TRANSITIVE TRIANGLE TILINGS IN ORIENTED GRAPHS

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Abstract. In this paper, we prove an analogue of Corrádi and Hajnal's result. There exists n_0 such that for every $n \in 3\mathbb{Z}$ when $n \geq n_0$ the following holds. If G is an oriented graph on n vertices and $\delta^0(G) \geq 7n/18$, then G contains a perfect TT_3-tiling, which is a collection of vertex disjoint transitive triangles covering every vertex of G. This result is best possible, as there exists an oriented graph G on n vertices without a perfect TT_3-tiling and $\delta^0(G) = \lfloor 7n/18 \rfloor - 1$.

1. Introduction

Let G be an oriented graph, that is a directed graph without loops such that between every two vertices there is at most one edge. We write xy for an edge directed from x to y. The *outdegree* $d^+_G(x)$ of a vertex x is the number of vertices y such that $xy \in E(G)$. Similarly, the *indegree* $d^-_G(x)$ of a vertex x is the number of vertices y such that $yx \in E(G)$. Define the *minimum outdegree* $\delta^+(G)$ of G to be the minimal $d^+_G(x)$ over all vertices x of G, and define the *minimum indegree* $\delta^-(G)$ of G similarly. Define the *minimum semidegree* $\delta^0(G)$ of G to be $\min\{\delta^+(G), \delta^-(G)\}$.

The oriented graph on $\{v_1, \dots, v_n\}$ with edge set $\{v_n v_1\} \cup \{v_i v_{i+1} : i \in \{1, \dots, n-1\}\}$ is the *directed cycle* of length n. An oriented graph in which there is an edge between every pair of vertices is called a *tournament*. A tournament that does not contain a directed cycle is *transitive*. Up to isomorphism, there are two tournaments on 3 vertices: The directed cycle of length 3, which we refer to as the *cyclic triangle*, and the transitive tournament on 3 vertices, which we refer to as the *transitive triangle* or as TT_3.

A *tiling* of G is a collection of vertex disjoint subgraphs called *tiles*. If every tile is isomorphic to some oriented graph H, then the tiling is an H*-tiling*. If every vertex in G is contained in a tile, then the tiling is *perfect*. The same definitions are applied to graphs and directed graphs.

In [5], Hajnal and Szemerédi proved that for any $k, r \in \mathbb{N}$ and for any graph G on kr vertices if the minimum degree of G is at least $(r-1)k$, then G has a perfect K_r-tiling. The case when $r = 3$ was proved earlier by Corrádi and Hajnal [1].

The problem of finding cyclic triangle tilings in an oriented graph was considered by Keevash and Sudakov [7], who proved a nearly optimal result: For some $\varepsilon > 0$ there exists n_0 such that if G is an oriented graph on $n \geq n_0$ vertices and $\delta^0(G) \geq (1/2 - \varepsilon)n$, then G contains a cyclic triangle tiling that covers all but at most 3 vertices. Furthermore, if $n \equiv 3 \pmod{18}$, then there is a tournament T such that $\delta^0(T) \geq (n-1)/2 - 1$ which does not have a perfect cyclic triangle tiling. They repeated the following question which was asked by both Cuckler [2] and Yuster [13].

Date: January 3, 2014.

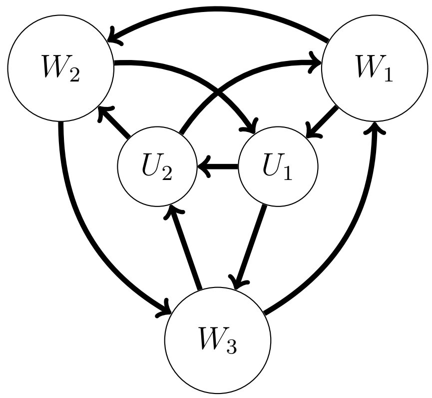

FIGURE 1. The extremal graph.

Question 1.1. *Does every tournament T on $n \equiv 3 \pmod 6$ vertices with $\delta^0(T) = (n-1)/2$ have a perfect cyclic triangle tiling?*

In this paper, we consider the problem of finding a perfect transitive triangle tiling, proving an analogue to Corrádi and Hajnal's result for oriented graphs.

Theorem 1.2. *There exists n_0 such that for every $n \in 3\mathbb{Z}$ when $n \geq n_0$ the following holds. If G is an oriented graph on n vertices and $\delta^0(G) \geq 7n/18$, then G contains a perfect TT_3-tiling.*

Treglown [10] conjectured that Theorem 1.2 is true for any $n \in 3\mathbb{Z}$.

The related problems for directed graphs have been considered (see [12], [4], [3] and [11]).

The following family of examples shows that Theorem 1.2 is tight. Let $n \in 3\mathbb{Z}$ and let G be the following oriented graph on n vertices. Let W_1, W_2, W_3, U_1, U_2 be a partition of $V(G)$ such that

$$|W_i| = \left\lfloor \frac{2n/3 + i}{3} \right\rfloor \text{ for } i \leq 3, \qquad |U_1| = \left\lfloor \frac{n-3}{6} \right\rfloor, \qquad |U_2| = \left\lceil \frac{n-3}{6} \right\rceil.$$

The edges of G are all possible directed edges from W_i to W_{i+1}, from U_1 to U_2, from $W_1 \cup W_2$ to U_1, from U_1 to W_3, from U_2 to $W_1 \cup W_2$ and from W_3 to U_2, see Figure 1. Note that $\delta^0(G) = \lfloor 2n/9 \rfloor + \lfloor (n-3)/6 \rfloor + 1$, which is achieved by considering a vertex in W_3 (the outdegree when $n \equiv 15 \pmod{18}$ and indegree in all other cases). Since every transitive triangle in G contains a vertex in $U_1 \cup U_2$ and $|U_1 \cup U_2| = n/3 - 1$, G does not contain a perfect TT_3-tiling.

1.1. **Outline of the paper.** We prove Theorem 1.2 using a stability approach and the absorption technique. We say that an oriented graph G on n vertices is *α-extremal* if there exists $W \subseteq V(G)$ such that $|W| \geq (2/3 - \alpha)n$ and $G[W]$ does not contain a transitive triangle.

In Section 2, we handle the case when G is not α-extremal.

Lemma 1.3. *For every $\alpha > 0$ there exists $\varepsilon = \varepsilon(\alpha) > 0$ and $n_0 = n_0(\alpha)$ such that when G is an oriented graph on $n \in 3\mathbb{Z}$ vertices and $n \geq n_0$ the following holds. If $\delta^0(G) \geq (7/18 - \varepsilon)n$, then G has a perfect TT_3-tiling or G is α-extremal.*

In Section 3, we prove Theorem 1.2 for oriented graphs G which are α-extremal.

Lemma 1.4. *There exists $\alpha > 0$ and n_0 such that when G is an oriented graph on $n \in 3\mathbb{Z}$ vertices and $n \geq n_0$ the following holds. If $\delta^0(G) \geq 7n/18$ and G is α-extremal, then there exists a perfect TT_3-tiling of G.*

Lemma 1.3 and Lemma 1.4 together clearly prove Theorem 1.2.

While proving Lemma 1.3 we prove the following result which may be of some interest because it applies for all n. Furthermore, it might be possible to extend the proof of this theorem to prove the main theorem for all n.

Theorem 1.5. *If G is an oriented graph on n vertices and $\delta^0(G) \geq 7n/18$, then there exists a TT_3-tiling of G that covers all but at most 11 vertices.*

1.2. **Notation.** Given a graph or digraph G, we write $V(G)$ for its vertex set, $E(G)$ for its edge set, and $e(G) = |E(G)|$ for the number of its edges. Given a collection $\mathcal{T}$ of subgraphs, we write $V(\mathcal{T})$ for $\bigcup_{T \in \mathcal{T}} V(T)$. When $\mathcal{W}$ is a collection of vertex subsets we will also use the notation $V(\mathcal{W})$ to denote $\bigcup_{W \in \mathcal{W}} W$.

Suppose that G is an oriented graph. If x is a vertex of G, then $N_G^+(x)$ denotes the *out-neighborhood* of x, i.e. the set of all those vertices y for which $xy \in E(G)$. Similarly, $N_G^-(x)$ denotes the *in-neighborhood* of x, i.e. the set of all those vertices y for which $yx \in E(G)$. Note that $d_G^+(x) = |N_G^+(x)|$ and $d_G^-(x) = |N_G^-(x)|$. We write $N_G(x) = N_G^+(x) \cup N_G^-(x)$ and $d_G(x) = d_G^+(x) + d_G^-(x)$. We write $\delta(G)$ and $\Delta(G)$ for the minimum degree and maximum degree of the underlying undirected graph of G, respectively. Given a vertex v of G and a set $A \subseteq V(G)$, we define $d_G^+(v, A) = |N_G^+(v) \cap A|$ and define $d_G^-(v, A)$ and $d_G(v, A)$ similarly. Given $A, B \subseteq V(G)$, let $\overrightarrow{E}_G(A, B)$ be the set of edges in G directed from A to B. Similarly, $E_G(A, B)$ denotes the set of edges with one endpoint in A and the other in B and let $e_G(A, B) = |E_G(A, B)|$. For a vertex v, we write $E_G(v)$ for $E_G(v, V(G))$. For a vertex set $A \subseteq V(G)$, we write $G[A]$ for the subgraph of G induced by A and let $e_G(A) = e(G[A])$. If G is known from the context, then we may omit the subscript. We let $G[A, B]$ be the bipartite graph in which $a \in A$ is adjacent to $b \in B$ if and only if $ab \in E(G)$ or $ba \in E(G)$. If $x, y, z \in V(G)$ we sometimes refer to $G[\{x, y, z\}]$ as xyz or as xe, where $e = yz$ or $e = zy$. If we refer to a directed path or cyclic triangle as xyz, then it must contain the edge set $\{xy, yz\}$ or the edge set $\{xy, yz, zx\}$, respectively.

