An extended lower bound on the number of $(\leq k)$-edges to generalized configurations of points and the pseudolinear crossing number of K_n

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Abstract

Recently, Aichholzer, García, Orden, and Ramos derived a remarkably improved lower bound for the number of $(\leq k)$-edges in an n-point set, and as an immediate corollary an improved lower bound on the rectilinear crossing number of K_n. We use simple allowable sequences to extend all their results to the more general setting of simple generalized configurations of points and slightly improve the lower bound on Sylvester's constant from 0.37963 to 0.379688. In other words, we prove that the pseudolinear (and consequently the rectilinear) crossing number of K_n is at least $0.379688\binom{n}{4} + O\left(n^3\right)$. We use this to determine the exact pseudolinear crossing numbers of K_n and the maximum number of halving pseudolines in an n-point set for $n = 17, 19$, and 21. All these values coincide with the corresponding rectilinear numbers obtained by Aichholzer et al.

1 Introduction

A *pseudoline* is a simple closed curve whose removal does not disconnect the real projective plane. A *simple arrangement of pseudolines* is a set of pseudolines where every two of them intersect exactly once (where they cross) and no three of them meet in a common point. A *simple generalized configuration of points* consists of a set S of points in the plane together with a simple arrangement of pseudolines, each spanning exactly two points of S. When these pseudolines are all straight lines, the generalized configuration is completely determined by the point-set S, which must be in general position. Such a simple generalized configuration corresponds then to a configuration of points in general position in the plane. Concepts like k-sets, halving lines, rectilinear crossing numbers, etc., that have been studied in the *geometric* setting, i.e., for configurations of points in general position in the plane, can be extended to the *pseudolinear* setting, i.e., for simple generalized configurations (see Section 2). Goodman and Pollack [11], [12, 13] showed a duality between the family of simple generalized configurations of points and *simple allowable sequences*, defined in the next section.

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We use simple allowable sequences to approach two important, closely related, open problems in discrete geometry: the determination of the minimum number of $(\leq k)$-edges in a configuration of n points, and the determination of the rectilinear crossing number of K_n, denoted by $\overline{cr}(K_n)$. The close relationship between these problems was independently unveiled by Ábrego and Fernández-Merchant [1], and by Lovász, Vesztergombi, Wagner, and Welzl [14]. For a concise, up-to-date overview on these and related problems see the monograph [10].

A recent major breakthrough from Aichholzer et al. [6] is an improved lower bound on the minimum number of $(\leq k)$-edges and, as an immediate corollary, the improved lower bound $\overline{cr}(K_n) \geq 0.37962\binom{n}{4} + O(n^3)$. We show that both results hold in the pseudolinear setting. Specifically, if $\chi_{\leq k}(n)$ denotes the minimum number of $(\leq k)$-*pseudoedges* in a simple generalized configuration on n points and $\widetilde{cr}(K_n)$ denotes the *pseudolinear crossing number of* K_n (precise definitions are given in Section 2), then we prove $\chi_{\leq k}(n) \geq 3\binom{k+2}{2} + 3\binom{k-\lfloor n/3\rfloor+2}{2} - O(n)$ and $\widetilde{cr}(K_n) \geq 0.37962\binom{n}{4} + O(n^3)$. Furthermore, using a result by Balogh and Salazar [9] we further improve this bound to $\widetilde{cr}(K_n) \geq 0.379688\binom{n}{4} + O(n^3)$. In an earlier version of [6], Aichholzer et al. used a weaker result by Lovász et al. [14], valid only for geometric configurations, to improve the 0.37962 to 0.37963. After learning from our work they have incorporated our improvement on their final version. Using different techniques the 0.379688 bound was proved in [3] for a subclass of simple generalized configurations called 3-decomposables. We conjectured in [3] that all crossing-optimal configurations are in fact 3-decomposable.

The value 0.379688 above is a lower bound on Sylvester's constant [16] defined as $v = \lim_{n\to\infty} \overline{cr}(K_n)/\binom{n}{4}$. Scheinerman and Wilf [15] proved that v is the infimum, over all open sets R with finite area, of the probability that four randomly chosen points in R are in convex position.

