Some Exact Ramsey-Turán Numbers

József Balogh [*]
University of Illinois at Urbana-Champaign
jobal@math.uiuc.edu

John Lenz
University of Illinois at Chicago
jlenz2@uic.edu

September 22, 2011

Abstract

Let r be an integer, $f(n)$ a function, and H a graph. Introduced by Erdős, Hajnal, Sós, and Szemerédi [8], the r-Ramsey-Turán number of H, $\mathbf{RT}_r(n, H, f(n))$, is defined to be the maximum number of edges in an n-vertex, H-free graph G with $\alpha_r(G) \leq f(n)$ where $\alpha_r(G)$ denotes the K_r-independence number of G.

In this note, using isoperimetric properties of the high dimensional unit sphere, we construct graphs providing lower bounds for $\mathbf{RT}_r(n, K_{r+s}, o(n))$ for every $2 \leq s \leq r$. These constructions are sharp for an infinite family of pairs of r and s. The only previous sharp construction was by Bollobás and Erdős [6] for $r = s = 2$.

1 Introduction

Let G be a graph and define the K_r*-independence number* of G as

$$\alpha_r(G) := \max \{|S| : S \subseteq V(G), G[S] \text{ is } K_r\text{-free}\}.$$

Define $\mathbf{RT}_r(n, H, f(n))$ to be the maximum number of edges in an H-free graph G on n vertices with $\alpha_r(G) \leq f(n)$ and let

$$\theta_r(H) = \lim_{\epsilon \to 0} \lim_{n \to \infty} \frac{1}{n^2} \mathbf{RT}_r(n, H, \epsilon n). \tag{1}$$

We write $\mathbf{RT}_r(n, H, o(n)) = \theta_r(H)n^2 + o(n^2)$. For $r = 2$, it is easy to show that the limit in (1) exists; for $r \geq 3$, its existence was proved when H is a complete graph in [7]. The r*-Ramsey-Turán number* of H is $\theta_r(H)$.

Turán's Theorem [18] states that the maximum number of edges in a K_r-free graph on n vertices is achieved by the complete $(r-1)$-partite graph. This extremal graph has independent sets with linear size, which motivated Erdős and Sós [10] to ask about the maximum number of edges in a K_r-free graph on n vertices with sublinear independence number. They solved this problem when r is an odd integer. The case when r is even has a more interesting history. Szemerédi [17] used an early version of the Szemerédi Regularity Lemma to upper bound $\theta_2(K_4)$ by $\frac{1}{8}$. This turned out to be sharp as four years later Bollobás and Erdős [6] constructed K_4-free graphs with $n^2/8 - o(n^2)$

[*]This material is based upon work supported by NSF CAREER Grant DMS-0745185, UIUC Campus Research Board Grants 11067, 09072 and 08086, and OTKA Grant K76099.

edges and sublinear independence number. Erdős, Sós, Hajnal, and Szemerédi [8] extended these results to determine $\theta_2(K_{2r})$ for all $r \geq 2$.

Another Ramsey-Turán result is an important and widely applicable theorem of Ajtai, Komlós, and Szemerédi [2]. They lower bounded the independence number of triangle-free, n-vertex graphs with m edges. Their result can be phrased as

$$\mathbf{RT}_2\left(n, K_3, \frac{cn^2}{m}\log\left(\frac{m}{n}\right)\right) < m \tag{2}$$

for some constant c. This result imples a sharp upper bound of $cn^2/\log n$ on the Ramsey number $R(3, n)$. Other applications of (2) include Ajtai, Komlós, and Szemerédi's [3] impovements on Erdős and Turán's [12] result on the existence of dense infinite Sidon sets. Recently, Fox [13] used (2) to find large clique-minors in graphs with independence number two. Hypergraph variants of (2) by Ajtai, Komlós, Pintz, Spencer and Szemerédi [1] have been applied by Komlós, Pintz and Szemerédi [14] in discrete computational geometry to provide a counterexample for Heilbronn's Conjecture. See [15] for a more detailed history of Ramsey-Turán numbers.

This paper focuses on the problem of determining $\theta_r(K_t)$ for $r \geq 3$, suggested by Erdős, Hajnal, Sós, and Szemerédi [8, p. 80] (see also [15, Problem 17]). Erdős, Hajnal, Simonovits, Sós, and Szemerédi [7] proved that $\theta_r(K_t) \leq \frac{1}{2}\left(1 - \frac{r}{t-1}\right)$ and this is best possible for all $t \equiv 1 \pmod{r}$. This left open the question when $t \not\equiv 1 \pmod{r}$, where they made partial progress for $s \leq \min\{5, r\}$.