For $m \in \mathbb{N}$ we write $[m] = \{1, \dots, m\}$. For any set V we let $\binom{V}{m}$ be the collection of subsets of V that are of order m. When it is clear that a variable i must remain in $[m]$ (e.g. when i is the index of $W_1, \dots, W_m$) we let $i + 1 = 1$ when $i = m$ and $i - 1 = m$ when $i = 1$.

1.3. **Preliminary lemmas and propositions.** Let G be an oriented graph. Note that if uw is an edge in $G[N^+(v)]$, then vuw is a transitive triangle. We get the following easy proposition.

Proposition 1.6. *Let G be a tournament on 4 vertices. Then every vertex of G is contained in a transitive triangle.*

Proposition 1.7. *Let G be an oriented graph on n vertices. Then*

(a) *every (directed) edge uv is contained in at least $3\delta^0(G) - n$ transitive triangles uvw such that $w \in N^-(v)$;*
(b) *every (directed) edge uv is contained in at least $3\delta^0(G) - n$ transitive triangles uvw such that $w \in N^+(u)$;*

(c) *for every directed path* uvw *on* 3 *vertices, there are at least* $2(3\delta^0(G) - n)$ *vertices* x *such that there exists a transitive triangle in* $G[\{u, v, w, x\}]$ *containing* x *and* v.

Proof. Let uv be an edge in G. Note that every vertex w in $N(u) \cap N^-(v)$ forms a transitive triangle with uv. Since $|N(u) \cap N^-(v)| \geq \delta(G) + \delta^-(G) - n \geq 3\delta^0(G) - n$, (a) follows. By a similar argument, (b) also holds.

Let uvw be a directed path on 3 vertices. By (a), there is a set $U \subseteq N^-(v)$ with $|U| \geq 3\delta^0(G) - n$ such that every $u' \in U$ forms a transitive triangle with uv. By (b), there is a set $W \subseteq N^+(v)$ with $|W| \geq 3\delta^0(G) - n$ such that every $w' \in W$ forms a transitive triangle with vw. Since $U \cap W = \emptyset$, (c) holds. □

2. Non-extremal case

2.1. **Absorbing structure.** In this section, we prove Lemma 2.2. Roughly speaking, the lemma states that there exists a small vertex set $U \subseteq V(G)$ such that $G[U \cup W]$ contains a perfect TT_3-tiling for every small $W \subseteq V(G) \setminus U$. Thus, in order to find a perfect TT_3-tiling in G, it is suffices to find a TT_3-tiling covering almost all vertices in $G[V(G) \setminus U]$. This technique was introduced by Rödl, Ruciński and Szemerédi [9] to obtain results on matchings in hypergraphs.

For any $r \in \mathbb{N}$ and any collection $\mathcal{H}$ of oriented graphs on $[r]$, define $\mathcal{F}(\mathcal{H}, G)$ to be the set of functions f from $[r]$ to $V(G)$ such that f is a directed graph homomorphism from some $H \in \mathcal{H}$ to G. Let $\mathcal{K}$ be the set of oriented graphs K on $\{1, \dots, 21\}$ such that both K and $K[\{1, \dots, 18\}]$ have a perfect TT_3-tiling. For any ordered triple $X = (x_1, x_2, x_3)$ of vertices in G, let $\mathcal{A}'(X)$ be the set of functions $f \in \mathcal{F}(\mathcal{K}, G)$ such that $f(19) = x_1$, $f(20) = x_2$ and $f(21) = x_3$. Let $\mathcal{A}(X)$ be the set of functions in $\mathcal{A}'(X)$ restricted to $[18]$. Clearly $|\mathcal{A}(X)| = |\mathcal{A}'(X)|$. Note we do not require the function in $\mathcal{F}(\mathcal{H}, G)$ to be injective, but at a later stage of the proof non-injective functions will essentially be discarded. We consider non-injective functions only to make the following arguments simpler.

Lemma 2.1. *For* $\varepsilon_0 = 1/250$ *and* $0 \leq \varepsilon \leq \varepsilon_0$, *there exists* $\tau = \tau(\varepsilon) > 0$ *and* $n_0 = n_0(\varepsilon)$ *such that the following holds. If* G *is an oriented graph on* $n \geq n_0$ *vertices and* $\delta^0(G) \geq (7/18 - \varepsilon)n$, *then* $|\mathcal{A}(X)| \geq \tau n^{18}$ *for every ordered triple* $X = (x_1, x_2, x_3)$ *of vertices in* G.

Proof. Let $0 < \beta < (1/249 - \varepsilon)/10$ and $\tau = \beta^{18}$. Let $\mathcal{T}$ be the set of functions from $\{1, 2, 3\}$ to $V(G)$ that are digraph homomorphisms from a transitive triangle on $\{1, 2, 3\}$ to G. In other words, $\mathcal{T}$ contains all functions from $\{1, 2, 3\}$ to $V(G)$ whose image induces a transitive triangle. If we let $f(1)$ be any vertex $a \in V(G)$ and let $f(2)$ be any $b \in N_G(a)$, by Proposition 1.7, there are $3\delta^0(G) - n \geq (1/6 - 3\varepsilon)n$ vertices we can assign to $f(3)$ so that $f \in \mathcal{T}$. This gives us that

$$|\mathcal{T}| \geq n \cdot (7/9 - 2\varepsilon)n \cdot (1/6 - 3\varepsilon)n > n^3/9 > (\beta n)^3.$$

For any $p \in \mathbb{N}$, let $\mathcal{L}_p$ be the set of oriented graphs L on $[3p + 1]$ such that both $L[\{2, \dots, 3p + 1\}]$ and $L[\{1, \dots, 3p\}]$ have perfect TT_3-tilings (see Figure 2 for some examples). For any $x, y \in V(G)$, let $\mathcal{C}_p(x, y)$ be the set of $f \in \mathcal{F}(\mathcal{L}_p, G)$ such that $f(1) = x$ and $f(3p+1) = y$. If $x = y$ we say that x and y are 0-linked. Otherwise, we say that x and y are p-linked if $|\mathcal{C}_p(x, y)| \geq (\beta n)^{3p-1}$.

Let $q \geq p$. We have that $|\mathcal{C}_q(x,y)| \geq |\mathcal{C}_p(x,y)||\mathcal{T}|^{q-p}$. Indeed, for any $f \in C_p(x,y)$ and $g_i \in \mathcal{T}$ for $i \in [q-p]$, the function

$$h(j) = \begin{cases} f(j) & \text{if } 1 \leq j \leq 3p \\ g_i(k) & \text{if } j = 3p + 3(i-1) + k \text{ for some } i \in [q-p] \text{ and } k \in [3] \\ y & \text{if } j = 3q+1 \end{cases}$$

is in $\mathcal{C}_q(x,y)$. Hence, the inequality holds. Recall that we do not require the functions in $\mathcal{C}_q(x,y)$ to be injective. So since $|\mathcal{T}| \geq (\beta n)^3$, if x and y are p-linked, then x and y are also q-linked.

Let $X = (x_1, x_2, x_3)$ be an ordered triple of vertices in G. We have that

$$|\mathcal{A}(X)| = |\mathcal{A}'(X)| \geq \sum_{f \in \mathcal{T}} \prod_{i \in 1}^{3} |\mathcal{C}_2(f(i), x_i)|.$$

Indeed, let $f \in \mathcal{T}$ and $g_i \in \mathcal{C}_2(f(i), x_i)$ for $i \in [3]$, and define

$$h(j) = \begin{cases} g_i(k) & \text{if } j = 6(i-1) + k \text{ for some } i \in [3] \text{ and } k \in [6] \\ x_i & \text{if } j = 18 + i \text{ for some } i \in [3]. \end{cases}$$

By the definition of $\mathcal{C}_2$ and the fact that the image $h(\{1,7,13\}) = f([3])$ induces a transitive triangle in G, it is not too hard to see that $h \in \mathcal{A}'(X)$ and that the inequality holds. Therefore, because $|\mathcal{T}| \geq (\beta n)^3$, we can complete the proof of the lemma by showing that every pair of vertices in $V(G)$ is 2-linked.

