ON THE TREE PACKING CONJECTURE

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Abstract. The Gyárfás tree packing conjecture states that any set of $n-1$ trees $T_1, T_2, \ldots, T_{n-1}$ such that T_i has $n-i+1$ vertices pack into K_n (for n large enough). We show that $t = \frac{1}{10} n^{1/4}$ trees $T_1, T_2, \ldots, T_t$ such that T_i has $n-i+1$ vertices pack into K_{n+1} (for n large enough). We also prove that any set of $t = \frac{1}{10} n^{1/4}$ trees $T_1, T_2, \ldots, T_t$ such that no tree is a star and T_i has $n-i+1$ vertices pack into K_n (for n large enough). Finally, we prove that $t = \frac{1}{4} n^{1/3}$ trees $T_1, T_2, \ldots, T_t$ such that T_i has $n-i+1$ vertices pack into K_n as long as each tree has maximum degree at least $2n^{2/3}$ (for n large enough). One of the main tools used in the paper is the famous spanning tree embedding theorem of Komlós, Sárközy and Szemerédi [15].

Key words. tree packing, packing, tree

AMS subject classifications. 05C05, 05C70

1. Introduction. A set of (simple) graphs $G_1, G_2, \ldots, G_n$ are said to *pack* into a graph H if $G_1, G_2, \ldots, G_n$ can be found as pairwise edge-disjoint subgraphs in H. In this paper we are concerned with the case when each G_i is a tree and H is a complete graph on n vertices, denoted by K_n. The famous tree packing conjecture (TPC) posed by Gyárfás (see [11]) states:

CONJECTURE 1. *Any set of $n-1$ trees $T_n, T_{n-1}, \ldots, T_2$ such that T_i has i vertices pack into K_n.*

Bollobás suggested a weakening of TPC in the Handbook of Combinatorics [10]:

CONJECTURE 2. *For every $k \geq 1$ there is an $n(k)$ such that if $n \geq n(k)$, then any set of k trees $T_1, T_2, \ldots, T_k$ such that T_i has $n-i+1$ vertices pack into K_n.*

A number of partial results concerning the TPC are known. The first results are by Gyárfás and Lehel [11] who proved that the TPC holds with the additional assumption that all but two of the trees are stars. Roditty [16] confirmed TPC in the case when all but three trees are stars (see also [9]). Gyárfás and Lehel [11] also showed that the TPC is true if each tree is either a path or a star. A second proof of this statement is by Zaks and Liu [18][1]. Bollobás [1] showed that any set of $k-1$ trees $T_k, T_{k-1}, \ldots, T_2$ such that T_i has i vertices pack into K_n if $k \leq \frac{\sqrt{2}}{2} n$. Bollobás also noted that the bound on k can be increased to $\frac{\sqrt{3}}{2} n$ if we assume the Erdős-Sós conjecture is true (see [8]). Packing many large trees seems to be difficult. Hobbs, Bourgeois and Kasiraj [13] showed that any three trees T_n, T_{n-1}, T_{n-2} such that T_i has i vertices pack into K_n. A series of papers by Dobson [5, 6, 7] concerns packing trees into K_n with restrictions on the structure of each tree.

Instead of packing trees into the complete graph, a number of papers have examined packing trees into complete bipartite graphs. Hobbs, Bourgeois and Kasiraj [13] conjectured that $n-1$ trees $T_n, T_{n-1}, \ldots, T_2$ such that T_i has i vertices pack into the complete bipartite graph $K_{n-1, \lfloor n/2 \rfloor}$. The conjecture is true if each of the trees is a star or path (see Zaks and Liu [18] and Hobbs [12]). Yuster [17] showed that $k-1$ trees $T_k, T_{k-1}, \ldots, T_2$ such that T_i has i vertices pack into $K_{n-1, \lfloor n/2 \rfloor}$ if $k \leq \lfloor \sqrt{5/8n} \rfloor$

[1]An incorrect version of this proof appears in [14].

(improving the previously best-known bound on k by Caro and Roditty [4]). Various generalizations of the tree packing conjecture were investigated by Gerbner, Keszegh and Palmer [9]. Recently, Böttcher, Hladký, Piguet and Taraz [3] proved an asymptotic version of the tree packing conjecture for trees with bounded maximum degree.

Notation will be standard (following e.g. [2]). A vertex of a graph of degree 1 is a *leaf*. A set of leaves in a graph are *independent* if the neighbors of the leaves are pairwise disjoint (i.e. the edges incident to a set of independent leaves form a matching). A *leaf edge* is an edge incident to a vertex of degree 1. The *neighborhood* of a set of vertices X is the set of vertices not in X with a neighbor in X (i.e. neighborhoods are not considered closed). The maximum degree of a graph G is denoted by $\Delta(G)$; the minimum degree by $\delta(G)$. For the sake of brevity, the set of vertices of a graph G will also be denoted by G.

The set of the first k natural numbers is denoted by $[k]$. We write $A - B$ for the difference of two sets A and B.

Clearly a set of graphs $G_1, G_2, \ldots, G_k$ pack into H if there is a k-edge-coloring of H where the graph induced by the edges of color i contains a G_i. Generally we will pack a set of trees by starting with an uncolored complete graph and k-coloring the edges in a series of steps. Thus we call an edge *uncolored* if it has not yet received a color.

We will suppress all integer part notation. In an effort to make the proofs easier many of the multiplicative constants stated in the paper are not optimal. Our main results are Theorems 3 and 5:

THEOREM 3. *Let n be sufficiently large and let $t = \frac{1}{4}n^{1/3}$. Then the trees $T_1, T_2, \ldots, T_t$ pack into K_n if for each i we have $|T_i| = n - i + 1$ and at least one of the following holds:*

   (1) T_i has a set of at most $n^{1/3}$ vertices such that the union of their neighborhoods contains at least $n^{2/3}$ leaves.