Theorem 1. *For $2 \leq s \leq \min\{5, r\}$, $\mathbf{RT}_r(n, K_{r+s}, o(n)) \leq \frac{s-1}{4r}n^2 + o(n^2)$.*

Our main result is to construct for every $2 \leq s \leq r$ an infinite graph family providing near-optimal lower bounds for $\mathbf{RT}_r(n, K_{r+s}, o(n))$. In particular, we show that Theorem 1 is sharp when $4r/(s-1)$ is a power of 2. Earlier the only sharp construction was by Bollobás and Erdős [6] for $r = s = 2$.

Theorem 2. *Let $2 \leq s \leq r$. Let ℓ be the largest positive integer such that $\lceil r \cdot 2^{-\ell} \rceil < s$. Then*

$$\mathbf{RT}_r(n, K_{r+s}, o(n)) \geq 2^{-\ell-2}n^2.$$

For example, it yields that $\theta_4(K_6) = 1/16$ and $\theta_4(K_7) = 1/8$. We suspect that Theorem 2 should be best possible for all s when $4r/(s-1)$ is a power of 2; towards this direction we have only the following partial result extending Theorem 1.

Proposition 3. $\theta_{10}(K_{16}) = \theta_{12}(K_{19}) = \frac{1}{8}$.

The authors [4] recently proved $\theta_r(K_{r+2}) > 0$ for every $r \geq 2$. This resolved one of the main open questions from [7]. In [4] hypergraphs were constructed to estimate Ramsey-Turán numbers of some hypergraphs. Taking the shadow graphs of the constructed hypergraphs implied the results for graphs. Our proof builds on the techniques developed in [6] and [4] combined with several new ideas.

The remainder of this paper is organized as follows: in Section 2 we describe the construction for the graphs used to prove Theorem 2, in Section 3 we prove Theorem 2, and in Section 4 we list several open problems. The appendix contains a sketch of the proof of Proposition 3.

2 Construction

The construction for Theorem 2 builds on the Bollobás-Erdős Graph [5]. The reader is encouraged to read Section 4 and the first few paragraphs of Section 5 from [4], which provide overviews some of the previous constructions.

First we briefly sketch a few properties of the unit sphere. For more details, see Section 3 of [4]. Let μ be the Lebesgue measure on the k-dimensional unit sphere $\mathbb{S}^k \subseteq \mathbb{R}^{k+1}$ normalized so that $\mu(\mathbb{S}^k) = 1$. Given any $\alpha, \beta > 0$, it is possible to select $\epsilon > 0$ small enough and then k sufficiently large so that Properties (P1) and (P2) are satisfied.

(P1) Let C be a spherical cap in $\mathbb{S}^k$ with height h, where $2h = \left(\sqrt{2} - \epsilon/\sqrt{k}\right)^2$ (this means all points of the spherical cap are within distance $\sqrt{2} - \epsilon/\sqrt{k}$ of the center). Then $\mu(C) \geq \frac{1}{2} - \alpha$.

(P2) Let C be a spherical cap with diameter $2 - \epsilon/(2r^2\sqrt{k})$. Then $\mu(C) \leq \beta$.

To prove Theorem 2, it suffices to prove that for all integers $n, r \geq 2$, every $2 \leq s \leq r$, and every $\alpha, \beta > 0$, there exists an N-vertex graph $G = G(n, r, s, \alpha, \beta)$ such that G is K_{r+s}-free, $N \geq n$, and

$$|E(G)| \geq \left(2^{-\ell-2} - \alpha\right) N^2 \quad \text{and} \quad \alpha_r(G) < \beta N,$$

where ℓ is the largest positive integer such that $\lceil r \cdot 2^{-\ell} \rceil < s$.

Assume n, r, s, α, β, and ℓ are given as above, we shall show how to construct $G = G(n, r, s, \alpha, \beta)$. For the given α and β, there exists $\epsilon > 0$ and $k \geq 3$ such that properties (P1) and (P2) hold. Define $\theta = \epsilon/\sqrt{k}$ and $z = 2n$. Partition the k-dimensional unit sphere $\mathbb{S}^k$ into z domains having equal measures and diameter at most $\theta/4$. Choose a point from each set and let P be the set of these points. Let $\phi : P \to \mathcal{P}(\mathbb{S}^k)$ map points of P to the corresponding domains of the sphere. Before defining G, we construct some auxiliary bipartite graphs $B_1, \ldots, B_\ell$ and hypergraphs $\mathcal{H}$ and $\mathcal{H}'$.