For any $U \subseteq V(G)$, let $N(U) = \bigcap_{u \in U} N_G(u)$. If $U = \{x, y\}$ is a 2-set, we often write $N(x,y)$ instead of $N(U)$. We have the following inequality

$$N_G(U) \geq |U|\delta(G) - (|U|-1)n \geq \left(\frac{9 - 2|U|}{9} - 2|U|\varepsilon\right) n. \tag{1}$$

For any pair $x, y \in V(G)$, let

$$N_1^1(x,y) = N_G^+(x) \cap N_G^+(y), \qquad N_2^1(x,y) = N_G^-(x) \cap N_G^+(y),$$
$$N_1^2(x,y) = N_G^+(x) \cap N_G^-(y), \qquad N_2^2(x,y) = N_G^-(x) \cap N_G^-(y)$$

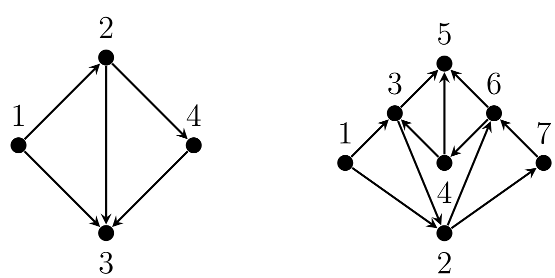

FIGURE 2. A graph in $\mathcal{L}_1$ and a graph in $\mathcal{L}_2$.

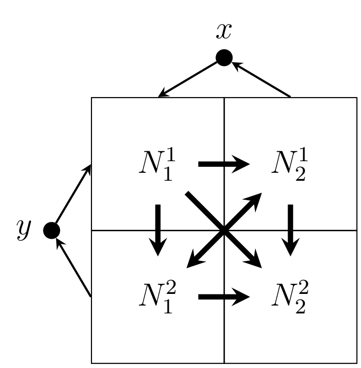

FIGURE 3. Orientation of any edges in $G'[N]$.

and $\mathcal{N}(x,y) = \{N_i^j(x,y) : 1 \le i,j \le 2\}$. Furthermore, let

$$F(x,y) = \overrightarrow{E}_G(N_2^2(x,y), N(x,y)) \cup \overrightarrow{E}_G(N(x,y), N_1^1(x,y)) \cup \bigcup_{A \in \mathcal{N}(x,y)} E(G[A]),$$

and note that for every $e \in F(x,y)$ both xe and ye are transitive triangles, so e corresponds to two distinct homomorphisms in $\mathcal{C}_1(x,y)$. Therefore, if $|F(x,y)| \ge (\beta n)^2/2$, then x and y are 1-linked.

Claim 1. *For any pair $x, y \in V(G)$, if there exists $A \in \mathcal{N}(x,y)$ such that $|A| \ge (2/9 + \beta + 2\varepsilon)n$, then x and y are 1-linked.*

Proof. We have that $|F(x,y)| \ge e_G(A) \ge |A|(\delta(G) + |A| - n)/2 > (\beta n)^2/2$. □

For a contradiction, assume that there exists a pair $x, y \in V(G)$ that are not 2-linked and define $F = F(x,y)$. Therefore, since this implies that x and y are not 1-linked, we have that $|F| < (\beta n)^2/2$. Let

$$N^0 = \{v \in N(x,y) : |E_G(v) \cap F| \ge \beta n\}$$

and note that $|N^0| < \beta n$. Let $G' = G - F$, $N = N(x,y) \setminus N^0$, $N_i^j = N_i^j(x,y) \setminus N^0$ for every $1 \le i,j \le 2$, and $\mathcal{N} = \{N_i^j : 1 \le i,j \le 2\}$ (see Figure 3).

Claim 2. *For every $u \in N$, there exist at least two distinct sets $A, B \in \mathcal{N}$ such that both $|N_{G'}(u,A)| \ge 2\beta n$ and $|N_{G'}(u,B)| \ge 2\beta n$.*

Proof. If we suppose there exists only one such set $A \in \mathcal{N}$, then, using (1), we have that

$$|A| > |N_G(\{u,x,y\})| - |N^0| - |E_G(u) \cap F| - d_{G'}(u, N \setminus A) > (1/3 - 6\varepsilon - 8\beta)n > n/4$$

which, by Claim 1, contradicts the fact that x and y are not 1-linked. □

Let $I^+ = \{v \in N : d_G^+(v,N) < 2\beta n\}$, $I^- = \{v \in N : d_G^-(v,N) < 2\beta n\}$ and $I = I^+ \cup I^-$. Note that $N_1^1 \subseteq I^-$ and $N_2^2 \subseteq I^+$.

Claim 3. *Every pair $u, w \in I$ is 1-linked.*

Proof. Both u and w have all but $2\beta n$ of one type of neighbors (either in or out) in $\overline{N}$, so there exists $A \in \mathcal{N}(u,w)$ such that

$$|A| \ge 2(\delta^0(G) - 2\beta n) - |\overline{N}| \ge (7/9 - 2\varepsilon - 4\beta)n - (4/9 + 4\varepsilon + \beta)n > n/4.$$

Applying Claim 1 completes the proof. □

Claim 4. *If $A \in \mathcal{N}$ and $|A| \ge (1/18 + 5\varepsilon + 3\beta)n$, then every pair $u, w \in A$ is 1-linked.*

Proof. Since u and w both have all but βn of their neighbors in $\overline{A}$,

$$N_G(u,w) \ge 2(\delta(G) - \beta n) - |\overline{A}| \ge (14/9 - 4\varepsilon - 2\beta)n - (17/18 - 5\varepsilon - 3\beta)n = (11/18 + \beta + \varepsilon)n.$$

Let $B = N_1^1(u,w) \cup N_2^2(u,w)$ and $v \in B$. We have that

$$|E_G(v) \cap F(u,w)| \ge \delta^0(G) + |N_G(u,w)| - n \ge \beta n.$$

Therefore, if $|B| \ge \beta n$, then $|F(u,w)| \ge (\beta n)^2/2$ and u and w are 1-linked. If $|B| < \beta n$, then there exists $i \in \{1,2\}$ such that

$$|N_i^{3-i}(u,w)| \ge (|N_G(u,w)| - |B|)/2 > n/4$$

and the result follows from Claim 1. □

Suppose that there are $(\beta n)^2$ pairs $(a, b) \in N_1^1 \times N_2^2$. By (1) and the fact that a and b are each incident to at most βn edges from F,

$$|N_{G'}(a, b) \cap (N_1^2 \cup N_2^1)| \geq |N_G(\{a, b, x, y\})| - |N^0| - 2\beta n > \beta n.$$

Pick $c \in N_{G'}(a, b) \cap (N_1^2 \cup N_2^1)$ in one of βn ways. If $c \in N_1^2$, then xac and ybc are transitive triangles and if $c \in N_2^1$, then xbc and yac are transitive triangles. By Claim 3, a and b are 1-linked which implies that x and y are 2-linked, a contradiction.

So we can assume that there exists $i \in \{1, 2\}$ such that $|N_{3-i}^{3-i}| < \beta n$. For the rest of the argument we will assume $i = 1$, a similar argument will work for the case when $i = 2$. By Claim 2 and the fact that $|N_2^2| < \beta n$, there are at least $(\beta n)^2$ edges in $E_{G'}(N_1^2, N_2^1)$. Let ab be one such edge and pick $j \in \{1, 2\}$ so that $a \in N_j^{3-j}$ and $b \in N_{3-j}^j$.

Assume there are at least βn out-neighbors c of b in N_j^{3-j}. If $j = 1$, then xab and ybc are transitive triangles and if $j = 2$, then xbc and yab are transitive triangles. By Claim 1,

$$|N_j^{3-j}| \geq |N(x, y)| - |N^0| - \left|N_2^2\right| - 2(2/9 + \beta + 2\varepsilon)n > n/12.$$

With Claim 4, this implies that a and c are 1-linked. Therefore, x and y are 2-linked, a contradiction.

So assume that there are less than βn out-neighbors of b in N_j^{3-j}. Recall that

$$\overrightarrow{E}_G(b, N \setminus (N_j^{3-j} \cup N_2^2)) \subseteq F,$$

so b has less than $2\beta n$ out neighbors in N and hence, $b \in I^+$. By Claim 3, b is 1-linked with every vertex in N_1^1, and by Claim 2, we can pick $c \in N_{G'}(a, N_1^1)$ in $2\beta n$ ways. If $j = 1$, then xab and yac are transitive triangles and if $j = 2$, then xac and yab are transitive triangles. Therefore, since b and c are 1-linked, x and y are 2-linked, a contradiction. ☐

Lemma 2.2 (Absorbing Lemma). *For every $0 \leq \varepsilon \leq 1/250$, there exists $\sigma_0 = \sigma_0(\varepsilon)$ such that for every $0 < \sigma < \sigma_0$, there exists $n_0 = n_0(\varepsilon, \sigma)$ such that the following holds. If G is an oriented graph on $n \geq n_0$ vertices with $\delta^0(G) \geq (7/18 - \varepsilon)n$, then G contains a vertex set $U \subseteq V(G)$ with $|U| \leq 3\sigma n$ and $|U| \in 3\mathbb{Z}$ such that, for every $W \subseteq V(G)\setminus U$ with $|W| \leq 3\sigma^2 n$ and $|W| \in 3\mathbb{Z}$, $G[U \cup W]$ contains a perfect TT_3-tiling.*

Proof. Let $\tau = \tau(\varepsilon)$ be the constant given by Lemma 2.1 and let $\sigma_0 = \tau/(72^2 + 1)$ and let $0 < \sigma < \sigma_0$. Let G be sufficiently large oriented graph with $\delta^0(G) \geq (7/18 - \varepsilon)n$. Let $\mathcal{F}$ be the set of functions from [18] to $V(G)$. Call a map $f \in \mathcal{F}$ *absorbing* if there exists an ordered triple X of vertices such that $f \in \mathcal{A}(X)$.