   (2) T_i has at least $n^{2/3}$ independent leaves.

Any tree with maximum degree at least $2n^{2/3}$ must satisfy (1) or (2) from Theorem 3 thus we have the following corollary.

COROLLARY 4. *Let n be sufficiently large and let $t = \frac{1}{4}n^{1/3}$. If $T_1, T_2, \ldots, T_t$ are trees such that $|T_i| = n - i + 1$ and $\Delta(T_i) \geq 2n^{2/3}$ for every i, then $T_1, T_2, \ldots, T_t$ pack into K_n.*

If we let the complete graph have one more vertex than allowed by Conjecture 1, then we can pack many trees without conditions on their structure.

THEOREM 5. *Let n be sufficiently large and $t = \frac{1}{10}n^{1/4}$. If $T_1, T_2, \ldots, T_t$ are trees such that $|T_i| = n - i + 1$ for every i, then $T_1, T_2, \ldots, T_t$ pack into K_{n+1}.*

Eliminating a single case from the proof of Theorem 5 gives the following proposition.

PROPOSITION 6. *Let n be sufficiently large and $t = \frac{1}{10}n^{1/4}$. If $T_1, T_2, \ldots, T_t$ are trees such that $|T_i| = n - i + 1$ and T_i is not a star for each i, then $T_1, T_2, \ldots, T_t$ pack into K_n.*

The remainder of the paper is organized as follows. In Section 2 we will prove some preliminary claims that will help with the proofs of both theorems. Sections 3 and 4 concern the proofs of Theorems 3 and 5 and Proposition 6.

2. Preliminaries. Before proving Theorems 3 and 5, we will need some preparation. Komlós, Sárközy and Szemerédi [15] proved the following.

THEOREM 7. *Let $\delta > 0$ be given. Then there exist constants c and n_0 with the following properties. If $n \geq n_0$, T is a tree on n vertices with $\Delta(T) \leq cn/\log n$, and*

G is a graph on n vertices with $\delta(G) \geq (\frac{1}{2} + \delta)n$, then T is a subgraph of G.

The following corollary is an immediate consequence of Theorem 7 (we fix $\delta = 1/6$ here).

COROLLARY 8. *Let n be sufficiently large and let $t = t(n)$ be a function such that $t(n)/\log n \to \infty$ as $n \to \infty$. If $T_1, T_2, \ldots, T_t$ are forests of order at most n and $\Delta(T_i) < \frac{1}{3}n/t$, then $T_1, T_2, \ldots, T_t$ pack into K_n.*

Corollary 8 will allow us to pack into K_n the forests that remain after removing vertices of "high" degree from each tree T_i. We will also need the following easy claims. The first follows from an application of the greedy algorithm.

CLAIM 9. *Fix k, a, b such that $k < a < b$. Let G be a bipartite graph with classes $A = \{v_1, v_2, \ldots, v_a\}$ and B such that $|B| = b$ and the degree of each vertex in A is at least $b - k$. Then for any non-negative integers $c_1, c_2, \ldots, c_a$ such that $\sum_{i=1}^{a} c_i \leq b - k$ we can pack a star forest into G such that for all i each vertex a_i is the center of a star with exactly c_i leaves.*

CLAIM 10. *Fix a, b, k such that $4k^2 < a < b - k$. Let G be a graph resulting from the removal of the edges of k forests from a complete bipartite graph $K_{a,b}$. Then G contains a matching of size $a - k$.*

Proof. Let A, B be the two vertex classes of $K_{a,b}$ such that $|A| = a$ and $|B| = b$. By the defect version of Hall's theorem, if there is no matching in G of size $a - k$, then there exists a nonempty set $S \subset A$ with neighborhood $N(S) \subset B$ such that $|S| - k > |N(S)|$. Clearly there is no edge between S and $B - N(S)$. Restrict each of the original forests to the complete bipartite graph between S and $B - N(S)$. This gives a packing of k forests into $K_{|S|,|B-N(S)|}$. As the average degree of the union of k forests is less than $2k$, we must have either $|S| < 2k$ or $|B| - |N(S)| < 2k$.

First assume $|S| < 2k$. Immediately we have $|N(S)| < |S| - k < k$. Observe that there are $|S|(|B| - |N(S)|)$ non-edges between S and $B - N(S)$. Furthermore, these non-edges form a subgraph of the forest(s) removed from $K_{a,b}$, so there are at most $k(|S| + |B| - |N(S)| - 1)$ such non-edges. Therefore $|S|(|B| - |N(S)|) \leq k(|S| + |B| - |N(S)| - 1)$. Solving for $|B|$ gives $|B| \leq \frac{k(|S| - |N(S)| - 1) + |S||N(S)|}{|S| - k} \leq 4k^2$, a contradiction.

