The number of the maximal triangle-free graphs

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Abstract

Paul Erdős suggested the following problem: Determine or estimate the number of maximal triangle-free graphs on n vertices. Here we show that the number of maximal triangle-free graphs is at most $2^{n^2/8+o(n^2)}$, which matches the previously known lower bound. Our proof uses among others the Ruzsa-Szemerédi triangle removal lemma, and recent results on characterizing of the structure of independent sets in hypergraphs.

The maximum triangle-free graph has $n^2/4$ edges [8]. Hence, the number of triangle-free graphs is at least $2^{n^2/4}$, which was shown to be the correct order of magnitude by Erdős, Kleitman and Rothschild [5] (see Balogh-Morris-Samotij [2] or Saxton-Thomason [12] for recent proofs). Moreover, almost every triangle-free graph is bipartite [5], even if there is a restriction on the number of edges (first shown by Osthus-Prömel-Taraz [10], extended by Balogh-Morris-Samotij-Warnke [3]; see [2] and [3] for a more detailed history of the problem). This suggests that most of those graphs are bipartite, and subgraphs of a complete bipartite graph, therefore most of them are not maximal. Erdős suggested the following problem (as stated in Simonovits [13]):

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Problem. *Determine or estimate the number of maximal triangle-free graphs on n vertices.*

The following folklore construction (see [9], but it was known much earlier) shows that $f(n) \geq 2^{n^2/8+o(n^2)}$. Let H be a graph on a vertex set $X \cup Y$ with $|X| = |Y| = n/2$ such that X induces a perfect matching, Y is an independent set, and there are no edges between X and Y. For each pair of a matching edge $x_1x_2 \in E(H[X])$ and a vertex $y \in Y$, we add one of the edges x_1y or x_2y to H. Since there are $n/4$ matching edges in $E(H[X])$ and $n/2$ vertices in Y, we obtain $2^{n^2/8}$ triangle-free graphs, most of which are maximal.

In this note we prove a matching upper bound.

Theorem 1. *The number of maximal triangle-free graphs with vertex set $[n]$ is at most $2^{n^2/8+o(n^2)}$.*

Our first tool is a corollary of recent powerful counting theorems of Balogh-Morris-Samotij [2, Theorem 2.2.], and Saxton-Thomason [12].

Theorem 2. *For each $\delta > 0$ there is $t < 2^{O(\log n \cdot n^{3/2})}$ and a set $\{G_1, \dots, G_t\}$ of graphs, each containing at most δn^3 triangles, such that for every triangle-free graph H there is $i \in [t]$ such that $H \subseteq G_i$, where n is sufficiently large.*

By the Erdős-Simonovits supersaturation theorem [6], each G_i has at most about $n^2/4$ edges.

Theorem 3. *For every $\gamma > 0$ there is $\delta(\gamma) > 0$ such that every n-vertex graph with $(1/4 + \gamma)n^2$ edges contains as least $\delta(\gamma)n^3$ triangles.*

We also use the Ruzsa-Szemerédi triangle-removal lemma [11].

Theorem 4. *For every $\varepsilon > 0$ there is $\delta(\varepsilon) > 0$ such that any graph G on n vertices with at most $\delta(\varepsilon)n^3$ triangles can be made triangle-free by removing at most εn^2 edges.*

Our next tool is the following theorem of Hujter and Tuza [7]. Recall that a set $I \in V(G)$ is an *independent set* if no two vertices in I are adjacent. An independent set I is a *maximal independent set* if $I \cup \{v\}$ contains an edge for every $v \in V(G) - I$. Note that we write $|G|$ for the number of vertices of G.

Theorem 5. *Every triangle-free graph G has at most $2^{|G|/2}$ maximal independent sets.*

In the next section we prove our main result, Theorem 1.

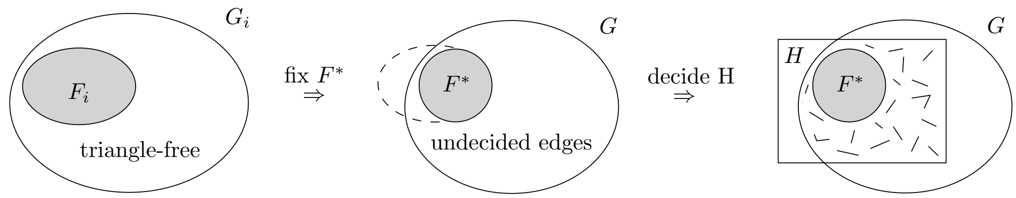

Figure 1: The structure of G_i and F_i.