Choose $\mathcal{U}' \subseteq \mathcal{F}$ by selecting each $f \in \mathcal{F}$ independently at random with probability $p = 2\sigma n^{-17}$. Call a pair $f, g \in \mathcal{F}$ *bad* if either f or g is not injective or the images of f and g intersect and note that there are less than $n \cdot \binom{36}{2} \cdot n^{34}$ bad pairs in $\mathcal{F}$. Therefore, the expected number of bad pairs in $\mathcal{U}'$ is less than $18 \cdot 35 \cdot 4\sigma^2 n$. Thus, using Markov's inequality, we derive that, with probability more than $1/2$, $\mathcal{U}'$ contains at most $(72\sigma)^2 n$ intersecting pairs.

By Chernoff's bound and Lemma 2.1, with positive probability the set $\mathcal{U}'$ also satisfies $|\mathcal{U}'| \leq 3\sigma n$ and $|\mathcal{A}(X) \cap \mathcal{U}'| \geq \tau\sigma n$ for each ordered triple X of vertices. By deleting every bad pair from $\mathcal{U}'$ and any $f \in \mathcal{U}'$ for which f is not absorbing, we get a $\mathcal{U} \subset \mathcal{U}'$ consisting of injective homomorphisms with pairwise disjoint images. Moreover, for each ordered triple X of vertices, there are at least $\tau\sigma n - (72\sigma)^2 n > \sigma^2 n$ functions in $\mathcal{A}(X) \cap \mathcal{U}$. Let U be the union of the images of every $f \in \mathcal{U}$. Since $\mathcal{U}$ consists only of absorbing functions, $G[U]$ has a perfect TT_3-tiling, so $|U| \in 3\mathbb{Z}$. For any set $W \subseteq V \setminus U$ of size $|W| \leq 3\sigma^2 n$ and $|W| \in 3\mathbb{Z}$, W can be partitioned into at most $\sigma^2 n$ sets of size 3. Each such set can be arbitrarily ordered to give a triple X which then can be successively paired up with a different absorbing homomorphism $f \in \mathcal{A}(X) \cap \mathcal{U}$. Therefore, $G[U \cup W]$ contains a perfect TT_3-tiling. □

2.2. **Almost TT_3-tilings.** In the following theorem and lemma, which we prove simultaneously, we show that if $\delta^0(G) \geq 7n/18$ then there is a TT_3-tiling on all but at most 11 vertices of G, and if $\delta^0(G)$ is slightly less than $7n/18$ then there is a TT_3-tiling on all but at most 14 vertices or G is α-extremal for some small $\alpha > 0$.

Lemma 2.3. *For any $\alpha > 0$ there exists $\varepsilon = \varepsilon(\alpha)$ such that the following holds. If G is an oriented graph on n vertices such that $\delta^0(G) \geq (7/18 - \varepsilon)n$, then either G has a TT_3-tiling on all but at most 14 vertices or G is α-extremal.*

Proof of Theorem 1.5 and Lemma 2.3. For the proof of Lemma 2.3, let $\varepsilon < \alpha/50$. For the proof of Theorem 1.5, let $\varepsilon = 0$. So, in either case, we have that $\delta^0(G) \geq (7/18 - \varepsilon)n$.

Let $\mathcal{M} = \mathcal{T} \cup \mathcal{P} \cup F \cup I$ be a collection of vertex disjoint subgraphs of G such that every vertex in G is contained in a subgraph of $\mathcal{M}$, every $T \in \mathcal{T}$ is a transitive triangle, every $P \in \mathcal{P}$ is a directed path on 3 vertices, every $e \in F$ is an edge and every $v \in I$ is a single vertex. Clearly such a set $\mathcal{M}$ exists. Assume that $\mathcal{M}$ is selected to maximize $(|\mathcal{T}|, |\mathcal{P}|, |F|)$ lexicographically. Let $X = V(\mathcal{T})$, $Y = V(\mathcal{P})$ and $Z = V(G) \setminus (X \cup Y)$.

We will show that if $\varepsilon = 0$, then $|\mathcal{P}| \leq 2$ and if $\varepsilon > 0$ and $|\mathcal{P}| \geq 4$, then G is α-extremal. We will also show that $|F| \leq 1$ and $|I| \leq 3$, and this will prove the theorem.

Let B be a $(\mathcal{P}, V(G))$ bipartite graph in which there is an edge between $P \in \mathcal{P}$ and $v \in V(G)$ if and only if $G[P \cup v]$ contains a transitive triangle. By Proposition 1.7, $d_B(P) \geq 2(3\delta^0(G) - n) \geq (1/3 - 6\varepsilon)n$. For every $P \in \mathcal{P}$, by the maximality of $|\mathcal{T}|$, $d_B(P, Y \cup Z) = 0$. Also by the maximality of $|\mathcal{T}|$, for every $T \in \mathcal{T}$ if there exists $x \in V(T)$ such that $d_B(x) \geq 2$, then $d_B(y) = 0$ for every $y \in V(T) - x$. Assume $|\mathcal{P}| \geq 3$ and note that we then have that $e_B(\mathcal{P}, V(T)) \leq |\mathcal{P}|$ for every $T \in \mathcal{T}$. Let

$$\mathcal{T}' = \{T \in \mathcal{T} : e_B(\mathcal{P}, V(T)) > 3\}.$$

We have that

$$n + (|\mathcal{P}| - 3)|\mathcal{T}'| > 3|\mathcal{T}| + (|\mathcal{P}| - 3)|\mathcal{T}'| \geq e_B(\mathcal{P}, V(G)) \geq (1/3 - 6\varepsilon)n|\mathcal{P}|.$$

Which, since $|\mathcal{T}'| < n/3$, is a contradiction when $\varepsilon = 0$, so in this case, we must have that $|\mathcal{P}| \leq 2$. If $\varepsilon > 0$ and $|\mathcal{P}| \geq 4$, then

$$|\mathcal{T}'| \geq \left(\frac{|\mathcal{P}|}{3(|\mathcal{P}| - 3)} - \frac{1}{|\mathcal{P}| - 3} - \frac{6|\mathcal{P}|\varepsilon}{|\mathcal{P}| - 3} \right) n \geq \left(\frac{1}{3} - 24\varepsilon \right) n.$$

For every $T \in \mathcal{T}'$, since $e_B(\mathcal{P}, T) \geq 4$, there exists $x_T \in V(T)$ such that $d_B(x_T) \geq 2$. Therefore, by the maximality of $|\mathcal{T}|$, $d_B(x_T) = e_B(\mathcal{P}, V(T)) \geq 4$. Let

$$W = Y \cup Z \cup \bigcup_{T \in \mathcal{T}'} (V(T) - x_T),$$

and note that $|W| > (2/3 - 48\varepsilon)\, n$. The graph $G[W]$ does not contain a transitive triangle. Indeed, if such a triangle T exists and we define $\mathcal{T}'' = \{T' \in \mathcal{T}' : V(T) \cap V(T') \neq \emptyset\}$ and $B' = B - \{P \in \mathcal{P} : V(P) \cap V(T) \neq \emptyset\}$, then for every $T' \in \mathcal{T}''$, we have that

$$d_{B'}(x_{T'}) \geq d_B(x_{T'}) - |Y \cap V(T)| \geq 4 - |Y \cap V(T)| > |X \cap V(T)| \geq |\mathcal{T}''|.$$

Therefore, there is a matching covering $\mathcal{T}''$ in B'. The edges in this matching correspond to $|\mathcal{T}''|$ disjoint transitive triangles in the graph induced by $(V(\mathcal{T}'')\cup Y\cup Z)\setminus V(T)$, contradicting the maximality of $|\mathcal{T}|$. Hence, G is α-extremal.

Assume that there exist two distinct edges ab and cd in F. For any set $U \subseteq V(G)$, define

$$w(U) = d_G^+(a, U) + \sum_{v\in\{b,c,d\}} d_G(v, U).$$

Note that by the maximality of $|\mathcal{P}|$, there are no triangles in $G[Z]$, so for every $z \in Z$, $w(z) \leq 2$. For any $P \in \mathcal{P}$, the maximality of $|\mathcal{T}|$ implies that there is no transitive triangle in the graph induced by $\{a, b, c, d\} \cup V(P)$. It is not hard to see that, with Proposition 1.6, this gives us that $d_G^+(a, V(P)) + d_G(b, V(P)) \leq 3$ and $e_G(cd, V(P)) \leq 4$, so $w(P) \leq 7$.

Claim. *For every $T \in \mathcal{T}$, $w(T) \leq 8$.*

Proof. Assume that there exists $T \in \mathcal{T}$ such that $w(T) \geq 9$. We will show that there exists a disjoint directed path on 3 vertices and a transitive triangle in the graph induced by $\{a, b, c, d\} \cup V(T)$, contradicting the maximality of $|\mathcal{P}|$.