Now assume $|B| - |N(S)| < 2k$. This gives $|B| - 2k < |N(S)| < |S| - k$ and thus $|S| > |B| - k > a$, a contradiction. $\square$

CLAIM 11. *Fix a and k such that $a \geq 2k$ and let G be a graph resulting from the removal of the edges of k matchings from a $K_{a,a}$. Then G has a perfect matching.*

Proof. If G does not have a perfect matching, then by Hall's theorem there exists a nonempty set S in one of the partite classes with neighborhood $N(S)$ such that $|S| > |N(S)|$. First observe that a vertex is not in $N(S)$ if every edge between it and S has been removed. Thus $|S| \leq k$. Furthermore, we have that the number of non-edges with an endpoint in S is at least $|S|(a - |N(S)|)$ and at most $|S|k$. Simplifying this inequality gives $k \leq a - k \leq |N(S)| < |S|$, a contradiction. $\square$

As stated in the introduction, Conjecture 1 is true when each tree is either a path or a star. By examining the packing of any set of paths and stars $T_n, T_{n-1}, \ldots, T_2$ where T_i has i vertices into K_n as given by Zaks and Liu [18] it is easy to see that the set of endpoints of the paths with at least $\frac{2}{3}n$ vertices are pairwise vertex-disjoint in K_n. This gives the following helpful claim.

CLAIM 12. *Let $T_1, T_2, \ldots, T_k$ be trees such that each T_i is either a path or a star and $|T_i| = 3k - i + 1$. Then there is a k-edge-coloring of K_{3k} such that for each i, the edges of color i span T_i and the set of endpoints of the paths are pairwise vertex-disjoint in K_{3k}.*

3. Proof of Theorem 3. Throughout the proof we will assume that n is sufficiently large for the appropriate inequalities to hold. For ease of notation put $h = \frac{3}{4}n^{2/3}$ and note that $8t = 2n^{1/3}$, thus $8t^2 = \frac{1}{2}n^{2/3} = h - \frac{1}{4}n^{2/3}$. We partition the vertex set of K_n into three parts of order $n - h - 8t$, h, and $8t$. We will refer to the three parts as K_{n-h-8t}, K_h, and K_{8t}.

If T_i satisfies condition (1) from the statement of Theorem 3 then we call it *type I*, otherwise it satisfies condition (2) and is called *type II*. We will partition the trees into parts corresponding to the partition of K_n.

Partition of type I trees: For each tree T_i of type I, define H_i to be the union of the set of vertices of degree greater than $n^{2/3}$ in T_i and a set of at most $n^{1/3}$ vertices in T_i such that the union of their neighborhoods contains at least $n^{2/3}$ leaves and an arbitrary set of vertices such that $|H_i| = 8t = 2n^{1/3}$.

Partition T_i into three parts: H_i, a set of $h - (i-1)$ leaves in the neighborhood of H_i and the remaining $n - (i-1) - |H_i| - (h - (i-1)) = n - h - 8t$ vertices, denoted by F_i.

Partition of type II trees: For each tree T_i of type II, let Y_i be a set of $t - 1$ independent leaves and denote the neighborhood of Y_i by X_i (thus $|X_i| = t - 1$). Define H_i to be the union of X_i and the set of vertices of degree greater than $n^{2/3}$ in T_i and an arbitrary set of vertices in $T_i - Y_i$ such that $|H_i| = 8t = 2n^{1/3}$.

Partition T_i into four parts: H_i, Y_i, a set of $h - |Y_i| - (i-1)$ independent leaves that are not adjacent to H_i (there are at most $|H_i| = 2n^{1/3} = 8t$ independent leaves adjacent to H_i), denoted by L_i, and the remaining $n - (i-1) - |H_i| - |Y_i| - (h - |Y_i| - (i-1)) = n - h - 8t$ vertices, denoted by F_i.

Packing into K_n: To pack the trees into K_n we first pack each F_i into K_{n-h-8t} and each H_i into K_h at the same time. Then we will embed the remaining parts of the trees one-by-one starting with type I trees and finishing with type II trees. Recall that each F_i is a forest on $n - h - 8t$ vertices and $\Delta(F_i) \leq n^{2/3} < \frac{4}{3}(n^{2/3} - \frac{3}{4}n^{1/3} - 2) = \frac{1}{3}(n - \frac{3}{4}n^{2/3} - 2n^{1/3})/t = \frac{1}{3}(n - h - 8t)/t$. Thus by Corollary 8 the forests $F_1, F_2, \ldots, F_t$ pack into K_{n-h-8t}. In other words there is an edge-coloring of K_{n-h-8t} such that for each i the edges of color i contain F_i.

We can pack each H_i vertex-disjointly into K_h as $\sum_{i=1}^{t} |H_i| = 8t^2 = \frac{1}{2}n^{2/3} < h = \frac{3}{4}n^{2/3}$. Because the sets H_i are disjoint in K_h, for each i the edges of T_i between H_i and F_i in K_n can be colored with i.

We will now complete the packing of the trees starting with type II trees followed by type I trees. We pack the trees of a given type from largest to smallest i.e. when completing the packing T_i we may assume that all trees of the same type among $T_1, T_2, \ldots, T_{i-1}$ have already been packed into K_n. A set of vertices in K_n are called *finished (in color i)* if each vertex has as many incident edges of color i as its degree in T_i. Otherwise it is *unfinished (in color i)*.

Packing of type II trees: Our goal will be to pack L_i into $(K_h \cup K_{8t}) - H_i$ and Y_i into $K_h - \cup_{j=1}^{i} H_j$. For each type II tree T_i let N_i be the set of neighbors of the independent leaves L_i, so $|N_i| = h - |Y_i| - (i-1)$. Observe that $N_i \subset F_i$, therefore the vertices in N_i are already packed in K_{n-h-8t}. To complete the packing of T_i we need to find a matching of uncolored edges between $N_i \cup X_i$ and a set of vertices with no incident edge of color i such that $N_i \cup X_i$ is covered. These edges will represent the leaf edges between $N_i \cup X_i$ (which are already packed into K_n) and $L_i \cup Y_i$ (which have not yet been packed into K_n). Thus coloring the edges of this matching with i will complete the packing of T_i.