The vertex set of the auxiliary bipartite graphs $B_1, \ldots, B_\ell$ is $[r]$, and the edges are built from the ℓ-dimensional hypercube Q_ℓ as follows. Blow up Q_ℓ into Q'_ℓ so that each vertex is blown up into an independent set of size $s-1$. Discard vertices of Q'_ℓ so that Q'_ℓ has exactly r vertices, discarding at most one vertex from each blow up class. (Note that ℓ was chosen so that Q_ℓ is the smallest hypercube with at least $r/(s-1)$ vertices.) Consider the vertices of Q_ℓ as labeled by binary words of length ℓ. If the $(2i+1)$-st discarded vertex is from the class labelled by $(a_1, \ldots, a_\ell)$, then the $(2i+2)$-nd vertex should be removed from the class labelled $(1 - a_1, \ldots, 1 - a_\ell)$. Denote by $A_{i,0}$ the subset of vertices of Q'_ℓ which come from a blowup of a vertex with its i-th coordinate zero. Similarly define $A_{i,1}$. The bipartite graph B_i is the complete bipartite graph with parts $A_{i,0}$ and $A_{i,1}$.

Now we define an r-uniform hypergraph $\mathcal{H}$ with vertex set P^ℓ, the family of ordered ℓ-tuples of elements of P. We let $\left\{\bar{x}^1, \ldots, \bar{x}^r\right\} \subseteq P^\ell$ be a hyperedge of $\mathcal{H}$ if for every $1 \leq i < j \leq r$ and $1 \leq a \leq \ell$ if $ij \in E(B_a)$ then $d(x^i_a, x^j_a) > 2 - \theta$. In other words, we form a hyperedge if the edges of B_a correspond to almost antipodal points on the sphere in the a-th vertex coordinate.

From $\mathcal{H}$, define a hypergraph $\mathcal{H}'$ by applying the following theorem to $\mathcal{H}$ with $\gamma = \beta$ and $k = r^3$.

Theorem 4 (Theorem 16 in [4]). *Let $\mathcal{H}$ be an r-uniform hypergraph on n vertices. Let $0 < \gamma < 1$ and let k be a positive integer. Then there exists a $t = t(\mathcal{H}, k, \gamma, r)$ and an r-uniform hypergraph $\mathcal{G}$ with vertex set $V(\mathcal{H}) \times [t]$ with the following properties.*

(i) *For all* $\{a_1, \ldots, a_r\} \in \mathcal{H}$ *and all sets* $U_i \subseteq \{a_i\} \times [t]$ *with* $|U_i| \geq \gamma t$ *for each* $1 \leq i \leq r$*, there exists at least one hyperedge of* $\mathcal{G}$ *with one vertex in each* U_i*.*

(ii) $\mathcal{G}$ *does not contain as a subhypergraph any* v*-vertex hypergraph* $\mathcal{F}$ *with* m *edges where* $v \leq k$ *and* $v + (1 + \gamma - r)(m - 1) < r$*.*

We are finally ready to define G. Let U and V be two distinct copies of $V(\mathcal{H}')$ and let the vertex set of G be $U \dot\cup V$. We place a copy of the shadow graph of $\mathcal{H}'$ on both $G[U]$ and $G[V]$. (The *shadow graph* of a hypergraph has the same vertex set and xy forms an edge of the shadow graph if x and y are contained together in some hyperedge.) Lastly, for $\bar{u} = \langle u_1, \ldots, u_\ell \rangle \in U$ and $\bar{v} = \langle v_1, \ldots, v_\ell \rangle \in V$ let $\bar{u}\bar{v}$ be an edge if $d(u_i, v_i) < \sqrt{2} - \theta$ for all $1 \leq i \leq \ell$.

This differs from the constructions in [4] in two important places. In [4], the cross-edges are defined when $d(u_i, v_j) < \sqrt{2} - \theta$ for all $1 \leq i, j \leq \ell$. By weakening this to only require $d(u_i, v_i) < \sqrt{2} - \theta$, the density of cross-edges is much larger. The cost is that here we need to work harder to show these new edges do not create copies of K_{r+s}. Secondly, where we used the auxiliary bipartite graphs B_i's in the construction, [4] used trees. The number of auxiliary graphs is ℓ, the number of coordinates in our vertices. The larger ℓ gets, the smaller the number of edges since each additional coordinate imposes more distance requirements on points. By switching from trees to bipartite graphs, we are able to use fewer coordinates. This makes $G[U]$ and $G[V]$ sparser, which forces a more complicated proof that G has small independence number.