Remove the edges into a from G to form G'. Note that this implies that for every $x \in V(T)$, if $e_{G'}(ab, x) = 2$, then abx is a transitive triangle. If, in addition to this, there exists $y \in V(T) - x$ such that $e_{G'}(cd, y) = 2$, then we have the desired P_3. This is the case when for one of the edges $e \in \{ab, cd\}$, $d_{G'}(e, V(T)) = 5$. Indeed, if $f \in \{ab, cd\} - e$, then $e_{G'}(f, V(T)) \geq 4$, so we can pick $x \in V(T)$ such that $e_{G'}(f, x) = 2$. Since then $e_{G'}(e, V(T) - x) \geq 3$, we can pick $y \in V(T) - x$ such that $e_{G'}(e, y) \geq 2$. Therefore, we are only left to consider the cases when one of ab or cd and $V(T)$ induce a tournament on 5 vertices in G'.

If $e_{G'}(ab, V(T)) = 6$, then $e_{G'}(cd, V(T)) \geq 3$ and for one of c or d, say c, $e_{G'}(c, V(T)) \geq 2$, so if x and y are the neighbors of c in $V(T)$, cxy is a triangle. If $z \in V(T) - x - y$, then zab is a transitive triangle.

If $e_{G'}(cd, V(T)) = 6$, then $e_{G'}(ab, V(T)) \geq 3$. We can assume that T is the unique transitive triangle in $G'[\{a, b\} \cup V(T)]$, because, if it was not, then the graph induced by vertices of $V(T)$ not also in this triangle and $\{c, d\}$ would contain a triangle, which contains a directed path on 3 vertices. This implies that $e_{G'}(ab, v) = 1$ for every $v \in V(T)$, and that $e_{G'}(a, V(T)) \leq 1$. This further implies that b has an out neighbor $x \in V(T)$, so abx is a directed path on 3 vertices. Since $G'[\{c, d\} \cup V(T) - x]$ is a tournament on 4 vertices, we have the desired transitive triangle by Proposition 1.6. □

Therefore,

$$\left(\frac{49}{18} - 7\varepsilon\right) n \leq 7\delta^0(G) \leq w(V(G)) \leq 2|Z| + 7|Y|/3 + 8|X|/3 \leq 8n/3,$$

a contradiction. Hence $|F| \leq 1$.

By the maximality of $|F|$, I is an independent set. Since there are no triangles in $G[Z]$, $e_G(I, e) \leq |I|$ for every $e \in F$. Let $T \in \mathcal{P} \cup \mathcal{T}$. If $e_G(I, V(T)) > 2|I| + 1$, then there exist vertices $v_1, v_2 \in I$ such that $e_G(v_1, V(T)) = e_G(v_2, V(T)) = 3$. Furthermore, in the graph induced by $\{v_1, v_2\} \cup V(T)$, if $T \in \mathcal{P}$, then there is a triangle and a disjoint edge, and if $T \in \mathcal{T}$, then there is a transitive triangle and a disjoint edge. Since both cases violate the maximality of $|F|$, we have that

$$|I|\left(\frac{7}{9}-2\varepsilon\right)n \le |I|\delta(G) \le e_G(I, V(G)\setminus I)$$
$$\le |I||F| + (2|I|+1)|\mathcal{P}\cup\mathcal{T}| \le \frac{|I||Z|}{2} + \frac{(2|I|+1)|X\cup Y|}{3} \le |I|\frac{2n}{3} + \frac{n}{3}.$$

Hence, $|I| \le 3 + 18\varepsilon$. □

Using Lemmas 2.2 and 2.3, we can prove Lemma 1.3.

Proof of Lemma 1.3. Let $\varepsilon = \min\{\varepsilon(\alpha)/2, 1/250\}$ where $\varepsilon(\alpha)$ is as in Lemma 2.3 and let $\sigma_0 = \sigma_0(\varepsilon)$ be as in by Lemma 2.2. Assume that n is sufficiently large. So, by Lemma 2.2, there exists $\sigma < \min\{\sigma_0, \varepsilon/3\}$ for which there exists $U \subseteq V(G)$ such that $|U| \le 3\sigma n < \varepsilon n$ and the conclusion of Lemma 2.2 holds. Let $G' = G - U$ and note that $\delta^0(G') \ge (7/18 - 2\varepsilon)n$. If we assume G is not α-extremal, then by Lemma 2.3 and the fact that $n \in 3\mathbb{Z}$, there exists a TT_3-tiling on all but a set W of size at most 12. Since $|W| < 3\sigma^2 n$ there exists a perfect TT_3-tiling of $G[U \cup W]$ completing the proof. □

3. The α-extremal case

In this section we prove Lemma 1.4. We start with some well-known and simple propositions regarding matchings in graphs.

Proposition 3.1. *Every graph G on n vertices has a matching of size at least* $\min\{\lfloor n/2 \rfloor, \delta(G)\}$.

Proof. Let M be a maximum matching in G and assume $|M| < \min\{\lfloor n/2 \rfloor, \delta(G)\}$. Let U be the set of vertices that are incident to an edge in M. Because $|M| \le n/2 - 1$, there exist distinct $x, y \in V(G) \setminus U$. Since M is a maximum matching, $e_G(\{x,y\}, V(G)\setminus U) = 0$ which implies

$$e_G(\{x,y\}, U) \ge 2\delta(G) > 2|M|.$$

So there exists $e \in M$ such that $e_G(\{x,y\}, e) \ge 3$. This contradicts the maximality of M. □

Proposition 3.2. *Let G be an (X,Y)-bipartite graph with $d_G(x) \ge a$ for every $x \in X$ and $d_G(y) \ge b$ for every $y \in Y$. If $|X| = |Y|$ and $a + b \ge |X|$, then G contains a perfect matching.*

Proof. We show that G satisfies Hall's condition. Let $X' \subseteq X$ be non-empty, let $x \in X'$ and let Y' be the set of vertices in Y that are adjacent to a vertex in X'. Clearly, $|Y'| \ge d_G(x) \ge a$, so assume $|X'| > a$. We have that $d_G(y) \ge b > |X \setminus X'|$ for every $y \in Y$. Hence, $y \in Y'$ and $|Y'| = |Y| \ge |X'|$. □

Let G be a (V_1, V_2)-bipartite graph. For $X_1 \subseteq V_1$ and $X_2 \subseteq V_2$ both non-empty, define $d_G(X_1, X_2) := \frac{e_G(X_1, X_2)}{|X_1||X_2|}$ to be the *density* of G. For constants $0 < \varepsilon, d < 1$, we say that G is (d,ε)-*regular* if

$$(1-\varepsilon)d \le d_G(X_1, X_2) \le (1+\varepsilon)d$$

whenever $|X_i| \ge \varepsilon|V_i|$ for $i = 1, 2$. We say that G is (d,ε)-*superregular* if G is (d,ε)-regular and $(1-\varepsilon)d|V_i| \le d_G(v, V_i) \le (1+\varepsilon)d|V_i|$ for every $v \in V_{3-i}$ and $i \in \{1,2\}$. Note that the preceding definitions match the corresponding definitions in [8].

Proposition 3.3. *For any $0 < \varepsilon < 1$, if G is a (V_1,V_2)-bipartite graph such that $|V_1| = |V_2| = n$ and $\delta(G) \ge (1-\varepsilon)n$ then G is $(1, \varepsilon^{1/2})$-superregular.*

Proof. It is clear that we only need to show that G is $(1, \varepsilon^{1/2})$-regular. Let $X_i \subseteq V_i$ such that $|X_i| \geq \varepsilon^{1/2} n$ for $i \in \{1, 2\}$. We have that

$$1 \geq d(X_1, X_2) \geq \frac{|X_1|(|X_2| - \varepsilon n)}{|X_1||X_2|} = 1 - \frac{\varepsilon n}{|X_2|} \geq 1 - \varepsilon^{1/2}. \qquad \square$$

The following theorem follows immediately from the Chernoff type bounds on the hypergeometric distribution (see Theorem 2.10 in [6]).

Theorem 3.4. *For every $0 < \eta < 1$ there exists $k = k(\eta) > 0$ such when V is a set, $X \subseteq V$ and m is a positive integer such that $m \leq |V|$ the following holds. If $|U|$ is selected uniformly at random from $\binom{V}{m}$, then with probability at least $1 - e^{-km}$*

$$\frac{|X|}{|V|} - \eta \leq \frac{|X \cap U|}{|U|} \leq \frac{|X|}{|V|} + \eta.$$

A partition of a set is *equitable* if the class sizes are differing by at most 1.

Proposition 3.5. *For every $\eta > 0$ there exist integers $k = k(\eta) > 0$ and $n_0 = n_0(\eta)$ such that when F is an (A, B)-bipartite graph with $|A| = |B| = n$ for $n \geq n_0$ the following holds. If an equitable partition $\{A_1, A_2\}$ of A and an equitable partition $\{B_1, B_2\}$ of B are both chosen uniformly at random from all such partitions, then with probability at least $1 - e^{-kn}$ we have*

$$d_F(A, B) - \eta \leq d_F(A_i, B_j) \leq d_F(A, B) + \eta$$

for every $1 \leq i, j \leq 2$.