Consider the bipartite graph between N_i and $\cup_{j=1}^{i-1} H_j$ in K_n. We intend to apply Claim 10 with $k = i - 1$, $a = |\cup_{j=1}^{i-1} H_j|$, and $b = |N_i|$. Observe that at this point any color from $[i-1]$ may have been used on the edges of this bipartite graph and that the edges in a single color class form a forest. Thus we have removed at most $k = i - 1$ forests from a complete bipartite graph with class sizes:

$$a = |\cup_{j=1}^{i-1} H_j| = (i-1)8t \geq 4(i-1)^2 = 4k^2$$

and

$$
\begin{aligned}
b = |N_i| = h - |Y_i| - (i-1) > h - 2t &= \frac{3}{4}n^{2/3} - \frac{1}{2}n^{1/3} \\
&> \frac{1}{2}n^{2/3} + (i-1) = 8t^2 + (i-1) \\
&> (i-1)8t + (i-1) = |\cup_{j=1}^{i-1} H_j| + (i-1) = a + k.
\end{aligned}
$$

So we may apply Claim 10 to the bipartite graph between N_i and $\cup_{j=1}^{i-1} H_j$ with $i-1$ forests removed. Therefore there is a matching of uncolored edges between N_i and $\cup_{j=1}^{i-1} H_j$ that misses only $i-1$ vertices of $\cup_{j=1}^{i-1} H_j$. Color this matching with i. The $i-1$ vertices missed by the matching will never have an incident edge of color i. Note that if $i = 1$ there is nothing to do in the previous step as $\cup_{j=1}^{i-1} H_j$ is empty.

Now we consider the bipartite graph between the unfinished vertices of N_i and $K_h - \cup_{j=1}^{i} H_j$ in K_n. By the previous step the number of unfinished vertices in N_i is $|N_i| - (|\cup_{j=1}^{i-1} H_j| - (i-1))$. We intend to apply Claim 10 with $k = t - 1$, $a = |K_h - \cup_{j=1}^{i} H_j|$, and $b = |N_i| - |\cup_{j=1}^{i-1} H_j| + (i-1)$. Observe that at this point any color except i may have been used on the edges between these two classes and that the edges in a single color class form a forest. Thus we have removed at most $k = t - 1$ forests from a complete bipartite graph with class sizes:

$$
\begin{aligned}
a = |K_h - \cup_{j=1}^{i} H_j| = h - i8t &\\
> h - 8t^2 = \frac{3}{4}n^{2/3} - \frac{1}{2}n^{2/3} &= \frac{1}{4}n^{2/3} = 4t^2 \\
&> 4(t-1)^2 = 4k^2
\end{aligned}
$$

and the number of unfinished vertices in N_i i.e.

$$
\begin{aligned}
b = |N_i| - |\cup_{j=1}^{i-1} H_j| + (i-1) &= h - |Y_i| - (i-1)8t \\
&= h - (t-1) - i8t + 8t \\
&> h - i8t + (t-1) \\
&= |K_h - \cup_{j=1}^{i} H_j| + (t-1) = a + k.
\end{aligned}
$$

So we may apply Claim 10 to the bipartite graph between the unfinished vertices of N_i and $K_h - \cup_{j=1}^{i} H_j$ with $t-1$ forests removed. Therefore there is a matching of uncolored edges between the unfinished vertices of N_i and $K_h - \cup_{j=1}^{i} H_j$ that misses only $t-1$ vertices of $K_h - \cup_{j=1}^{i} H_j$. Color this matching with i.

Now we consider the set of $t-1$ vertices in $K_h - \cup_{j=1}^{i} H_j$ not incident to an edge of color i. In the packing of T_j for $j < i$ each edge in K_h that is colored with j is incident to H_j (this will be clear once we complete the packing of a type II tree). Therefore no edge between H_i and $K_h - \cup_{j=1}^{i} H_j$ has been colored and so we can embed Y_i into

the $t-1$ vertices and color with i the edges between X_i and Y_i (these are leaf edges of T_i).

Finally, there remains $2n^{1/3} = 8t$ unfinished vertices in N_i. For each color $j \in [i-1]$ there is at most one edge of color j incident to each vertex in K_{8t}. Thus by Claim 11 there is a matching of uncolored edges between the unfinished vertices of N_i and K_{8t}. Coloring this matching with i completes the packing of T_i. Note that in packing T_i, every edges in K_h of color i is incident to H_i.

Packing of type I trees: To complete the packing of T_i we need to color edges incident to H_i that correspond to the $h-(i-1)$ leaf edges removed from T_i. Recall that these leaf edges form a star forest with each center vertex in H_i. From the packing of type II trees it is clear that each vertex in H_i is incident to at most one edge of color $j \neq i$ when T_j is a type II tree. When we complete the packing of T_i it will become clear that each edge in H_i is incident to at most one edge of color $j < i$ when T_j is a type I tree. Therefore each vertex in H_i is incident to at most one edge of color $j \in [i-1]$ with an endpoint in $\cup_{j=1}^{i-1} H_j$. Furthermore, all edges between H_i and $K_h - \cup_{j=1}^{i} H_j$ are uncolored as are all edges between H_i and K_{8t}. Thus the bipartite graph with classes H_i and $(K_h - H_i) \cup K_{8t}$ is such that each vertex in H_i is incident to at most $i-1$ colored edges. Therefore each vertex in H_i is incident to at least $|(K_h - H_i) \cup K_{8t}| - (i-1) = h - 8t + 8t - (i-1) = h - (i-1)$ uncolored edges between H_i and $(K_h - H_i) \cup K_{8t}$. So we may apply Claim 9 with $k = i-1$ to the bipartite graph of uncolored edges between H_i and $(K_h - H_i) \cup K_{8t}$ to find the appropriate star forest removed from T_i. Coloring this star forest with i completes the packing of T_i. Note that the coloring of this star forest gives that for each $j > i$, the vertices of H_j are incident to at most one edge of color i. □

4. Proof of Theorem 5. The proof of Theorem 5 follows the general structure of the proof of Theorem 3. However, we must introduce a new type of tree. Because we have no maximum degree condition we will need a class of graphs with lower maximum degree than in Theorem 3. This new class introduces a conflict with trees which are stars. This conflict forces us to pack into K_{n+1} instead of K_n if there are stars present in the sequence of trees. However, the extra vertex does allow several steps in the proof to be less delicate and thus simpler than their counterpart in the proof of Theorem 3.