1 Proof of Theorem 1

We show that for every $\varepsilon > 0$ and for every $\gamma > 0$, the number of maximal triangle-free graphs with vertex set $[n]$ is $2^{(1/8+2\varepsilon+\gamma)n^2}$ for sufficiently large n. We fix arbitrarily small constants $\varepsilon, \gamma > 0$. First we apply Theorem 4 with this ε, and Theorem 3 with this γ, which provides us $\delta(\varepsilon)$ and $\delta(\gamma)$. We define $\delta = \min\{\delta(\varepsilon), \delta(\gamma)\}$ and then apply Theorem 2 with this δ. For every $i \in [t]$, we count the number of maximal triangle-free graphs H that satisfy $H \subseteq G_i$. Denote $\mathcal{H}$ the set of maximal triangle-free graphs with vertex set $[n]$, and let $\mathcal{H}_i = \{H \in \mathcal{H} : H \subseteq G_i\}$.

Since $t \leq 2^{\varepsilon n^2}$ for sufficiently large n, we have

$$|\mathcal{H}| \leq \sum_{i=1}^{t} |\mathcal{H}_i| \leq 2^{\varepsilon n^2} \max_{i\in[t]} |\mathcal{H}_i|.$$

Fix an arbitrary $i \in [t]$. By Theorem 4 applied on G_i, there is $F_i \subseteq E(G_i)$ such that $|F_i| \leq \varepsilon n^2$ and $G_i - F_i$ is triangle-free. For each G_i we fix one such F_i. For every $F^* \subseteq F_i$ define $\mathcal{H}_i(F^*) = \{H \in \mathcal{H}_i : E(H) \cap F_i = F^*\}$.

Now we show that for every choice of F^* we have $|\mathcal{H}_i(F^*)| \leq 2^{e(G_i)/2}$. Fix F^*, and let

$$G := G_i - (F_i - F^*) - \{e \in E(G_i) : \exists f, g \in F^* \text{ such that } e, f, g \text{ form a triangle}\}.$$

So, G is obtained from G_i by removing edges that are in none of $H \in \mathcal{H}_i(F^*)$. We can assume that F^* is triangle-free since otherwise $\mathcal{H}_i(F^*) = \emptyset$. We now count the number of ways to add edges of $E(G) - F^*$ to F^* such that the resulting graph is maximal triangle-free. We construct an auxiliary graph T as follows:

$$V(T) := E(G) - F^* \quad \text{and} \quad E(T) := \{ef \mid \exists d \in F^*\colon \{d, e, f\} \text{ spans a triangle in } G\}.$$

Claim 1. T is triangle-free.

Proof. Suppose not. Let e, f, g be vertices of a triangle in T. Then $e, f, g \in E(G) - F^*$ and there are $d_1, d_2, d_3 \in F^*$ such that the 3-sets $\{d_1, e, f\}$, $\{d_2, e, g\}$, and $\{d_3, f, g\}$ span triangles in G. As $G_i - F_i$ is triangle-free and $G - F^* \subseteq G_i - F_i$, it follows that the edges e, f, g share a common endpoint in G, and that $\{d_1, d_2, d_3\}$ spans a triangle. This is a contradiction since F^* is triangle-free. □

Claim 2. If $H \in \mathcal{H}_i(F^*)$, then $E(H) - F^*$ spans a maximal independent set in T.

Proof. Let $H \in \mathcal{H}_i(F^*)$. We first show that $E(H) - F^*$ spans an independent set in T. If not, then there is an edge ef in $E(T)$ with $e, f \in E(H) - F^*$. By the definition of $E(T)$, there is $d \in F^*$ such that the edges d, e, f form a triangle in G, which is clearly in H.

Suppose now that $E(H) - F^*$ is an independent set in T that is not maximal. So, there is $x \in E(G) - E(H)$ such that for every $y \in E(H) - F^*$ and for every $z \in F^*$, the edges x, y, z do not span a triangle in G. This means that $H \cup \{x\}$ is triangle-free. Hence, H is not maximal. □

By Theorem 5, the number of maximal independent sets in T is at most $2^{|T|/2}$. Thus

$$|\mathcal{H}_i(F^*)| \leq 2^{|T|/2} \leq 2^{e(G_i)/2} \leq 2^{(n^2/4+\gamma)/2},$$

where the last inequality follows from Theorem 3.