Proof. Choose partitions $\{A_1, A_2\}$ and $\{B_1, B_2\}$ as in the proposition and let $k = 5k(\eta/2)$, where $k(\eta/2)$ is as in in Theorem 3.4, and assume that n is sufficiently large. Let $1 \leq i, j \leq 2$. By Theorem 3.4, for any $v \in A$ with probability at least $1 - e^{-5k|B_j|} \geq 1 - e^{-2kn}$

$$\frac{d_F(v, B)}{|B|} - \frac{\eta}{2} \leq \frac{d_F(v, B_j)}{|B_j|} \leq \frac{d_F(v, B)}{|B|} + \frac{\eta}{2}, \tag{2}$$

and the analogous statement holds for every $v \in B$. So with probability at least

$$1 - 2ne^{-2kn} \geq 1 - e^{-kn}$$

(2) holds for every $v \in V(G)$. Therefore,

$$\begin{aligned} d_F(A_i, B_j) &= \frac{\sum_{v \in A_i} d_F(v, B_j)}{|A_i||B_j|} \geq \frac{\sum_{v \in A_i} d_F(v, B)}{|A_i||B|} - \frac{\eta}{2} = \frac{\sum_{v \in B} d_F(v, A_i)}{|A_i||B|} - \frac{\eta}{2} \\ &\geq \frac{\sum_{v \in B} d_F(v, A)}{|A||B|} - \eta = d_F(A, B) - \eta. \end{aligned}$$

By a similar computation, the upper bound also holds. $\square$

Theorem 3.6 (Kühn and Osthus [8]). *For all positive constants $d, \xi_0, \eta \leq 1$ there is a positive $\varepsilon = \varepsilon(d, \xi_0, \eta)$ and an integer $n_0 = n_0(d, \xi_0, \eta)$ such that the following holds for all $n \geq n_0$ and all $\xi \geq \xi_0$. Let G be a (d, ε)-superregular bipartite graph whose vertex classes both have size n and let F be a subgraph of G with $e(F) = \xi e(G)$. Choose a perfect matching M uniformly at random in G. Then with probability at least $1 - e^{-\varepsilon n}$ we have*

$$\xi - \eta \leq \frac{|M \cap E(F)|}{|M|} \leq \xi + \eta.$$

Proposition 3.7. *Let G be an oriented graph and let $x \in V(G)$ and let $a, b, c \in N_G(x)$. If abc is a cyclic triangle in G, then xe is a transitive triangle for at least two edges $e \in \{ab, bc, ca\}$.*

Proof. Let $i = d^+_G(x, \{a, b, c\})$. By symmetry, there are four cases depending on the value of i. Furthermore, by reversing the edges of G it is easy to see that the cases when $i = j$ are equivalent to the cases when $i = 3 - j$. It is easy to verify the statement when $i = 3$ and when $i = 2$, we omit the details. □

We will use the following lemma to finish the proof of Lemma 1.4. Lemma 3.8 essentially states that if a graph looks very similar to the graph depicted in Figure 1 except that $|W_1| + |W_2| + |W_3| = 2(|U_1| + |U_2|)$, then there exists a perfect TT_3-tiling of G. As a final step, we show that if $\delta(G) \geq 7n/18$ and G is α-extremal, then we can remove a small number of vertex disjoint transitive triangles to leave a graph that looks like Figure 1 with the property that $|W_1| + |W_2| + |W_3| = 2(|U_1| + |U_2|)$.

We say that G is a *blow-up of a cyclic triangle* if G is an oriented graph on n vertices such that there exists a vertex partition W_1, W_2, W_3 with $|W_i| = \lfloor (n + i - 1)/3 \rfloor$ for $i \leq 3$ and every vertex in W_i contains W_{i+1} in its out-neighborhood.

For a vertex set V such that $|V| = 18m$ for some $m \in \mathbb{N}$, let $\mathcal{C}(V)$ be the collection of oriented graphs C on V for which there is a partition $\{U, W\}$ of V such that $|U| = 6m$, $|W| = 12m$, U is an independent set, $C[W]$ is a blow-up of a cyclic triangle and $d^+_C(w, U) = d^-_C(w, U) = 3m$ for every $w \in W$.

Lemma 3.8. *There exists a constant $\beta > 0$ and an integer n_0 such that for any $m \in \mathbb{N}$ where $n = 18m \geq n_0$ and any oriented graph G on n vertices the following holds. If there exists $C \in \mathcal{C}(V(G))$ such that $\Delta(C - G) \leq \beta m$, then G contains a perfect TT_3-tiling.*

Proof. Let $\eta = 1/12$, $\varepsilon = \varepsilon(1, \eta/2, \eta/2)$, $\beta = \min\{\varepsilon^2, 1/24\}$ where $\varepsilon(1, \eta/2, \eta/2)$ is as in Theorem 3.6. Assume m is sufficiently large. Let $\{U, W\}$ be a partition of $V(G)$ for which $|W| = 12m$ and $C[W]$ is a blow-up of a cyclic triangle with parts W_1, W_2 and W_3 each of order $4m$ such that every vertex in W_i has W_{i+1} in its out-neighborhood and $d^+_C(w, U) = d^-_C(w, U) = 3m$ for every $w \in W$. We may assume $G \subseteq C$, so there are no transitive triangles in $G[W]$. Let $F = E(G[W])$ and define a bipartite graph B with classes U, F as follows. A vertex $u \in U$ and an edge $xy \in F$ forms an edge in B if uxy is a transitive triangle in G.

Clearly $|F| \leq 3(4m)^2 = 48m^2$.

Claim. *For every $u \in U$, $d_B(u) \geq (2/3 - \beta)48m^2$.*

Proof. Let $P(u)$ be the set of pairs of the form (e, abc) where $a \in W_1 \cap N_G(u)$, $b \in W_2 \cap N_G(u)$ and $c \in W_3 \cap N_G(u)$, abc is a cyclic triangle and $e \in \{ab, bc, ac\} \cap N_B(u)$. By Proposition 3.7, for every $(e, abc) \in P(u)$ the cyclic triangle abc appears at least twice as the second element of a pair in $P(u)$. Therefore, because $\Delta(C - G) \leq \beta m$

$$|P(u)| \geq 2 \cdot (4 - \beta)m \cdot (4 - 2\beta)m \cdot (4 - 3\beta)m > (1 - 3\beta/2)128m^3.$$

Since any edge can appear as the first element in at most $4m$ of the pairs in $P(u)$,

$$d_B(u) \geq |P(u)|/(4m) \geq (1 - 3\beta/2)32m^2 = (2/3 - \beta)48m^2.$$ □

For every $u \in U$, let $F(u)$ be the graph on W with edge set $N_B(u)$. By Proposition 3.5 there exists an equitable partition of $\{W_i^1, W_i^2\}$ of W_i for each $i \in [3]$, such that for every $u \in U$,

$$d_{F(u)}(W_i^k, W_j^\ell) \geq d_{F(u)}(W_i, W_j) - \eta/2$$

for all $1 \leq k, \ell \leq 2$ and $j \in \{1,2,3\} - i$. Let $G_1 = G[W_1^1, W_2^1]$, $G_2 = G[W_2^2, W_3^2]$ and $G_3 = G[W_3^1, W_1^2]$. Note that $\delta(G_i) \geq (2-\beta)m$ for every $i \in [3]$, so G_i is $(1, \beta^{1/2})$-superregular by Proposition 3.3. Therefore, by Theorem 3.6, there exists a perfect matching M_i of G_i such that

$$\frac{|M_i \cap E(F(u))|}{2m} \geq \frac{|E(G_i) \cap E(F(u))|}{e(G_i)} - \frac{\eta}{2} \geq d_{F(u)}(W_i, W_{i+1}) - \eta$$

for every $u \in U$ and $i \in [3]$. Note that Theorem 3.6, does not apply when $|E(G_i) \cap E(F(u))|/e(G_i) \leq \eta/2$, but in that case the inequality is vacuously true. Observe $M = M_1 \cup M_2 \cup M_3$ is a perfect matching of $G[W]$ and for $B' = B[U, M]$, and every $u \in U$

$$\begin{aligned}\frac{d_{B'}(u)}{|M|} &= \frac{|M \cap E(F(u))|}{6m} = \frac{1}{3}\frac{\sum_{i=1}^{3}|M_i \cap E(F(u))|}{2m} \\ &\geq \frac{1}{3}\sum_{i=1}^{3}\big(d_{F(u)}(W_i, W_{i+1}) - \eta\big) = \frac{d_B(u)}{48m^2} - \eta \geq \frac{2}{3} - (\beta + \eta).\end{aligned}$$

We also have that for every $e \in M$, $d_{B'}(e) \geq (3 - 2\beta)m > (1/2 - \beta)6m$. Note that since

$$2/3 - (\beta + \eta) + 1/2 - \beta \geq 7/6 - 2\beta - \eta \geq 1,$$

Proposition 3.2 implies that B' has a perfect matching. This perfect matching corresponds to a perfect TT_3-tiling of G. □

Proof of Lemma 1.4. Let β be as in Lemma 3.8. Let $\tau = \beta/288$ and let $\alpha = \tau^3$.