Finally, as it was done for the geometric case [6], we can determine the values $\widetilde{cr}(K_{17}) = 798, \widetilde{cr}(K_{19}) = 1318$, and $\widetilde{cr}(K_{21}) = 2055$. In [3] we were able to show that $\widetilde{cr}(K_n) = \overline{cr}(K_n)$ for $n \in \{10, 11, 12, 13, 15\}$ and the new exact values support our conjecture that the rectilinear and the pseudolinear crossing numbers of K_n coincide for all n. Furthermore, as a consequence, we can determine the maximum number $\widetilde{h}(n)$ of *halving pseudolines* ($\lfloor(n-2)/2\rfloor$-pseudoedges) spanned by a simple generalized configuration on n points for $n = 17, 19$, and 21.

Besides the generalizations and improvements mentioned above, this paper is an attempt to show the simplicity that allowable sequences bring to these problems. Surely the best example of these simplifications is provided by comparing the proofs of Theorem 10 in [6] and its generalization, our Proposition 5 (which are the key results in both papers).

2 Simple allowable sequences, $(\leq k)$-pseudoedges, and $\widetilde{\mathrm{cr}}(K_n)$

A *simple allowable sequence* $\mathbf{\Pi}$ is a doubly-infinite sequence $(..., \pi_{-1}, \pi_0, \pi_1, ...)$ of permutations on n elements, such that any two consecutive permutations differ by a transposition of neighboring elements, and such that for every i, π_i is the reverse permutation of $\pi_{i+\binom{n}{2}}$. Thus $\mathbf{\Pi}$ has period $2\binom{n}{2}$, and any of its *halfperiods* contains all necessary information to reconstruct $\mathbf{\Pi}$.

Let $\Pi = \{\pi_0, \pi_1, \ldots, \pi_{\binom{n}{2}}\}$ be a halfperiod of a simple allowable sequence $\mathbf{\Pi}$. Thus, for each $i \geq 1$, π_{i-1} differs from π_i by a transposition of adjacent elements, whose *initial* and *final* permutations are π_{i-1} and π_i, respectively. A transposition of Π (or of any subsequence of a halfperiod of $\mathbf{\Pi}$) is a *k-transposition* if it swaps elements in positions k and $k+1$, and a ($\leq k$)-*transposition* if it is an i-transposition for either some $i \leq k$ or some $i \geq n-k$. Let $\mathcal{N}_k(\Pi)$ (respectively $\mathcal{N}_{\leq k}(\Pi)$) denote the set of all k-transpositions (respectively, ($\leq k$)-transpositions) in Π. Let $N_k(\Pi) := |\mathcal{N}_k(\Pi)|$, and $N_{\leq k}(\Pi) := |\mathcal{N}_{\leq k}(\Pi)|$.

A drawing of a graph G is *pseudolinear* if each edge can be extended to a pseudoline in such a way that the resulting set is a simple arrangement of pseudolines. The *pseudolinear* crossing number $\widetilde{cr}(G)$ of G is the minimum number of edge crossings in a pseudolinear drawing of G. In a simple generalized configuration S of n points, a j-*pseudoedge*, for $0 \leq j \leq \left\lfloor \frac{n-2}{2} \right\rfloor$, is a pseudoline (in S) spanned by points $p, q \in S$, that divides $S \setminus \{p,q\}$ into two sets, one of size j and one (obviously) of size $n-j-2$. A ($\leq k$)- *pseudoedge* is a j-pseudoedge with $j \leq k$. We denote by $\chi_{\leq k}(S)$ the number of ($\leq k$)-pseudoedges in S. Goodman [11] established the duality between simple allowable sequences and simple generalized configurations of points (see also the classic papers [12, 13]). Under this setting, each ($\leq k$)-pseudoedge corresponds to a ($\leq k+1$)-transposition. That is, if Π is a halfperiod of the simple allowable sequence generated by S then $\chi_{\leq k}(S) = N_{\leq k+1}(\Pi)$.

Ábrego and Fernández-Merchant [1] (and although not explicitly stated there, this also follows from [14]) used this correspondence to derive an expression for the number of crossings in a pseudolinear drawing in terms of its corresponding allowable sequence. They proved that if S is a simple generalized configuration of n points, and $\mathcal{D}$ is the pseudolinear drawing of K_n induced by S, then the number $\widetilde{\text{cr}}(\mathcal{D})$ of crossings in $\mathcal{D}$ satisfies $\widetilde{\text{cr}}(\mathcal{D}) = \sum_{1\leq k\leq (n-2)/2}(n-2k-3)\chi_{\leq k-1}(S) + \Theta(n^3)$. If Π is any halfperiod of the simple allowable sequence $\mathbf{\Pi}$ defined by S, then $\widetilde{\text{cr}}(\mathcal{D}) = \sum_{1\leq k\leq (n-2)/2}(n-2k-1)N_{\leq k}(\Pi) + \Theta(n^3)$. As $\widetilde{\text{cr}}(K_n)$ is the minimum of $\widetilde{\text{cr}}(\mathcal{D})$ over all pseudolinear drawings $\mathcal{D}$ of K_n, it follows that

$$\widetilde{\text{cr}}(K_n) \geq \min_{\Pi} \sum_{1\leq k\leq (n-2)/2} (n-2k-1)N_{\leq k}(\Pi) + \Theta(n^3), \tag{1}$$

where the minimum is taken over all halfperiods Π on n points.