3 Verifying properties of G

To complete the proof of Theorem 2, we need to prove three properties of G: G has at least $\left(2^{-\ell-2} - \alpha\right) N^2$ edges, G is K_{r+s}-free, and the K_r-independence number of G is smaller than βN.

3.1 The number of edges of G.

First we compute the number of vertices of G. The hypergraph $\mathcal{H}$ has z^ℓ vertices and each vertex in $\mathcal{H}$ is blown up into a set of size t so $\mathcal{H}'$ has tz^ℓ vertices. Thus G has $2tz^\ell$ vertices. To estimate the number of edges of G, we fix some vertex $x' \in U$; we will compute a lower bound on its degree in V. There exists a vertex x in $\mathcal{H}$ such that x' is contained in the blowup of x. For $y' \in V$ to be adjacent to x', we must have $d(x_i, y_i) \leq \sqrt{2} - \theta$ for all i. By Property (P1), there are at least $\left(\frac{1}{2} - \alpha\right) |P|$ points y_i that are within distance $\sqrt{2} - \theta$ of x_i. Thus there are at least $2^{-\ell} |P| - C\alpha |P|$ choices for y where C is some constant depending only on ℓ. Since each y is blown up into a set of size t, the degree of x' is at least $2^{-\ell} tz^\ell - C\alpha tz^\ell$. Thus

$$|E(G)| \geq \frac{|V(G)|}{2} \left(2^{-\ell} tz^\ell - C\alpha tz^\ell\right).$$

Since $tz^\ell = \frac{|V(G)|}{2}$,

$$|E(G)| \geq 2^{-\ell-2} |V(G)|^2 - C\alpha |V(G)|^2 / 2 = 2^{-\ell-2} |V(G)|^2 (1 - C\alpha/2).$$

Since C depends only on ℓ and $\alpha > 0$ can be chosen arbitrarily small, this gives the required bound.

3.2 G is K_{r+s}-free

First we need a couple of short lemmas.

Lemma 5. *$\alpha(\cup B_i) < s$.*

Proof. Fix any two vertices $x', y' \in V(Q'_\ell)$ and let x and y be the vertices of Q_ℓ such that x' and y' are contained in the blowups of x and y respectively. If $x \neq y$, then their binary labels differ in at least one position so there will be some B_i where x' and y' appear in different classes of the bipartition of B_i. Thus the independent sets in $\cup B_i$ are subset of the blowup of some vertex in Q_ℓ. Using that each vertex in Q_ℓ is blown up into a set of size at most $s-1$, the proof is complete. □

Lemma 6. *Let K_w be a complete w-vertex subgraph of $G[U]$. Then there exists a hyperedge E in $\mathcal{H}'$ such that $V(K_w) \subseteq E$.*

Proof. Let $K_w \subseteq G[U]$ and $V(K_w) = \{x_1, \ldots, x_w\}$. Since K_w is complete, for every i, j there exists some hyperedge $E_{i,j}$ of $\mathcal{H}'$ such that $E_{i,j}$ contains both x_i and x_j. If the $E_{i,j}$'s are not all the same hyperedge, then *(ii)* of Theorem 4 is violated. □

Lemma 7. *$G[U]$ (and similarly $G[V]$) is K_{r+1}-free.*

Proof. This is an immediate corollary of Lemma 6. Hyperedges in $\mathcal{H}$ have size at most r, so $G[U]$ does not contain any K_{r+1}. □

We now need the following property of the unit sphere observed by Bollobás and Erdős [5].

Theorem 8 (Bollobás-Erdős Rombus Theorem)**.** *For any $0 < \gamma < \frac{1}{4}$, it is impossible to have $p_1, p_2, q_1, q_2 \in \mathbb{S}^k$ such that $d(p_1, p_2) \geq 2 - \gamma$, $d(q_1, q_2) \geq 2 - \gamma$, and $d(p_i, q_j) \leq \sqrt{2} - \gamma$ for all $1 \leq i, j \leq 2$.*

Recall that from the hypergraph $\mathcal{H}$ we formed the hypergraph $\mathcal{H}'$ by blowing up each vertex in $\mathcal{H}$ into a strong independent set in $\mathcal{H}'$. Also recall that the vertices in $G[U]$ are vertices of $\mathcal{H}'$, so vertices in $G[U]$ correspond to blowups of vertices in $\mathcal{H}$. We define a function Ξ between $V(G)$ and $V(\mathcal{H})$: for $x \in V(G)$, let $\Xi(x)$ be the vertex of $V(\mathcal{H})$ such that x is contained in the blowup of $\Xi(x)$.

