1 now follows easily.

Proof of Theorem 1. Let \mathcal{P} be a hereditary property of graphs with $\chi_c(\mathcal{P}) = r$, and let $k = k(\mathcal{P}) \in \mathbb{N}$ be sufficiently large, and $\alpha = \alpha(\mathcal{P}, k)$, $\gamma = \gamma(\mathcal{P}, \alpha, k)$, $\delta = \delta(\mathcal{P}, \gamma, \alpha, k) > 0$ and $\varepsilon = \varepsilon(\mathcal{P}, \gamma, \delta, \alpha, k) > 0$ be sufficiently small. By Lemma 16, almost every graph $G \in \mathcal{P}$ has a BBS-partition for $(\varepsilon, \delta, \gamma)$. So let $G \in \mathcal{P}$, let P be a BBS-partition of G for $(\varepsilon, \delta, \gamma)$, and let B be a maximal (2α) -bad set for (G, P) .

Now, by Lemmas 20 and 21, for almost every such G there exists an α -adjustment $P' = (S'_1, \dots, S'_r)$ of (G, P) with respect to B . Let $U(G, P', k)$ denote the set given by the algorithm. By Lemma 23, $U(G, P', k) \leq n^{1-\alpha}$ for almost every such G .

Let $A = B \cup U(G, P', k)$, and let $S_j := S'_j \setminus A$ for each $j \in [r]$. Then S_j is $U(k)$ -free for each $j \in [r]$, and Theorem 1 follows. \square

8. A SHARPER BOUND FOR BIPARTITE GRAPHS

In this section we refine the methods of the proof of Theorem 2, giving a close to sharp upper bound on

$$f(n, n, U(k)) := |\{G \text{ bipartite on } A \cup B : |A| = |B| = n \text{ and } G \text{ is } U(k)\text{-free}\}|.$$

The exponent in our bound will be within a polylog-factor of the best that we could hope for, i.e., the exponent in the extremal result of Alon, Krivelevich and Sudakov [3] (see the more general Theorem 6.1 in their paper), that a $U(k)$ -free graph with class sizes n cannot have more than $O(n^{2-1/(k-1)})$ edges.

We remark that to remove all of the seemingly unnecessary log-factors looks extremely hard. Corresponding results are known only for the monotone case, and even then only in special cases (for C_4 by Kleitman and Winston [25], for C_6 and C_8 by Kleitman and Wilson [24], for $\{C_4, C_6, \dots, C_{2k}\}$ by Kohayakawa, Kreuger and Steger [26], and for $K_{s,t}$ by Balogh and Samotij [13, 14]).

Theorem 24. *For every $3 \leq k \in \mathbb{N}$, and every sufficiently large n , we have*

$$f(n, n, U(k)) \leq \exp\left(n^{2-1/(k-1)}(\log n)^{k+1}\right).$$

We first prove the following lemma, the proof of which uses the methods of Section 6. Let G be a $U(k)$ -free bipartite graph with classes U and V where $|U| = |V| = n$. For each $u, v \in U$, we define $\Delta(u, v) := |\Gamma(u) \Delta \Gamma(v)|$, the ‘distance’ between the two vertices.

Lemma 25. *Let $m, n, x, k \in \mathbb{N}$, and let G be a $U(k)$ -free bipartite graph with classes U and V , where $|U| = m$ and $|V| = n$. Let $U' \subset U$, and suppose that, for any $u, v \in U'$, we have $\Delta(u, v) \geq x$. Then*

$$|U'| \leq \left(\frac{n}{x}\right)^{k-1} 3^k (\log m)^{k-1}.$$

Proof. The proof is very similar to that of Lemma 10. Indeed, let $c = |U'|$, and with foresight, let $p = \frac{3 \log c}{x}$. We claim that there exists a subset $X \subset V$, with $|X| = pn$, such that

$$\Gamma(u) \cap X \neq \Gamma(v) \cap X$$

for each $u, v \in U'$ with $u \neq v$. Indeed, if we choose a random subset $X \subset V$ of size pn , then

$$\mathbb{P}\left(\Gamma(x) \cap X = \Gamma(y) \cap X\right) \leq \binom{n-x}{pn} / \binom{n}{pn} \leq \left(1 - \frac{x}{n}\right)^{pn}$$

for each such pair $\{x, y\}$, and so

$$\mathbb{E} \left| \left\{ \{x, y\} \in \binom{U'}{2} : \Gamma(x) \cap X = \Gamma(y) \cap X \right\} \right| \leq \binom{c}{2} \left(1 - \frac{x}{n}\right)^{pn} \leq c^2 e^{-px} < 1.$$