In the geometric setting, if S is a configuration of n points in general position in the plane and $0 \leq k \leq \left\lfloor \frac{n-2}{2} \right\rfloor$, then a k-*edge* is a segment spanned by two points of S such that exactly k points of S lie in one open halfplane defined by the segment. The *rectilinear crossing number* of a graph G, denoted by $\overline{cr}(G)$, is the minimum number of crossings in a drawing of G where the set of vertices is in general position and the edges are straight line segments. As expected, in the particular case in which a simple generalized configuration of points corresponds to a configuration of n points in general position in the plane (as described in the previous section), k-pseudoedges agree with k-edges. Also $\widetilde{cr}(G) \leq \overline{cr}(G)$ and thus any lower bound for $\widetilde{cr}(G)$ is a lower bound for $\overline{cr}(G)$.

3 The main result

Our main result is that the bound obtained in [6] for ($\leq k$)-edges still holds in the more general context of simple generalized configurations of points.

Theorem 1 *For any simple generalized configuration S of n points, the number $\chi_{\leq k}(S)$ of ($\leq k$)-pseudoedges of S satisfies*

$$\chi_{\leq k}(S) \geq 3\binom{k+2}{2} + 3\binom{k-\lfloor n/3\rfloor+2}{2} - 3\max\left\{0, \left(k - \left\lfloor\frac{n}{3}\right\rfloor + 1\right)\left(\frac{n}{3} - \left\lfloor\frac{n}{3}\right\rfloor\right)\right\}.$$

We remark that a routine manipulation shows that the lower bound in Theorem 1 equals the lower bound $3\binom{k+2}{2}+\sum_{j=\lfloor n/3\rfloor}^{k}(3j-n+3)$ in [6].

Since any ($\le k$)-pseudoedge corresponds to a ($\le k+1$)-transposition, Theorem 1 is equivalent to the following.

Theorem 2 (Equivalent to Theorem 1) *For any halfperiod* Π *on* n *points, the number* $N_{\le k}(\Pi)$ *of* $(\le k)$*-transpositions of* Π *satisfies*

$$N_{\le k}(\Pi) \ge 3\binom{k+1}{2} + 3\binom{k-\lfloor n/3\rfloor+1}{2} - 3\max\left\{0, \left(k-\left\lfloor\frac{n}{3}\right\rfloor\right)\left(\frac{n}{3}-\left\lfloor\frac{n}{3}\right\rfloor\right)\right\}.$$

Using (1) and Theorem 2 we can accurately estimate the $\binom{n}{4}$ coefficient of $\widetilde{\mathrm{cr}}(K_n)$ by means of a definite integral.

$$\begin{aligned}\widetilde{\mathrm{cr}}(K_n) &\ge \binom{n}{4}\int_0^{1/2}(1-2x)\left(\frac{3}{2}x^2+\frac{3}{2}\max\left\{0,\frac{x-1}{3}\right\}^2\right)dx+\Theta(n^3)\\ &\ge \frac{41}{108}\binom{n}{4}+\Theta(n^3) = 0.37\overline{962}\binom{n}{4}+\Theta(n^3).\end{aligned}$$

The reader can consult [14] and [9] where similar calculations have been carried out in more detail. To get the strongest possible result we incorporate the lower bound for $N_{\le k}(\Pi)$ obtained by Balogh and Salazar (see Theorem 8 and Proposition 9 in [9]). This bound is better than the bound from Theorem 2 when $k > 0.4864n$.

Theorem 3 $\widetilde{\mathrm{cr}}(K_n) > 0.379688\binom{n}{4}+\Theta(n^3)$.