The number of ways to choose $F^* \subseteq F_i$ for a given F_i is at most $2^{|F_i|} \leq 2^{\varepsilon n^2}$, so we can conclude that for sufficiently large n,

$$|\mathcal{H}| \leq 2^{\varepsilon n^2} \max_{i \in [t]} |\mathcal{H}_i| \leq 2^{\varepsilon n^2} \sum_{F^* \subseteq F_i} |\mathcal{H}_i(F^*)| \leq 2^{\varepsilon n^2} 2^{\varepsilon n^2} \max_{F^* \subseteq F} |\mathcal{H}_i(F^*)|$$

$$\leq 2^{2\varepsilon n^2} 2^{(n^2/4+\gamma n^2)/2} \leq 2^{(1/8+2\varepsilon+\gamma)n^2}.$$

2 Concluding remarks

It would be interesting to have similar results for K_{r+1} as well. A straightforward generalization of the construction for maximal triangle-free graphs implies that there are at least $r^{n^2/(4r^2)}$ maximal K_{r+1}-free graphs. Unfortunately, not all steps of our upper bound method work when $r > 2$. We are able to get only the following modest improvement on the trivial $2^{(1-1/r+o(1))n^2/2}$ bound: for every r there is a positive constant c_r such that the number of maximal K_{r+1}-free graphs is at most $2^{(1-1/r-c_r)n^2/2}$ for n sufficiently large. More precisely, if we let $s = 2^{\binom{r+1}{2}-1}$, then the number of maximal K_{r+1}-free graphs is at most $(s-1)^{n^2/(r(r+2))+o(n^2)}$.

A similar question was raised by Cameron and Erdős in [4], where they asked how many maximal sum-free sets are contained in $[n]$. They were able to construct $2^{n/4}$ such sets. An upper bound $2^{3n/8+o(n)}$ was proved by Wolfovitz [14]. Our proof method instantly improves this upper bound to $3^{n/6+o(n)}$, as observed in [1]. Balogh-Liu-Sharifzadeh-Treglown [1] pushed the method further to prove a matching upper bound, $2^{n/4+o(n)}$. As [1] contains all the details, we omit further discussion here.

Recent development

Remark 6. Alon pointed out that if the number of K_r-free graphs is $2^{c_r n^2+o(n^2)}$, then c_r is monotone (though not clear if strictly monotone) increasing in r.

Remark 7. A discussion with Alon and Łuczak at IMA led to the following construction that gives $2^{(1-1/r)n^2/4+o(n^2)}$ maximal K_{r+1}-free graphs: partition the vertex set $[n]$ into r equal classes, place a perfect matching into $r-1$ of them. Between the classes we have the following connection rule: between two matching edges place exactly 3 edges, and between a vertex (from the class which is an independent set) and a matching edge put exactly 1 edge.

Remark 8. In yet to be published work, Balogh, Liu, Petříčková, and Sharifzadeh proved that almost every maximal triangle-free graph G admits a vertex partition $X \cup Y$ such that $G[X]$ is a perfect matching and Y is an independent set, as in the construction.

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References

[1] J. Balogh, H. Liu, M. Sharifzadeh, and A. Treglown, *The number of maximal sum-free subsets of integers*, accepted to Proceedings of AMS, arXiv:1409.5661.

[2] J. Balogh, R. Morris, and W. Samotij, *Independent sets in hypergraphs*, to appear in JAMS, arXiv:1204.6530.

[3] J. Balogh, R. Morris, W. Samotij, and L. Warnke, *The typical structure of sparse K_{r+1}-free graphs*, accepted to Transactions of AMS, arXiv:1307.5967.

[4] P. J. Cameron and P. Erdős, *Notes on sum-free and related sets*, Recent trends in combinatorics (Mátraháza, 1995), Combin. Probab. Comput. 8(1–2) (1999) 95–107.

[5] P. Erdős, D. J. Kleitman and B. L. Rothschild, *Asymptotic enumeration of K_n-free graphs*, Colloquio Internazionale sulle Teorie Combinatorie (Rome, 1973), Tomo II, Atti dei Convegni Lincei 17, Accad. Naz. Lincei (1976) 19–27.

[6] P. Erdős and M. Simonovits, *Supersaturated graphs and hypergraphs*, Combinatorica 3 (1983) 181–192.

[7] M. Hujter and Z. Tuza, *The Number of Maximal Independent Sets in Triangle-Free Graphs*, SIAM J. Discrete Math. 6(2) (1993) 284–288.

[8] W. Mantel, *Problem 28*, Wiskundige Opgaven 10 (1907) 60–61.

[9] http://mathoverflow.net/q/160595.

[10] D. Osthus, H.J. Prömel and A. Taraz, *For which densities are random triangle-free graphs almost surely bipartite?*, Combinatorica 23 (2003) 105–150.

[11] I. Z. Ruzsa and E. Szemerédi, *Triple systems with no six points carrying three triangles*, Combinatorics (Keszthely, 1976), Coll. Math. Soc. J. Bolyai 18, Vol. II, (1978) 939–945.

[12] D. Saxton and A. Thomason, *Hypergraph containers*, arXiv:1204.6595.

[13] M. Simonovits, *Paul Erdős' influence on extremal graph theory*, The Mathematics of Paul Erdős II (Springer-Verlag, Berlin, 1996) 148–192.

[14] G. Wolfovitz, *Bounds on the number of maximal sum-free sets*, European J. Combin. 30 (2009) 1718–1723.