Lemma 9. *G is K_{r+s}-free.*

Proof. Towards a contradiction, assume that $K = K_{r+s}$ is a subgraph of G and let $K_u = K[V(K) \cap U]$ and $K_v = K[V(K) \cap V]$. Since U and V are symmetric in the definition of G, we may assume without loss of generality that $|K_u| \geq |K_v|$. By Lemma 7 and since $s \geq 2$,

$$\lceil r/2 \rceil + 1 \leq \left\lceil \frac{r+s}{2} \right\rceil \leq |V(K_u)| \leq r. \tag{3}$$

This implies that

$$|V(K_v)| = r + s - |V(K_u)| \geq s. \tag{4}$$

By Lemma 6 and (4), there exist $x_1, \ldots, x_s \in V(K_v)$ and a hyperedge E in $\mathcal{H}'$ such that $x_1, \ldots, x_s \in E$. Since $x_1, \ldots, x_s$ are all in E and edges of $\mathcal{H}$ were built from the auxiliary bipartite graphs $B_1, \dots, B_\ell$, we can think of $\Xi(x_1), \dots, \Xi(x_s)$ as vertices in $\cup B_i$. By Lemma 5, there exists some B_i and two vertices, say $\Xi(x_1)$ and $\Xi(x_2)$, such that the ith coordinate of $\Xi(x_1)$ and the ith coordinate of $\Xi(x_2)$ are almost antipodal. Fix this i for the remainder of this proof.

By (3) there exist at least $\lceil r/2 \rceil + 1$ vertices in $V(K_u)$, say $y_1, \dots, y_{\lceil r/2 \rceil + 1}$. By Lemma 6 there is a hyperedge F in $\mathcal{H}'$ containing them. Similar to the previous paragraph, we can think of $\Xi(y_1), \dots, \Xi(y_{\lceil r/2 \rceil + 1})$ as vertices in B_i (recall that i has already been chosen.) The parts of B_i have size at most $\lceil r/2 \rceil$ so there exist two vertices, say $\Xi(y_1)$ and $\Xi(y_2)$, such that the i-th coordinates are almost antipodal.

Consider the i-th coordinates of $\Xi(x_1)$, $\Xi(x_2)$, $\Xi(y_1)$, and $\Xi(y_2)$. The cross-distances between the x's and y's are all at most $\sqrt{2} - \theta$, since x_1, x_2, y_1, and y_2 all came from the clique K. Hence we have four points violating Theorem 8. ☐

3.3 The K_r-independence number of G

First we need an elementary statement about distances of points on a sphere.

Lemma 10. *Let $k \geq 2$ and $1 \leq h \leq \lfloor k/2 \rfloor$ be any positive integers and fix a positive $a < 1/(16h^4)$. Let $x_1, \dots, x_k \in \mathbb{S}^k$ such that for every i we have $d(x_i, x_{i+1}) \geq 2 - a$. Then $d(x_1, x_{2h}) > 2 - 4h^2 a$.*

Proof. For $u \in \mathbb{S}^k$ denote by $u' \in \mathbb{S}^k$ the antipodal point to u. Note that for every u, v trivially $d(u, u') = 2$ and $d(u, v) = d(u', v')$. First, we bound $d(x'_i, x_{i+1})$ for every i. The points x_i, x'_i, and x_{i+1} form a right triangle since x_i and x'_i are antipodal (the right angle is at the point x_{i+1}). Thus

$$d^2(x'_i, x_{i+1}) = d^2(x_i, x'_i) - d^2(x_i, x_{i+1}) \leq 4 - (2-a)^2 \leq 4a - a^2 \leq 4a.$$

Thus $d(x'_i, x_{i+1}) \leq 2\sqrt{a}$ for all i. Using the triangle inequality, we obtain

$$d(x'_1, x_{2h}) \leq d(x'_1, x_2) + d(x_2, x'_3) + \ldots + d(x'_{2h-1}, x_{2h}) \leq 2(2h-1)\sqrt{a}.$$