Let $\gamma > 0$ and let $\mathcal{W} = \{W_1, W_2, W_3\}$ be a collection of three disjoint vertex subsets. We say that $v \in V(G)$ is (i,γ)-*cyclic* for the triple $\mathcal{W}$ if

$$d_G^+(v, W_{i-1}) + d_G(v, W_i) + d_G^-(v, W_{i+1}) \leq \gamma n,$$

and that v is γ-*cyclic* for $\mathcal{W}$ if v is (i,γ)-cyclic for some i. The triple $\mathcal{W}$ is γ-*cyclic* if every vertex in W_i is (i,γ)-cyclic for every $i \in [3]$. A vertex is γ-*bad* for $\mathcal{W}$ if it is not γ-cyclic. The following claim follows from the preceding definition.

Claim 1. *For any $1 > \gamma > \gamma' \geq 0$, if a vertex v is γ-bad for $\{W_1, W_2, W_3\}$ and $|X| \leq \gamma' n$, then v is $(\gamma - \gamma')$-bad for $\{W_1 \setminus X, W_2 \setminus X, W_3 \setminus X\}$.*

For any λ, we say that $\mathcal{W}$ is λ-*equitable* if

$$||W_i| - |W_j|| \leq \lambda n \text{ for every } i, j \in [3]$$

and $|V(\mathcal{W})| \geq (2/3 - \lambda)n$. Note that this implies that $|W_i| \geq (2/9 - \lambda)n$ for every $i \in [3]$.

Let $\mathcal{W} = \{W_1, W_2, W_3\}$ be a λ-equitable triple and let $v \in V(G)$ be (i,γ)-cyclic for $\mathcal{W}$. By the degree condition,

$$\begin{aligned}d_G^-(v, W_{i-1}) + d_G^+(v, W_{i+1}) &= d_G(v, V(\mathcal{W})) - \big(d_G^+(v, W_{i-1}) + d_G(v, W_i) + d_G^-(v, W_{i+1})\big) \\ &\geq |V(\mathcal{W})| - 2n/9 - \gamma n.\end{aligned}$$

Therefore, since $|W_1|, |W_2|, |W_3| \geq (2/9 - \lambda)n$, we have the following:

$$\begin{aligned}d_G^-(v, W_{i-1}) &\geq |W_{i-1}| + |W_i| - 2n/9 - \gamma n \geq |W_{i-1}| - (\gamma + \lambda)n, \\ d_G^+(v, W_{i+1}) &\geq |W_{i+1}| - (\gamma + \lambda)n \text{ and} \\ d_G^-(v, W_{i-1}), d_G^+(v, W_{i+1}) &\geq (2/9 - 2\lambda - \gamma)n.\end{aligned} \tag{3}$$

Claim 2. *Let $0 < \gamma < 1/27$ and let $\mathcal{W} = \{W_1, W_2, W_3\}$ be such that $\mathcal{W}$ is both γ-cyclic and γ-equitable. If $v \in V(G)$ such that there are no transitive triangles in $G[v \cup W]$ that contain v, then v is 0-cyclic for $\mathcal{W}$.*

Proof. Since $|V(\mathcal{W})| \geq (2/3 - \gamma)n > 11n/18$, there exists an $x \in N^+_G(v, W_{i+1})$ for some $i \in [3]$. Let $I_x = N^-_G(x, W_i)$. By (3), $|I_x| \geq (2/9 - 3\gamma)n$. Suppose that v is not $(i, 0)$-cyclic, i.e. there exists

$$y \in N^+_G(v, W_{i-1} \cup W_i) \cup N^-_G(v, W_i \cup W_{i+1}).$$

If $y \in N^+_G(v, W_{i-1} \cup W_i)$, then let $I_y = N^-_G(y, W_{i+1} \cup W_{i-1})$ and if $y \in N^-_G(v, W_i \cup W_{i+1})$, then let $I_y = N^+_G(y, W_{i+1} \cup W_{i-1})$. Again by (3), we have that $|I_y| \geq (2/9 - 3\gamma)n$. Note that v has no neighbors in $I_x \cup I_y$, because any such neighbor would imply a transitive triangle containing v in $G[v \cup W]$. Since I_x and I_y are disjoint,

$$|W| + \delta(G) - n \leq d_G(v, W) \leq |W| - |I_x| - |I_y| \leq |W| - (4/9 - 6\gamma)n < |W| - 2n/9$$

a contradiction. ☐

Recall, since G is α-extremal there exists $W \subseteq V(G)$ such that $|W| \geq (2/3 - \alpha)n$ and $G[W]$ does not contain any transitive triangles.

Claim 3. *There exists a 0-cyclic partition $\mathcal{W} = \{W_1, W_2, W_3\}$ of W such that for every $i \in [3]$*

$$(2/9 - \alpha)n \leq |W_i| \leq 2n/9.$$

Proof. Let $G' = G[W]$ and note that

$$\delta(G') \geq \delta(G) + |W| - n \geq |W| - 2n/9. \tag{4}$$

Since G' is TT_3-free, for every $v \in W$ the sets $N^+_{G'}(v)$ and $N^-_{G'}(v)$ are independent. This with (4) implies that both sets are of order at most $2n/9$ and hence that

$$\delta^0(G') \geq \delta(G') - 2n/9 \geq |W| - 4n/9. \tag{5}$$

Since G' is TT_3-free there exists a cyclic triangle $w_1w_2w_3$ in G'. This also implies that, for any $i \in [3]$, the set $\widetilde{W}_i = N^+_{G'}(w_{i-1}) \cup N^-_{G'}(w_{i+1})$ is disjoint from $N_{G'}(w_i)$. Hence, by (4), $|\widetilde{W}_i| \leq 2n/9$. Define $\widehat{W}_i = N^+_{G'}(w_{i-1}) \cap N^-_{G'}(w_{i+1})$. Then we have that $\widehat{W}_i$ is an independent set and, by (5),

$$2n/9 \geq |\widehat{W}_i| \geq d^+_{G'}(x_{i-1}) + d^-_{G'}(x_{i+1}) - |\widetilde{W}_i| \geq 2|W| - 10n/9 \geq (2/9 - 2\alpha)n.$$

Note that

$$\overrightarrow{E}_{G'}(\widehat{W}_{i-1}, \widehat{W}_{i+1}) \subseteq \overrightarrow{E}_{G'}(N^-_{G'}(w_i), N^+_{G'}(w_i)) = \emptyset.$$

This gives us that $\widehat{\mathcal{W}} = (\widehat{W}_1, \widehat{W}_2, \widehat{W}_3)$ is 0-cyclic.

Let $X = W \setminus V(\widehat{\mathcal{W}})$. By repeatedly applying Claim 2, we can iteratively add each $x \in X$ to a set $\widehat{W}_i$ for which x is $(i, 0)$-cyclic for $\widehat{\mathcal{W}}$. Let $\mathcal{W} = \{W_1, W_2, W_3\}$ be the resulting collection. For every $i \in [3]$, the set W_i is independent, so $|W_i| \leq 2n/9$ by (4) and moreover $|W_i| \geq (2/9 - \alpha)n$. So $\mathcal{W}$ is the desired partition of W. ☐

Let $U = V(G) \setminus W$. If $v \in V(G)$ is (i, γ)-cyclic for $\mathcal{W}$, then

$$\begin{aligned} d^+_G(v, U), d^-_G(v, U) &\geq \delta^0(G) - \max\{|W_{i+1}|, |W_{i-1}|\} - \gamma n \\ &\geq (1/6 - \gamma)n \geq |U|/2 - (\alpha/2 + \gamma)n. \end{aligned} \tag{6}$$

We also have that,

$$d_G(v, U) \geq \delta(G) - (|W \setminus W_i| + d_G(v, W_i)) \geq 7n/9 - 4n/9 - \gamma n = |U| - (\alpha + \gamma)n. \quad (7)$$

By Claim 3, we can apply (7) with $\gamma = 0$ to give us that

$$e_G(W, U) \geq (|U| - \alpha n)|W| > |U||W| - \alpha n^2.$$

Defining $Z = \{u \in U : d_G(u, W) < |W| - \tau n\}$, we have that, since $\tau^3 = \alpha$,

$$|Z| < \tau^2 n. \quad (8)$$

Let $Z(i)$ be the set of vertices in Z that are (i, τ)-cyclic for $\mathcal{W}$. Clearly $Z(1), Z(2)$ and $Z(3)$ are disjoint. Let $Z'' = \bigcup_{i=1}^{3} Z(i)$, $W_i' = W_i \cup Z(i)$, $\mathcal{W}' = (W_1', W_2', W_3')$, $W' = V(\mathcal{W}') = W \cup Z''$, $U' = U \setminus Z''$ and $Z' = Z \setminus Z''$. Note that, for every $i \in [3]$, $(2/9 - \alpha)n \leq |W_i'| \leq 2n/9 + |Z|$ so $\mathcal{W}'$ is $(2\tau^2)$-equitable and that every vertex in W_i' is (i, τ)-cyclic for $\mathcal{W}$. Since $|W' \setminus W| \leq |Z|$, this implies that $\mathcal{W}'$ is (2τ)-cyclic. We also have that for every $u \in U' \setminus Z'$,

$$d_G(u, W') \geq |W| - \tau n \geq |W'| - |Z| - \tau n \geq |W'| - 2\tau n. \quad (9)$$

We will now find three collections $\mathcal{T}_1$, $\mathcal{T}_2$, $\mathcal{T}_3$ of disjoint transitive triangles. We define $X_i = V(\mathcal{T}_i)$ and $Y_i = \bigcup_{j=1}^{i} X_j$. The collections will be constructed so that the sets X_1, X_2, X_3 are disjoint. The collections will also have the following properties:

(P1) $|W' \setminus Y_i| = 2|U' \setminus Y_i|$ for $i \in \{1, 2, 3\}$,
(P2) $Z' \subseteq X_2$,
(P3) $|Y_3| \leq \tau n$,
(P4) $|W_1' \setminus Y_3| = |W_2' \setminus Y_3| = |W_3' \setminus Y_3|$ and
(P5) $|V(G) \setminus Y_3|$ is divisible by 18.