Thus such a set X must exist, as claimed. Now, if

$$c > 2 \left(\frac{3n \log c}{x} \right)^{k-1} = 2(pn)^{k-1} \geq \sum_{i=0}^{k-1} \binom{|X|}{i},$$

then, by Sauer's Lemma, there exist sets $U'' \subset U'$ and $X' \subset X$, with $|X'| = k$, such that $U'' \rightarrow X'$. But this is a contradiction, since G is $U(k)$ -free. Thus

$$|U'| = c \leq 2 \left(\frac{3n \log c}{x} \right)^{k-1} \leq 3^k \left(\frac{n}{x} \right)^{k-1} (\log m)^{k-1}$$

as required. \square

We are now ready to prove Theorem 24.

Proof of Theorem 24. Let $|U| = |V| = n$, and suppose that G is a bipartite $U(k)$ -free graph with classes U and V . We are required to show that the number of choices for the edge set of G is at most $\exp \left(n^{2-1/(k-1)} (\log n)^{k+1} \right)$.

The idea is to partition U into $t+2$ parts, and consider the edges from each part to V in turn. Indeed, let $t \in \mathbb{N}$, let $n > x_0 > \dots > x_t > n^{1-1/(k-1)}$, and let

$$U_0 \subset \dots \subset U_t \subset U$$

be maximal subsets satisfying $\Delta(u, v) \geq x_i$ for each $u, v \in U_i$ and $0 \leq i \leq t$. Moreover, and with foresight, let $t = 10 \log \log n$, and let

$$x_i := n^{1-1/(k-1)+1/(k-1)^{i+2}}$$

for each $0 \leq i \leq t$. Note that we have at most 2^n choices for the sets U_i .

Now, by Lemma 25,

$$|U_i| \leq \left(\frac{n}{x_i} \right)^{k-1} 3^k (\log n)^{k-1}$$

for each $0 \leq i \leq t$, so we have at most

$$\exp \left(n \left(\frac{n}{x_0} \right)^{k-1} 3^k (\log n)^{k-1} \right)$$

choices for the graph $G[U_0, V]$. Now let $0 \leq i \leq t-1$, and assume that the sets U_i and U_{i+1} and the graph $G[U_i, V]$ have already been chosen.

Claim: We have at most

$$\exp \left(x_i \left(\frac{n}{x_{i+1}} \right)^{k-1} 3^{k+1} (\log n)^k \right)$$

choices for the edges between $U_{i+1} \setminus U_i$ and V .

Proof of claim. Since U_i is maximal, for every $v \in U_{i+1} \setminus U_i$ there is a $u \in U_i$ such that $\Delta(u, v) < x_i$. Thus, the number of choices for the edges between v and V is at most

$$2|U_i| \binom{n}{x_i},$$

and so the number of choices for the graph $G[U_{i+1} \setminus U_i, V]$ is at most

$$\left(2|U_i| \binom{n}{x_i}\right)^{|U_{i+1}|} \leq (n^{x_i+1})^{(n/x_{i+1})^{k-1} 3^k (\log n)^{k-1}} \leq \exp\left(x_i \left(\frac{n}{x_{i+1}}\right)^{k-1} 3^{k+1} (\log n)^k\right),$$

as claimed. \square

Finally, given U_t and $G[U_t, V]$, the number of choices for the edge between $U \setminus U_t$ and V is at most

$$\left[2|U_t| \binom{n}{x_t}\right]^n \leq \exp\left((x_t + 1)n \log n\right) \leq \exp\left(2n^{2-1/(k-1)} \log n\right).$$

since for each u there is a vertex $v \in U_t$ such that $\Delta(u, v) \leq x_t$. There are at most n choices for v and $2 \binom{n}{x_t}$ choices for the symmetric difference.