The points x_1, x'_1, and x_{2h} form a right triangle since x_1 and x'_1 are antipodal. Thus

$$d^2(x_1, x_{2h}) = d^2(x_1, x'_1) - d(x'_1, x_{2h}) \geq 4 - 4(2h-1)^2 a = 4 - 16h^2 a + 16ha - 4a \geq 4 - 16h^2 a + a.$$

Since $a < 1/(16h^4)$ implies $16h^4 a^2 < a$ we have

$$d^2(x_1, x_{2h}) \geq 4 - 16h^2 a + a > 4 - 16h^2 a + 16h^4 a^2 = (2 - 4h^2 a)^2.$$

☐

We now need one lemma from [4]. There is a subtle point here: in [4] the statement of the lemma uses "$d(p_i, p_j) \geq 2 - \theta$". But the variable θ used in this paper and the θ used in [4] are slightly different constants. The θ used in the statement of [4, Lemma 13] comes from the statement of [4, Property (P3)] which matches our Property (P2). So the θ in [4, Lemma 13] is replaced with the constant from our Property (P2) when we cite that lemma below.

Lemma 11. *(Lemma 13 in [4]) If $A_1, \dots, A_r \subseteq P$ with $|A_i| \geq 2^r \beta z$ and T is a tree on vertex set $[r]$, then there exist $p_1 \in A_1, \dots, p_r \in A_r$ such that if $ij \in E(T)$ then $d(p_i, p_j) \geq 2 - \epsilon / \left(2r^2\sqrt{k}\right)$.*

One of the key improvements in this paper compared to [4] is improving the above lemma by replacing trees with complete bipartite graphs.

Lemma 12. *If $A_1, \ldots, A_r \subseteq P$ with $|A_i| \geq 2^r \beta z$ and B is a complete bipartite graph on vertex set $[r]$, then there exist $p_1 \in A_1, \ldots, p_r \in A_r$ such that if $ij \in E(B)$ then $d(p_i, p_j) \geq 2 - \theta$.*

Proof. Let T be a path on vertex set $[r]$. Apply Lemma 11 to find $p_1 \in A_1, \ldots, p_r \in A_r$ such that if $ij \in E(T)$ then $d(p_i, p_j) \geq 2 - \theta$. Since T is a path, this implies that $d(p_i, p_{i+1}) \geq 2 - \epsilon / \left(2r^2\sqrt{k}\right) = 2 - \theta/r^2$ for all i. We can then apply Lemma 10 to show that $d(p_{2i+1}, p_{2j}) > 2 - \theta$ for all i and j (set $x_1 = p_{2i+1}$ and $x_{2h} = p_{2j}$.) □

Lemma 13. $\alpha(\mathcal{H}) \leq r^\ell 2^{\ell+r} \beta z^\ell$.

Proof sketch. The proof is identical to the proof of [4, Lemma 14], except where [4, Lemma 14] uses Lemma 11 on trees, we instead use Lemma 12. □

Lemma 14. $\alpha_r(G) \leq r^\ell 2^{\ell+r+2} \beta z^\ell t$.

Proof sketch. The proof is identical to the proof of [4, Theorem 9 *(iv)*]. The statement of Lemma 13 is identical to the lemma used by the proof of [4, Theorem 9 *(iv)*]. □

4 Concluding Remarks

We conjecture that our construction is best possible when $4r/(s-1)$ is a power of 2; we know this only when $s \leq 5$ and for some additional cases (see Proposition 3). Probably, some mixture of a more involved application of the Szemerédi Regularity Lemma and some proof techniques from weighted Turán theory could help to prove our conjecture.

It seems very hard to decide if our constructions are best possible when $4r/(s-1)$ is not a 2-power. The simplest open cases are

$$\frac{1}{16} \leq \theta_3(K_5) \leq \frac{1}{12}, \qquad \frac{1}{8} \leq \theta_3(K_6) \leq \frac{1}{6}, \qquad \frac{1}{8} \leq \theta_4(K_8) \leq \frac{3}{16}.$$

The upper bounds are from Theorem 1 and the lower bounds from Theorem 2.

Theorems 1 and 2 focus on $\theta_r(K_t)$ for $t \leq 2r$. What happens when $t > 2r$? The construction from Theorem 2 can be easily extended to cliques larger than K_{2r} as follows. For a lower bound on $\theta_r(K_{qr+s})$ with $2 \leq s \leq r$, let G be the construction from Theorem 2 and join it to a complete $(q-1)$-partite graph with almost equal part sizes. Into each part insert a K_{r+1}-free graph with small K_r-independence number (such a graph exists by the Erdős-Rogers Theorem [9].) Erdős, Hajnal, Simonovits, Sós, and Szemerédi [7] conjectured that this type of construction provides the correct answer (see also [15, Conjecture 18].) Can Theorem 1 be extended to $t > 2r$ and if so, does it match our lower bound?