Assume we have such collections. We claim that $G - Y_3$ then satisfies the conditions of Lemma 3.8 with $\beta = 16 \cdot 18\tau$. To see this, first note that every vertex in $W_i' \setminus Y_3$ is (i, τ)-cyclic for $\mathcal{W}$; the triple $(W_1' \setminus Y_3, W_2' \setminus Y_3, W_3' \setminus Y_3)$ is (2τ)-cyclic and (2τ)-equitable; and $(U' \setminus Y_2) \cap Z = \emptyset$. The conclusion then follows from (6), (3), and (9).

We begin the construction by finding a collection $\mathcal{T}_1$ such that $|W' \setminus X_1| = 2|U' \setminus X_1|$. Call a transitive triangle T *standard* if $|V(T) \cap W'| = 2$ and $|V(T) \cap U'| = 1$. Every transitive triangle $T \in \mathcal{T}_2 \cup \mathcal{T}_3$ will be standard and this will give us Property (P1).

Let $c = |W'| - 2n/3$ and note that $-\alpha n \leq c \leq |Z''| \leq \tau^2 n$, so $|c| \leq \tau^2 n$. Simple computations show that the following claim gives the desired collection $\mathcal{T}_1$. Indeed, if $c > 0$, then $|W'| - 3c = 2(n/3 - c) = 2|U'|$, and if $c < 0$, then $|W'| - |c| = 2(n/3 - |c|) = 2(|U'| - 2|c|)$.

Claim 4. *There exists a collection $\mathcal{T}_1$ of $|c|$ disjoint transitive triangles such that for every $T \in \mathcal{T}_1$,*

- *if $c > 0$, $T \subseteq G[W']$; and*
- *if $c < 0$, $|V(T) \cap W'| = 1$ and $|V(T) \cap (U \setminus Z)| = 2$.*

Proof. First assume $c > 0$ and let $I = \{i \in [3] : |W_i'| > 2n/9\}$ and $c_i = |W_i'| - \lfloor 2n/9 \rfloor$ for $i \in I$. Note that $c_i < \tau^2 n$. For every $i \in I$, by the degree condition, we have that $\delta(G[W_i']) \geq c_i$. Therefore, Proposition 3.1 implies that there exists a matching M_i of size c_i in $G[W_i']$. For every $xy \in M_i$, x and y are (i, τ)-cyclic and $\mathcal{W}'$ is an (τ^2)-equitable triple. So, by (3),

$$|N_G^-(x, W_{i-1}') \cap N_G^-(y, W_{i-1}')| \geq |W_{i-1}'| - 2(\tau + \tau^2)n,$$

and, similarly, $|N_G^+(x, W'_{i+1}) \cap N_G^+(y, W'_{i+1})| \geq |W'_{i+1}| - 2(\tau + \tau^2)n$. Therefore, we can easily match the edges $\bigcup_{i \in I} M_i$ to vertices in W' so that the matching corresponds to disjoint transitive triangles in G. Since $\sum_{i \in I} c_i \geq c$ we have the desired collection $\mathcal{T}_1$.

Now assume $c < 0$. Let $U'' = U \setminus Z = U' \setminus Z'$. By (8), we have that $|U''| \geq (1/3 - \tau^2)n$ so by the degree condition, $\delta(G[U'']) \geq (1/9 - \tau^2)n$ and there exists a matching M of order $|c| \leq \tau^2 n$ in $G[U'']$. By Proposition 1.7, every $e \in E(G)$ has $n/6$ vertices v such that ev is a transitive triangle. Therefore, we can match each edge $e \in M$ to a vertex $v_e \in V(G) \setminus Z$ so that the ev_e are disjoint transitive triangles. Let $\mathcal{T}'_1$ be this collection of transitive triangles. Suppose that there exists $T \in \mathcal{T}'_1$ such that $V(T) \subseteq U''$. By Proposition 1.6 and the fact that, by (9), $|\bigcap_{v \in V(T)} N_G(v, W')| \geq |W'| - 6\tau n$, it is trivial to replace T with a transitive triangle that has one vertex in W' and an edge from $E(T)$ and is also disjoint from $V(\mathcal{T}'_1 - T)$. By replacing every such triangle in this manner, we can create the desired collection $\mathcal{T}_1$. □

We now aim to find a collection $\mathcal{T}_2$ of standard transitive triangles that satisfies Property (P2). Note that, by the definition of Z', every vertex in Z' is τ-bad for $\mathcal{W}$ and hence is τ-bad for $\mathcal{W}'$. The following claim then follows from Claim 1 and Claim 2.

Claim 5. *There exists a collection $\mathcal{T}_2$ of $|Z' \setminus X_1|$ disjoint standard transitive triangles in $G - X_1$ such that $|T \cap Z'| = 1$ for every $T \in \mathcal{T}_2$.*

Proof. Let $\mathcal{T}_2$ be a collection of disjoint standard transitive triangles in $G - X_1$ such that for every $T \in \mathcal{T}_2$, $|V(T) \cap Z'| = 1$. Let $Y_2 = V(\mathcal{T}_2) \cup X_1$. Suppose that $|\mathcal{T}_2|$ is maximal among all such collections and that there exists $z \in Z' \setminus Y_2$. Since z is τ-bad for $\mathcal{W}'$, by Claim 1 and the fact that $|Y_2| < |X_1| + 3|Z'| < 6\tau^2 < \tau n$, z is 0-bad for $\{W'_1 \setminus Y_2, W'_2 \setminus Y_2, W'_3 \setminus Y_2\}$. Hence, by Claim 2, there exists a transitive triangle containing z and two vertices in $V(\mathcal{W}') \setminus Y'_2$. Adding T to $\mathcal{T}_2$ contradicts the maximality of $|\mathcal{T}_2|$. □

Let $W''_i = W'_i \setminus Y_2$ for every $i \in [3]$. Since $\mathcal{W}'$ is $(2\tau^2)$-equitable and $|Y_2| \leq |X_1| + 3|Z'| \leq 6\tau^2$, the collection $\mathcal{W}'' = \{W''_1, W''_2, W''_3\}$ is $(8\tau^2)$-equitable.

Because $|\mathcal{T}_1 \cup \mathcal{T}_2| \leq 2|Z| \leq 2\tau^2 n$, if we we can find a collection $\mathcal{T}_3$ of at most $17\tau^2 n \leq \tau n/3 - 2\tau^2 n$ disjoint standard transitive triangles in $G - Y_2$ that satisfies (P4) and (P5), we will also satisfy Property (P3). This is quite easy to do, as we now describe.

Let π be a permutation of [3] such that $|W''_{\pi(1)}| \leq |W''_{\pi(2)}| \leq |W''_{\pi(3)}|$. Let M_1, M_2 and M_3 be disjoint edge sets such that their union is a matching and

- $|M_1| = |W''_{\pi(3)}| - |W''_{\pi(2)}|$, $|M_2| = |W''_{\pi(3)}| - |W''_{\pi(1)}|$,
- $M_1 \subseteq E_G(W''_{\pi(3)}, W''_{\pi(1)})$, $M_2 \subseteq E_G(W''_{\pi(3)}, W''_{\pi(2)})$ and
- M_3 consists of three edges, one from each of $E(W''_i, W''_{i+1})$ for $i \in [3]$.

Let $M' = M_1 \cup M_2$ and $M = M' \cup M_3$. Note that since $\mathcal{W}''$ is $(8\tau^2)$-equitable, $|M'| < |M| \leq 2(8\tau^2 n) + 3 \leq 17\tau^2 n$. Let $vv' \in M$. Since v and v' are both τ-cyclic for $\mathcal{W}$, (6) and (7) give us that the number of vertices $x \in U$ such that xvv' is a transitive triangle is at least

$$|N_G^-(v, U) \cap N_G(v', U)| \geq n/6 - \alpha n - 2\tau n.$$

Therefore, we can find the desired collection $\mathcal{T}_3$ by matching edges in either M or M' to unused vertices in U' in the graph B. We can clearly satisfy Property (P4). Note that Properties (P1) and (P4) imply that $|V(G) \setminus Y_3| \in 9\mathbb{Z}$. So we can satisfy Property (P5) by picking M or M' appropriately. □

Acknowledgement. We are thankful to Wojciech Samotij, who participated in many fruitful discussions in the beginning of the project.

This research was carried out whilst the second author was visiting the Department of Mathematics of the University of Illinois at Urbana–Champaign. This author would like to thank the department for the hospitality he received.

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