Proof. Throughout the proof we will assume that n is sufficiently large for the appropriate inequalities to hold. Let $t = \frac{1}{10}n^{1/4}$. We begin by partitioning the set of trees into three classes.

(1) We call T_i *type I* if there is a set of at most $n^{1/4}$ vertices such that the union of neighborhoods of these vertices contains at least $n^{1/2}$ leaves.
(2) We call T_i *type II* if there is a set of at least $n^{1/2}$ independent leaves (and T_i is not type I).
(3) We call T_i *path-like* otherwise.

Note that if a tree T_i has a vertex of degree $2n^{1/2}$ then it must be of type I or type II.

CLAIM 13. *If T_i is path-like, then T_i contains a path of length $3t + 2 = \frac{3}{10}n^{1/4} + 2$ such that the internal vertices of the path have degree 2 in T_i. Furthermore, T_i contains a set of $n^{1/2}$ vertices of degree 2 such that any two vertices have distance greater than 2 from each other and from the endpoints of the path.*

Proof. If T_i is path-like, then T_i has less than $n^{1/2}$ independent leaves and no set of $n^{1/4}$ vertices whose neighborhood contains $n^{1/2}$ vertices of degree 1. Thus if

we partition the set of neighbors of the independent leaves into sets of size $n^{1/4}$ we see that the total number of leaves is less than $n^{1/4}n^{1/2} = n^{3/4}$. If ℓ is the number of leaves in T_i, then clearly $\sum_{x:d(x)>2}(d(x) - 2) < \ell < n^{3/4} - 2$. This sum is at least the number of vertices of degree greater than 2. Thus if we remove all vertices of degree greater than 2 from T_i we are left with a forest of paths with at least $n - n^{3/4}$ vertices and less than $n^{3/4}$ components. Thus there is a component of size at least $\frac{n-n^{3/4}}{n^{3/4}} = n^{1/4} - 1 > 3t$.

The tree T_i has at most $n^{3/4}$ vertices of degree 1 and at most $n^{3/4}$ vertices of degree greater than 2, thus T_i has at least $n - 2n^{3/4}$ vertices of degree 2. Thus for n large enough we can find a path and vertices of degree 2 as required by the claim. □

For ease of notation put $h = \frac{1}{2}n^{1/2}$ and recall that $t = \frac{1}{10}n^{1/4}$, thus $25t^2 = \frac{1}{4}n^{1/2} = \frac{1}{2}h$. For each i define t_i', t_i'', and p_i to be the number of type I, type II, and path-like trees, respectively, among $T_1, T_2, \ldots, T_{i-1}$.

We partition the vertex set of K_{n+1} into four parts of order $n - h - 25t$, h, $25t$, and 1. We will refer to these four parts as $K_{n-h-25t}$, K_h, K_{25t}, and K_1. Now we partition the trees into parts corresponding roughly to the partition of K_{n+1}.

Partition of type I trees: For T_i type I there is a set of at most $n^{1/4}$ vertices such that the union of their neighborhoods contains at least $n^{1/2}$ leaves. Thus there is a vertex, x_i, with $n^{1/4}$ leaves in its neighborhood. Let S_i and Y_i be disjoint sets of leaf neighbors of x_i of sizes $3t - (t_i' + p_i) - 1$ and $2t$, respectively. If T_i is not a star, then there is a vertex, u_i, different from x_i that has a leaf neighbor, denoted by v_i. If T_i is a star, then let v_i be a leaf neighbor of x_i disjoint from S_i and Y_i.

Define H_i as the union of x_i, u_i (if it exists), the set of vertices of degree greater than $n^{3/4}$, a set of at most $n^{1/4}$ vertices in T_i such that the union of their neighborhoods contains at least $n^{1/2}$ leaves, and an arbitrary set of vertices in $T_i - S_i - Y_i - \{v_i\}$ such that $|H_i| = 25t$.

Partition T_i into six parts: H_i, S_i, Y_i, v_i, a set of $h - |S_i| - |Y_i| - 1 - (i - 1)$ neighbors of H_i of degree 1, denoted by L_i, and the remaining $n - (i - 1) - |H_i| - |S_i| - |Y_i| - 1 - (h - |S_i| - |Y_i| - 1 - (i - 1)) = n - h - 25t$ vertices, denoted by F_i.

Partition of type II trees: For T_i type II, let Y_i be a set of $2t$ independent leaves and let X_i be the set of neighbors of Y_i (note that $|X_i| = 2t$). Define H_i as the union of X_i, the set of vertices of degree greater than $n^{3/4}$, and an arbitrary set of vertices in $T_i - Y_i$ such that $|H_i| = 25t$.

Partition T_i into four parts: H_i, Y_i, a set of $h - (i - 1) - |Y_i|$ independent leaves that are not adjacent to H_i, denoted by L_i, and the remaining $n - (i - 1) - |H_i| - |Y_i| - (h - |Y_i| - (i - 1)) = n - h - 25t$ vertices, denoted by F_i.