In the area of the Ramsey-Turán theory, one of the major open problems is to prove a generalization of the Erdős-Stone Theorem [11] by proving that $\mathbf{RT}(n, H, o(n)) = \mathbf{RT}(n, K_s, o(n))$ where $s = s(H)$ is equal to some parameter depending only on H. Erdős, Hajnal, Sós, and Szemerédi [8] proved an upper bound using a parameter closely related to the arboricity. That is, one can take s to be the minimum s such that $V(H)$ can be partitioned into $\lceil s/2 \rceil$ sets $V_1, \ldots, V_{\lceil s/2 \rceil}$ such that $V_1, \ldots, V_{\lfloor s/2 \rfloor}$ span forests and if s is odd $V_{\lceil s/2 \rceil}$ spans an independent set. This is known to be sharp for odd s. In several papers, Erdős mentioned the problem of solving the simplest open case when $H = K_{2,2,2}$, where $s(K_{2,2,2}) = 4$, i.e., one would like to have a lower bound of $\mathbf{RT}(n, K_{2,2,2}, o(n)) \leq \mathbf{RT}(n, K_4, o(n)) = \frac{1}{8}n^2 + o(n^2)$. Even the question of determining if

$\mathbf{RT}(n, K_{2,2,2}, o(n)) = \Omega(n^2)$ is still open (see [15, Problem 4], [8, p. 72], [16, Problem 1.3] among others).

References

[1] M. Ajtai, J. Komlós, J. Pintz, J. Spencer, and E. Szemerédi. Extremal uncrowded hypergraphs. *J. Combin. Theory Ser. A*, 32:321–335, 1982.

[2] M. Ajtai, J. Komlós, and E. Szemerédi. A note on Ramsey numbers. *J. Combin. Theory Ser. A*, 29:354–360, 1980.

[3] M. Ajtai, J. Komlós, and E. Szemerédi. A dense infinite Sidon sequence. *European J. Combin.*, 2:1–11, 1981.

[4] J. Balogh and J. Lenz. The Ramsey-Turán Numbers of Graphs and Hyerpgraphs. to appear in Israel Journal of Mathematics.

[5] B. Bollobás. An extension of the isoperimetric inequality on the sphere. *Elem. Math.*, 44:121–124, 1989.

[6] B. Bollobás and P. Erdős. On a Ramsey-Turán type problem. *J. Combinatorial Theory Ser. B*, 21:166–168, 1976.

[7] P. Erdős, A. Hajnal, M. Simonovits, V. T. Sós, and E. Szemerédi. Turán-Ramsey theorems and K_p-independence numbers. *Combin. Probab. Comput.*, 3:297–325, 1994.

[8] P. Erdős, A. Hajnal, V. T. Sós, and E. Szemerédi. More results on Ramsey-Turán type problems. *Combinatorica*, 3:69–81, 1983.

[9] P. Erdős and A. Rogers. The construction of certain graphs. *Canad. J. Math.*, 14:702–707, 1962.

[10] P. Erdős and V. T. Sós. Some remarks on Ramsey's and Turán's theorem. In *Combinatorial theory and its applications, II (Proc. Colloq., Balatonfüred, 1969)*, pages 395–404. North-Holland, Amsterdam, 1970.

[11] P. Erdős and A. H. Stone. On the structure of linear graphs. *Bull. Amer. Math. Soc.*, 52:1087–1091, 1946.

[12] P. Erdős and P. Turán. On a problem of Sidon in additive number theory, and on some related problems. *J. London Math. Soc.*, 16:212–215, 1941.

[13] J. Fox. Complete minors and independence number. *SIAM J. Discrete Math.*, 24:1313–1321, 2010.

[14] J. Komlós, J. Pintz, and E. Szemerédi. A lower bound for Heilbronn's problem. *J. London Math. Soc. (2)*, 25:13–24, 1982.

[15] M. Simonovits and V. T. Sós. Ramsey-Turán theory. *Discrete Math.*, 229:293–340, 2001.

[16] B. Sudakov. A few remarks on Ramsey-Turán-type problems. *J. Combin. Theory Ser. B*, 88:99–106, 2003.