Using Theorem 2 for $n = 17, 19, 21$, we see that the vector of ($\le k$)-transpositions is bounded below, entry wise, by the vectors $(3, 9, 18, 30, 45, 64, 89, 136)$, $(3, 9, 18, 30, 45, 63, 86, 115, 171)$, and $(3, 9, 18, 30, 45, 63, 84, 111, 144, 210)$, respectively. The geometric constructions obtained by Aichholzer et al. [4, 5] match these lower bounds. Thus $\widetilde{\mathrm{cr}}(K_{17}) = 798$, $\widetilde{\mathrm{cr}}(K_{19}) = 1318$, and $\widetilde{\mathrm{cr}}(K_{21}) = 2055$. It follows that the number of halving pseudolines $\widetilde{h}(\Pi)$ in a halfperiod Π of a simple allowable sequence with n points is given by $\widetilde{h}(\Pi) = \binom{n}{2} - N_{\le\lfloor(n-2)/2\rfloor}(\Pi)$. Thus, as a consequence of Theorem 2, $\widetilde{h}(17) \le 47$, $\widetilde{h}(19) \le 56$, and $\widetilde{h}(21) \le 66$. Again the constructions by Aichholzer et al. [4, 5] match these bounds.

4 Proof of Theorems 2 and 3

If $\pi_i = (\pi_i(1), \dots, \pi_i(s), \dots, \pi_i(t), \dots, \pi_i(n))$, and $1 \le a < b \le n$, then we let $\pi_i[a, b]$ denote the subpermutation $(\pi_i(a), \dots, \pi_i(b))$, and π_i^{-1} is the permutation $(\pi_i(n), \pi_i(n-1), \dots, \pi_i(1))$.

We extend the definition of $\mathcal{N}_k(\Pi)$ to any subsequence Π' of Π: $\mathcal{N}_k(\Pi')$ is the set of k-transpositions of Π *whose final permutation is in* Π'. Clearly there is no conflict with this definition if we regard Π as a subsequence of itself. Moreover, if Π is partitioned into $\Pi_0, \Pi_1, \dots, \Pi_r$, so that Π is the concatenation $\Pi_0\Pi_1\dots\Pi_r$, then $\mathcal{N}_k(\Pi)$ equals the disjoint union $\bigcup_i \mathcal{N}_k(\Pi_i)$.

An *extreme point* of Π is one that occupies positions 1 or n in some π_i.

If Π and $\overline{\Pi}$ are halfperiods (of possibly different simple allowable sequences on n points) such that $N_{\le k}(\overline{\Pi}) \le N_{\le k}(\Pi)$ for every $k = 1, \dots, \lfloor n/2\rfloor$, then we write $\overline{\Pi} \preceq \Pi$.

In order to give a self-contained proof of Theorem 2, we include a proof of the following result, which was proved in [8].

Lemma 4 *Any halfperiod minimal with respect to $\preceq$ has exactly 3 extreme points.*

Proof. Let $\Pi = (\pi_0, \pi_1, \ldots, \pi_{\binom{n}{2}})$ be a halfperiod of a simple allowable sequence $\mathbf{\Pi}$, minimal with respect to $\preceq$, where $\pi_i = (\pi_i(1), \pi_i(2), \ldots, \pi_i(n))$. We note it suffices to show that $\pi_0(1)$ and $\pi_0(n)$ swap either when $\pi_0(1)$ is in position 1 or when $\pi_0(n)$ is in position n.

We may assume without any loss of generality (otherwise work instead with the halfperiod $(\pi_0^{-1}, \pi_1^{-1}, \ldots, \pi_{\binom{n}{2}}^{-1})$) that in Π the element $\pi_0(1)$ reaches position $\lceil n/2 \rceil$ (say in permutation π_ℓ) before $\pi_0(n)$ reaches position $\lfloor n/2 \rfloor + 1$. We claim that $\pi_0(1)$ swaps with the elements $\pi_\ell(1), \pi_\ell(2), \ldots, \pi_\ell(\lceil n/2 \rceil - 1)$ in the given order. Seeking a contradiction, suppose this is not the case. Let x, y be the first pair that swaps after $\pi_0(1)$ has swapped (in this order) with both x and y. Note that Π may be modified, if necessary, without losing its $\preceq$-minimality, so that the swap between $\pi_0(1)$ and x is put on hold until y is a neighbor of x. So we may assume that, in Π, just before $\pi_0(1)$ swapped with either x or y, x and y were neighbors. If we had swapped x and y back then, and kept Π otherwise unchanged, the result would be a halfperiod strictly $\preceq$-smaller than Π, a contradiction.