Partition of path-like trees: Note that a path-like tree has no vertex of degree greater than $n^{3/4}$. For T_i path-like, let P_i be a path on $3t - (t_i' + p_i)$ vertices that is contained in a path on $3t - (t_i' + p_i) + 2$ vertices such that all vertices in P_i are degree 2 in T_i (note that this means the endpoints of P_i are degree 2 in T_i). Such a path exists by Claim 13. Let Y_i be a set of $8t$ vertices of degree 2 that are pairwise of distance greater than 2 from each other and from the endpoints of P_i and let X_i be the set of neighbors of Y_i (note that $|X_i| = 16t$). Define H_i to be the union of the set of the two neighbors of the endpoints of P_i, the set X_i, and an arbitrary set of vertices in $T_i - P_i - Y_i$ such that $|H_i| = 25t$.

Partition T_i into five parts: H_i, P_i, Y_i, a set of $h - |P_i| - |Y_i| - (i - 1)$ vertices of degree 2 that are pairwise of distance greater than 2 from each other and not adjacent

to H_i or P_i, denoted by L_i, and the remaining $n - (i-1) - |H_i| - |P_i| - |Y_i| - (h - |P_i| - |Y_i| - (i-1)) = n - h - 25t$ vertices, denoted by F_i.

Packing into K_{n+1}: Observe that for each i, F_i is a forest on $n - h - 25t$ vertices and $\Delta(F_i) \leq n^{3/4} < \frac{10}{3}(n^{3/4} - \frac{1}{2}n^{1/4} - \frac{25}{10}) = \frac{1}{3t}(n - \frac{1}{2}n^{1/2} - \frac{25}{10}n^{1/4}) = \frac{1}{3t}(n - h - 25t)$. Thus by Corollary 8 we can pack the forests into $K_{n-h-25t}$. In other words there is an edge-coloring of $K_{n-h-25t}$ such that the edges of color i contain F_i.

We can pack each H_i vertex-disjointly into K_h as $\sum_{i=1}^{t} |H_i| = 25t^2 = \frac{25}{100}n^{1/2} < \frac{1}{2}n^{1/2} = h$. Because the sets H_i are disjoint in K_h, for each i the edges of T_i between H_i and F_i in K_{n+1} can be colored with i.

For each type I tree T_i, the set $S_i \cup \{x_i\}$ is a star on $3t - (t_i' + p_i)$ vertices and for each path-like tree T_i, the set P_i is a path on $3t - (t_i' + p_i)$ vertices. These stars and paths form a set of trees on $3t, 3t-1, \ldots, 3t - (t' + p) + 1$ vertices (where $t' + p$ is the total number of type I plus path-like trees). By Claim 12 these stars and paths pack into an auxiliary complete graph K_{3t} such that the endpoints of the paths are pairwise disjoint in this K_{3t}. Note that the x_is must necessarily be pairwise disjoint in K_{3t}. Now we embed K_{3t} (with the packing above) into K_h in such a way that for each type I tree T_i the center of $S_i \cup \{x_i\}$ in the auxiliary K_{3t} corresponds to the vertex x_i packed into H_i and the other vertices of the K_{3t} are arbitrary vertices of K_h that are disjoint from $\cup_{j=1}^{t} H_j$ in K_h (this can be done as $|\cup_{j=1}^{t} H_j| = 25t^2 < \frac{1}{2}h$). Note that this implies that for each type I tree T_i we have $K_{3t} \cap H_i = \{x_i\}$. For each type I tree T_i color with i the edges of $S_i \cup \{x_i\}$ in the K_{3t}. Now for each path-like tree T_i color with i the edges of P_i in the K_{3t} and the edge between each endpoint of P_i in the K_{3t} and its neighbor that is in H_i (by construction of H_i). Now for each type I tree T_i let us distinguish two simple cases:

Case A: If T_i is not a star, then consider the vertex x_i in K_{3t}. By the the application of Claim 12 there may be a path P_j in K_{3t} of color $j \neq i$ with endpoint x_i in K_{3t}. By the previous step P_j was have been extended to some vertex $z \in H_j$. In this case embed v_i to the vertex z and color with i the edge between u_i in H_i and z in H_j. If the vertex x_i does not correspond to the end of a path P_j, then embed u_i to an arbitrary vertex in $K_h - \cup_{i=1}^{t} H_i - K_{3t}$ and color with i the edge between u_i and v_i.

Case B: If T_i is a star, then there is no vertex u_i, so we will embed v_i into K_1 and color the edge between x_i and v_i with color i.

We remark that **Case B** is the only time when K_1 is needed to complete the packing of the trees. Thus if there is no tree which is a star, then we are able to pack into K_n.

At this point for each tree T_i the edges of color i in K_{n+1} induce a subgraph of T_i formed from T_i minus some of the vertices of degree 1 or 2 (that are pairwise non-adjacent in T_i).

We will complete the packing of the trees in three rounds by type in the following order: type II, path-like, type I. We will pack the trees from largest size to smallest i.e. when packing T_i we may assume that all trees of the same type among $T_1, T_2, \ldots, T_{i-1}$ have already been packed into K_{n+1}. At this point for each tree T_i we have packed F_i, H_i, and S_i and P_i (when they exist) into K_{n+1}. To complete the packing of the remaining trees we now will be focused on embedding vertices of degree 1 and 2 (from L_i and Y_i) into K_{n+1}. A set of vertices in K_{n+1} are called *finished (in color i)* if each vertex has as many incident edges of color i as its degree in T_i. Otherwise it is *unfinished (in color i)*.