Putting these bounds together, the number of choices for the graph $G[U, V]$ is at most

$$\exp\left(n \left(\frac{n}{x_0}\right)^{k-1} 3^k (\log n)^{k-1} + 2n^{2-1/(k-1)} \log n + \sum_{i=0}^{t-1} x_i \left(\frac{n}{x_{i+1}}\right)^{k-1} 3^{k+1} (\log n)^k\right).$$

But, recalling that $x_i = n^{1-1/(k-1)+1/(k-1)^{i+2}}$, we have

$$n \left(\frac{n}{x_0}\right)^{k-1} = x_i \left(\frac{n}{x_{i+1}}\right)^{k-1} = n^{2-1/(k-1)}$$

for each $0 \leq i \leq t-1$. Since $t = O(\log \log n)$, this gives an upper bound of

$$\exp\left(Cn^{2-1/(k-1)} (\log n)^k \log \log n\right),$$

as required. \square

We cannot hope to obtain very sharp results from such a (relatively) simple application of Sauer's Lemma. However, our results are close to optimal, if we do not care about poly-log factors in the exponent. For example, Theorem 24 for $k=3$ gives an $n^{3/2}$ in the exponent, and $3/2$ is best possible, as $U(3)$ contains a C_4 . In general, $U(k)$ contains a graph with average degree at least $2(k-3)$, and so we have the bounds

$$\exp\left(n^{2-1/(k-3)}\right) \leq f(n, n, U(k)) \leq \exp\left(n^{2-1/(k-1)} (\log n)^{k+1}\right).$$

It would be interesting to eliminate (if possible) the $\log n$ -factors from the exponent; this would yield Kleitman-Winston-type results for many different bipartite graphs.

9. QUESTIONS

The most obvious disadvantage of Theorem 1 is that we know almost nothing about the structure of a typical $U(k)$ -free graph.

Question 1. *What is the structure of a typical $U(k)$ -free graph? In particular, are almost all $U(k)$ -free graphs either dense or sparse?*

As we remarked in the Introduction, there has recently been some important progress on hereditary properties of hypergraphs. In particular, we noted the following theorems of Dotson and Nagle [18], Ishigami [22] and Person and Schacht [32].

Given a hereditary property of k -uniform hypergraphs \mathcal{P} , the extremal number of \mathcal{P} is defined to be

$$\text{ex}(n, \mathcal{P}) := \max \left\{ |\mathcal{A}| : \mathcal{A} \subset \binom{[n]}{k}, \text{ and there exists } \mathcal{M} \subset \binom{[n]}{k} \setminus \mathcal{H} \text{ such that} \right. \\ \left. \mathcal{M} \cup \mathcal{A}' \in \mathcal{P} \text{ for every } \mathcal{A}' \subset \mathcal{A} \right\}.$$

In other words, it is the maximum dimension of a subspace of \mathcal{P}_n , in the product space $\{0, 1\}^{\binom{[n]}{k}}$.

Theorem 26 (Dotson and Nagle [18], Ishigami [22]). *Let $k \in \mathbb{N}$ and let \mathcal{P} be a hereditary property of k -uniform hypergraphs. Then*

$$|\mathcal{P}_n| = 2^{\text{ex}(n, \mathcal{P}) + o(n^k)}.$$

Theorem 27 (Person and Schacht [32]). *Almost every Fano-plane-free 3-uniform hypergraph is bipartite.*

These results give reason to be optimistic that the following question, which until recently would have seemed very far out of reach, may now be approachable.

Question 2. *What is the structure of a typical member of a hereditary property of k -uniform hypergraphs?*

Finally, we note that Theorem 1 is considerably weaker than Conjecture 1, since the set A can be very large, and because our bounds on k are likely far from best possible. The following problem asks for some progress towards the conjecture.

Problem 1. *In the statement of Theorem 1, improve the upper bound on $|A|$, and give good bounds on the constant $k(\mathcal{P})$.*

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