By the same argument, $\pi_0(1)$ must swap with the elements in $\pi_\ell(\lceil n/2 \rceil + 1), \pi_\ell(\lceil n/2 \rceil + 2), \ldots, \pi_\ell(n)$ in the given order. Thus, if $\pi_0(n) = \pi_\ell(n)$, we are done. Otherwise, $\pi_0(n) = \pi_\ell(i)$ for some i, $\lceil n/2 \rceil + 1 < i < n$. The argument in the previous paragraph shows that $\pi_0(n)$ must swap with the elements in $\pi_\ell(n), \pi_\ell(n-1), \ldots, \pi_\ell(i-1)$ in the given order. If instead of allowing $\pi_0(n)$ to move, we leave it in position n, so that it swaps there with $\pi_0(1)$, and then let it swap with all the elements in $\pi_\ell(n), \pi_\ell(n-1), \ldots, \pi_\ell(i-1)$, the result is a halfperiod strictly $\preceq$-smaller than Π, a contradiction. ∎

If $\Pi = (\pi_0, \ldots, \pi_{\binom{n}{2}})$ is a halfperiod, and s, t are nonnegative integers such that $s \le t \le \binom{n}{2}$, then we let $\Pi[s,t]$ denote the subsequence $(\pi_s, \ldots, \pi_t)$.

We now prove our version of the main ingredient for the improved bound in [6].

Proposition 5 *Let Π be a halfperiod on n points. Let s, t be integers, $0 \le s \le t \le \binom{n}{2}$, and $k < n/2$. Then $N_k(\Pi[0,s]) + N_{n-k}(\Pi[s+1,t]) + N_k(\Pi[t+1, \binom{n}{2}]) \ge 3k - n$.*

Proof. Let $U := \pi_0[1,k] \cap \pi_s[k+1,n]$, and $V := \pi_0[n-k+1,n] \cap \pi_s[n-k+1,n]$. It is straightforward to see that $N_k(\Pi[0,s]) \ge |U|$, $N_{n-k}(\Pi[s+1,t]) + N_k(\Pi[t+1, \binom{n}{2}]) \ge |V|$, and $|V| \ge k - ((n-2k) + |U|)$. The claimed inequality follows. ∎

Proof of Theorem 2. We proceed by induction on n. The base cases $n \le 3$ are trivial. If $k = 1$ the result is also trivially true. Let $n \ge 4$ and $k \ge 2$. We may assume $\Pi = (\pi_0, \ldots, \pi_{\binom{n}{2}})$ is $\preceq$-minimal. By Lemma 4, Π has exactly 3 extreme points, say p, q, and r. Let $m := \binom{n}{2}$. By considering, if necessary, another halfperiod of the doubly-infinite sequence generated by Π, without loss of generality q moves from position k to position $k-1$ from π_{m-1} to π_m, while r, p are at positions 1 and n, respectively. That is, $\pi_{m-1}(1) = \pi_m(1) = r, \pi_{m-1}(n) = \pi_m(n) = p$, and $\pi_{m-1}(k) = \pi_m(k-1) = q$. Since $\pi_m = (\pi_0)^{-1}$, then $\pi_0(1) = p, \pi_0(n) = r$, and $\pi_0(n-k+2) = q$. Thus the swaps between p, q, and r occur as follows: first q and r (at positions $n-1$ and n), then p and r (at positions 1 and 2), and finally p and q (at positions $n-1$ and n).

Let π_s be the permutation in which r first enters position $k-1$ (that is, $\pi_{s-1}(k) = \pi_s(k-1) = r$), and let π_t be the permutation in which p first enters position $n-k+2$ (that is, $\pi_{t-1}(n-k+1) = \pi_t(n-k+2) = p$). Clearly, $s < t$. Note that $\pi_s(1) = p, \pi_s(n) = q$, $\pi_t(1) = r$, and $\pi_t(n) = q$.

A transposition in Π that involves p, q, or r is a (p,q,r)-*transposition.*

Let $\Lambda := \lambda_0, \lambda_1, \ldots, \lambda_{\binom{n-3}{2}}$ be the halfperiod on $(n-3)$ obtained by removing from Π the $3n-6$ permutations that result from a (p,q,r)-transposition, and then removing p, q, and r from each of the remaining $\binom{n-3}{2}$ permutations. Let I denote the natural injection from Λ to Π (thus, for instance, $I(\lambda_0) = \pi_0$), and define $\iota : \{0, \ldots, \binom{n}{2}\} \to \{0, \ldots, \binom{n-3}{2}\}$ by the condition $\iota(i) = j$ iff $I(\lambda_i) = \pi_j$. Let s' be the largest i such that $\iota(i) < s$, and let t' be the largest j such that $\iota(j) < t$.