Packing of type II trees: Recall that L_i and Y_i are partition classes of T_i containing $h - |Y_i| - (i-1)$ and $2t$ independent leaves, respectively. Also recall that X_i is the set of neighbors of Y_i and that $X_i \subset H_i$ which is already embedded into K_h. To complete the packing of T_i we will first embed the leaves L_i and then Y_i into $K_h \cup K_{25t} - H_i - K_{3t}$. Let N_i be the set of neighbors of L_i in T_i, so $|N_i| = h - |Y_i| - (i-1)$. Observe that $N_i \subset F_i$ therefore the vertices N_i are packed into $K_{n-h-25t}$. To complete the packing of T_i we need to find a matching of uncolored edges between $N_i \cup X_i$ and a set of vertices in K_{n+1} with no incident edge of color i. These edges will represent the edges between $N_i \cup X_i$ (which are already packed into K_{n+1}) and $L_i \cup Y_i$ (which have not been packed into K_{n+1}). Thus coloring the edges of this matching with i will complete the packing of T_i.

Consider the bipartite graph between N_i and $K_h - H_i$ in K_{n+1}. We intend to apply Claim 10 with $k = t-1$, $a = |K_h - H_i|$, and $b = |N_i|$. Observe that at this point any color but i have been used on the edges of this bipartite graph and that the edges in a single color class form a forest. Thus we have removed at most $t-1$ forests from a complete bipartite graph with class sizes:

$$a = |K_h - H_i| = h - 25t > 4(t-1)^2 = 4k^2$$

and

$$b = |N_i| = h - |Y_i| - (i-1) > h - 3t > |K_h - H_i| + (t-1) = a + k.$$

So we may apply Claim 10 to the bipartite graph between N_i and $K_h - H_i$ with $t-1$ forest removed. Therefore there is a matching of uncolored edges between N_i and $K_h - H_i$ that misses only $t-1$ vertices of $K_h - H_i$. Color the edges of the matching with i such that $2t + (i-1)$ vertices of $K_h - H_i$ are not incident to an edge of color i.

Now we consider the set of $2t + (i-1)$ vertices in $K_h - H_i$ not incident to an edge of color i. For each type I tree T_j with $j < i$, there is exactly one vertex x_j in $K_{3t} \subset K_h$. Thus there is a set of $2t + (i-1) - t_i' \geq 2t$ vertices in K_h that are not incident to an edge of color i and are disjoint from the vertices x_j for $j < i$. Embed Y_i into $2t$ of these vertices and consider the bipartite graph between X_i and Y_i. For now assume that each vertex in X_i (and H_i in general) is incident to at most one edge of each color $j \neq i$ (this fact will become clear at the completion of the packing of T_i). Thus the bipartite graph of uncolored edges between X_i and Y_i is the graph obtained by removing at most $t-1$ matchings from a complete bipartite graph. Thus by Claim 11 there is a perfect matching between X_i and Y_i. Coloring the edges of this matching with i embeds Y_i into K_{n+1}. Observe that this coloring implies that for $j > i$ each vertex in H_j will be incident to at most one edge of color i.

Finally, there remains $25t = \frac{25}{10} n^{1/4}$ unfinished vertices in N_i. For each color $j \in [i-1]$ there is at most one edge of color j incident to each vertex in K_{25t}. So we have colored at most $i-1$ matchings from a complete bipartite graph with class sizes $25t$. Thus by Claim 11 there is a perfect matching of uncolored edges between the unfinished vertices of N_i and K_{25t}. Coloring this perfect matching with i completes the packing of T_i.

Packing of path-like trees: Recall that L_i and Y_i are partition classes of T_i that contain $h - |P_i| - |Y_i| - (i-1)$ and $8t$ vertices of degree 2 that are pairwise of distance greater than 2 from each other, respectively. Also recall that X_i is the set of neighbors of Y_i and that $X_i \subset H_i$ which is already embedded into K_h. To complete the packing of T_i we will to embed the degree 2 vertices in $L_i \cup Y_i$ into

$K_h \cup K_{25t} - H_i - K_{3t}$. First we embed L_i then Y_i. Let N_i be the set of neighbors of L_i in T_i. Observe that $N_i \subset F_i$. Each vertex in L_i has exactly two neighbors in N_i and no two vertices in L_i share a neighbor in N_i, so $|N_i| = 2|L_i| = 2(h - |P_i| - |Y_i| - (i-1))$. Furthermore, each vertex in L_i can be associated with two unique vertices in N_i. We call two such vertices in N_i a *pair*.

Recall that K_{3t} only intersects H_i if T_i is type I, so for a path-like tree T_i we have that H_i and K_{3t} are disjoint. Now we consider the bipartite graph between N_i and $K_h - H_i - K_{3t}$. At this point any color except i may have been used on the edges between N_i and $K_h - H_i - K_{3t}$ and that the edges in a single color class form a forest. First let us contract each pair in N_i such that if any of the two edges identified together are already used in the embedding of a tree, then the resulting edge is considered colored in the contraction. Denote the contraction of N_i by N_i'. If we contract each pair in N_i, then each colored forest between N_i and $K_h - H_i - K_{3t}$ becomes the union of two colored forests between N_i' and $K_h - H_i - K_{3t}$. We intend to apply Claim 10 with $k = 2(t-1)$, $a = |K_h - H_i - K_{3t}|$, and $b = |N_i'|$. The complete bipartite graph between N_i' and $K_h - H_i - K_{3t}$ has class sizes:

$$a = |K_h - H_i - K_{3t}| > h - 25t - 3t \geq 4(2(t-1))^2 = 4k^2$$

and

$$b = |N_i'| = \frac{1}{2}|N_i| = h - |P_i| - |Y_i| - (i-1)$$
$$> h - 12t > |K_h - H_i - K_{3t}| + 2(t-1) = a + k.$$

The colored edges of this complete bipartite graph form $2(t-1)$ forests, so we may apply Claim 10 to the bipartite graph of uncolored edges between N_i' and $K_h - H_i - K_{3t}$. Therefore there is a matching of uncolored edges between N_i' and $K_h - H_i - K_{3t}$ that misses only $2(t-1)$ vertices of $K_h - H_i - K_{3t}$. Now we consider a subgraph of this matching such that exactly $8t + t_i''$ vertices of $K_h - H_i - K_{3t}$ are not incident to an edge of the subgraph. If we return to the uncontracted set N_i, then for each edge of the subgraph of the matching we have two edges between a pair in N_i and a vertex in $K_h - H_i - K_{3t}$. We color these edges with i.

Now there are $8t + t_i''$ vertices in $K_h - H_i - K_{3t}$ that are not incident to an edge of color i. Now we embed Y_i into $8t$ of these vertices and consider the bipartite graph between Y_i and $X_i \subset H_i$. As before, the vertices in X_i can be arranged as pairs of neighbors of degree 2 vertices of T_i. Each vertex in X_i is incident to at most $2(t-1)$ edges with colors other than i (as only vertices in L_j, Y_j or u_j for $j \neq i$ have been embedded in H_i). In each color $j \neq i$ the edges of color j incident to X_i are never incident with each other outside of X_i by the construction of the sets L_j, Y_j and u_j. Thus if we contract the pairs in X_i as before to get X_i', then each vertex in X_i' is incident to at most $4(t-1)$ edges with colors other than i and these edges can be partitioned into $4(t-1)$ matchings. Thus by Claim 11 there is a perfect matching of uncolored edges between X_i' and Y_i. Returning to the uncontracted set X_i, for each edge of the perfect matching we have two edges between a pair in X_i and a vertex in Y_i. Color these edges with i. Observe that there are t_i'' vertices in $K_h - K_i - K_{3t}$ not incident to an edge of color i and $t_i' + p_i$ vertices in K_{3t} not incident to an edge of color i. Thus there are exactly $t_i'' + t_i' + p_i = i - 1$ vertices in K_h not incident to an edge of color i.

Finally, there remain $25t = \frac{25}{10}n^{1/4}$ unfinished pairs in N_i. For each color $j \neq i$ there are at most two edges of color j incident to each vertex in K_{25t} (as only vertices

in L_j, Y_j or u_j have been embedded into K_{25t}). If we contract the unfinished pairs of N_i as before, we are left with a complete bipartite graph with $4(t-1)$ matchings colored with colors other than i. Thus by Claim 11 there is a perfect matching of uncolored edges between the contracted unfinished pairs of N_i and K_{25t}. Uncontracting these pairs and coloring the edges corresponding to the perfect matching with color i completes the packing of T_i.

Packing of type I trees: Recall that L_i and Y_i are partition classes of T_i containing $|L_i| = h - |S_i| - |Y_i| - 1 - (i-1) \le h - 5t$ and $2t$ leaves, respectively. Also recall the vertices of Y_i are adjacent to x_i which is already embedded into K_h. To complete the packing of T_i we need to embed the leaves $L_i \cup Y_i$ into K_n. First we embed L_i then Y_i. Recall that these leaf edges form a star forest with each center vertex in H_i. Observe that each vertex in H_i is incident to at most 2 edges of each color other than i (colored in the previous steps). Furthermore if T_i is not a star, then $u_i \in H_i$ exists and is incident to a single edge of color i in $(K_h - H_i - K_{3t}) \cup K_{25t}$. Recall that since T_i is type I we have $H_i \cap K_{3t} = \{x_i\}$. Therefore each vertex in H_i is incident to at least $|(K_h - H_i - K_{3t}) \cup K_{25t}| - 2t - 1 \ge h - 25t - 3t + 1 + 25t - 2t - 1 = h - 5t$ uncolored edges with an endpoint in $(K_h - H_i - K_{3t}) \cup K_{25t}$. So we may apply Claim 9 with $k = 2t$ to the bipartite graph of uncolored edges between H_i and $(K_h - H_i - K_{3t}) \cup K_{25t}$ to find the appropriate star forest (spanning the vertices in L_i) removed from T_i. Coloring this star forest with i finishes each vertex in $H_i - \{x_i\}$. Now in $K_h \cup K_{25t}$ there are $h + 25t - |H_i| - |L_i| - |S_i| - 1 = |Y_i| + (i-1) = 2t + (i-1)$ vertices that are not incident to an edge of color i. Exactly $t_i' + p_i$ of them are in K_{3t} thus there are $2t + t_i''$ of them in $(K_h - H_i - K_{3t}) \cup K_{25t}$. We can embed Y_i into $2t$ of these vertices and color the edges between x_i and Y_i to complete the packing of T_i (as we never colored edges incident to x_i in $K_h - H_i - K_{3t}$ in the embedding of type II and path-like trees). Note that this leaves exactly $t_i' + p_i + t_i'' = i - 1$ vertices in K_n that are not incident to an edge of color i. □

Proof. [Proposition 6] We repeat the proof of Theorem 5 and observe that if there is no star in the set of trees, then **Case B** above never occurs. This is the only situation when the vertex in K_1 is used. Thus we are only packing the trees into K_n which is what is claimed by Proposition 6. □

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