It is straightforward to check that if λ_i is a final permutation of a transposition in $\mathcal{N}_{\le k-2}(\Lambda) \cup \mathcal{N}_{\le k-1}(\Lambda[0, s']) \cup \mathcal{N}_{\le n-k-2}(\Lambda[s'+1, t']) \cup \mathcal{N}_{\le k-1}(\Lambda[t'+1, \binom{n-3}{2}])$, then $I(\lambda_i)$ is a non-(p,q,r)-transposition in $\mathcal{N}_{\le k}(\Pi)$. There are exactly $6k-3$ transpositions of the type (p,q,r) in $\mathcal{N}_{\le k}(\Pi)$, and so $N_{\le k}(\Pi) \ge N_{\le k-2}(\Lambda) + N_{\le k-1}(\Lambda[0, s']) + N_{\le n-k-2}(\Lambda[s'+1, t']) + N_{\le k-1}(\Lambda[t'+1, \binom{n-3}{2}]) + (6k-3)$.

By the induction hypothesis it follows that

$$N_{\le k-2}(\Lambda) \ge 3\binom{k-1}{2} + 3\binom{k - \lfloor n/3 \rfloor}{2} - 3\max\left\{0, \left(k - 1 - \left\lfloor \frac{n}{3} \right\rfloor\right)\left(\frac{n}{3} - \left\lfloor \frac{n}{3} \right\rfloor\right)\right\},$$

and by Proposition 5, $N_{k-1}(\Lambda[0, s']) + N_{n-k-2}(\Lambda[s'+1, t']) + N_{k-1}(\Lambda[t'+1, \binom{n-3}{2}]) \ge 3(k-1) - (n-3) = 3k - n$. Thus

$$\begin{aligned} N_{\le k}(\Pi) \ge\ & 3\binom{k+1}{2} + 3\binom{k - \lfloor n/3 \rfloor}{2} \\ & - 3\max\left\{0, \left(k - 1 - \left\lfloor \frac{n}{3} \right\rfloor\right)\left(\frac{n}{3} - \left\lfloor \frac{n}{3} \right\rfloor\right)\right\} + \max\{0, 3k - n\}. \end{aligned}$$

If $k < \lfloor n/3 \rfloor + 1$ then $N_{\le k}(\Pi) \ge 3\binom{k+1}{2}$, otherwise $k \ge \lfloor n/3 \rfloor + 1$ and

$$N_{\le k}(\Pi) \ge 3\binom{k+1}{2} + 3\binom{k + 1 - \lfloor n/3 \rfloor}{2} - 3\left(k - \left\lfloor \frac{n}{3} \right\rfloor\right)\left(\frac{n}{3} - \left\lfloor \frac{n}{3} \right\rfloor\right). \ \blacksquare$$

Proof of Theorem 3. Let $\widetilde{s}(x) = \lfloor (1/2)(1 + \sqrt{(1+6x)/(1-2x)}) \rfloor$ and

$$\widetilde{f}(x) = \left(2 - \frac{1}{\widetilde{s}(x)}\right)x^2 - \left(\frac{(\widetilde{s}(x)-1)^2}{\widetilde{s}(x)}\right)x\,(1-2x) + \left(\frac{\widetilde{s}(x)^4 - 7\widetilde{s}(x)^2 + 12\widetilde{s}(x) - 6}{12\widetilde{s}(x)}\right)(1-2x)^2.$$

Balogh and Salazar [9] proved that $N_k(\Pi) \ge \widetilde{f}(k/n) + O(n)$. Using (1) and Theorem 2 we get

$$\begin{aligned} \widetilde{\mathrm{cr}}(K_n) &\ge \binom{n}{4} \int_0^{1/2} (1-2x) \max\left\{\widetilde{f}(x), \frac{3}{2}x^2 + \frac{3}{2}\max\left\{0, \left(\frac{x-1}{3}\right)^2\right\}\right\} dx + \Theta(n^3) \\ &\ge \binom{n}{4} \int_0^{0.4864} (1-2x)\left(\frac{3}{2}x^2 + \frac{3}{2}\max\left\{0, \left(\frac{x-1}{3}\right)^2\right\}\right) dx \\ &\quad + \binom{n}{4} \int_{0.4864}^{1/2} (1-2x)\widetilde{f}(x) dx + \Theta(n^3) > 0.379688\binom{n}{4} + \Theta(n^3). \blacksquare \end{aligned}$$

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