[17] E. Szemerédi. On graphs containing no complete subgraph with 4 vertices. *Mat. Lapok*, 23:113–116 (1973), 1972.

[18] P. Turán. Eine Extremalaufgabe aus der Graphentheorie. *Mat. Fiz. Lapok*, 48:436–452, 1941.

A Upper bounds

Erdős, Hajnal, Simonovits, Sós, and Szemerédi [7] proved that $\theta_r(K_{r+s}) \le \frac{s-1}{4r}$ for $s \le 5$. They also proved two upper bounds: $\theta_3(K_8) \le \frac{3}{11}$ and $\theta_3(K_9) \le \frac{3}{10}$. Extending on their techniques, we sketch a proof that $\theta_{10}(K_{16}), \theta_{12}(K_{19}) \le \frac{1}{8}$. Combined with Theorem 2, this proves Proposition 3. Because of the similarity of the proofs, we do not give all the details of this proof; we only sketch the places where the argument differs from Section 4 in [7].

Assume G is a counterexample. Apply Szemerédi's Regularity Lemma to G to obtain a cluster graph H. Consider H as a weighted graph, where the weight of an edge is the density of the pair of corresponding clusters. If any vertex has weighted degree less than $(1/4 + \epsilon/4)n$, delete it from H.

The remainder of the proof is focused on finding a copy of K_{r+s} in G. Let $w : E(H) \to [0, 1]$ be the weight function on H. The proof technique is to force a bad configuration in H. A *configuration* is a weighted K_t. If t and the weights are chosen properly, the existence of the configuration in H will imply G contains K_{r+s}. The proof then comes down to a series of claims showing that H must contain at least one bad configuration.

As an example, consider the following argument: Assume H contains a copy of K_s with an edge of weight at least $1/r + \epsilon$ (the weights on the other edges can be anything.) Let A and B be the corresponding clusters of the partition of G with density $1/r + \epsilon$. By the low K_r-independence number of G, the class B must contain a copy of K_r. By the lower bound on the density between A and B, there are two vertices, say x and y, in this K_r of B which have a common neighborhood in A of size at least $\epsilon n/2$. Again, the sublinear K_r-independence number of G implies that we can find a copy of K_r in this common neighborhood. Thus we have found a copy of K_{r+2} in $G[A \cup B]$. The Embedding Lemma can be used to extend this K_{r+2} by adding $s - 2$ more vertices since we originally found a K_s in H.

We introduce some shorthand notation for configurations. $(K_t; a)$ means a copy of K_t in H with one edge with weight at least $a + \epsilon$, $(K_t; a, b)$ means a copy of K_t with one edge with weight at least $a + \epsilon$ and another edge with weight at least $b + \epsilon$, $(K_t; a, \ldots, a)$ means a copy of K_t with all edges with weight at least $a + \epsilon$. We use the symbol $\rightsquigarrow$ to mean that the existence of one configuration implies the existence of another. A bad configuration is a configuration whose existence implies a copy of K_{r+s} in G. It was proved in [7] that the following configurations are bad.

Lemma 15. *[7] For any non-negative integer a,* $(K_{s-a+1}; \frac{a}{r})$ *is a bad configuration.*

A simple corollary of this is that every edge has weight at most $(s - 2)/r + \epsilon$. The following lemmas are our main new tools, showing more configurations are bad.

Lemma 16. *For any non-negative integer a and any reals b, c satisfying* $b \ge c$ *and* $b + (a+1)c/r > s - 1$, $(K_3; \frac{a}{r}, \frac{b}{r}, \frac{c}{r})$ *is a bad configuration.*

Proof sketch. Assume H contains $(K_3; \frac{a}{r}, \frac{b}{r}, \frac{c}{r})$ and let X, Y, Z be the parts in the regularity partition with $d(X,Y) = \frac{a}{r}$, $d(X,Z) = \frac{b}{r}$, and $d(Y,Z) = \frac{c}{r}$. By the K_r-independence number, Y contains a copy of K_r. Because of the density between X and Y, this copy of K_r contains at least $a+1$ vertices with linear common neighborhood in X. This common neighborhood contains a copy of K_r, so we have found a copy of K_{r+a+1} in $G[X \cup Y]$. By averaging, there are s vertices of this K_{r+a+1} with linear common neighborhood in Z. This common neighborhood contains a copy of K_r, so we have found a copy of K_{r+s}. □

Lemma 17. $(K_2; \frac{s-3}{r}) \rightsquigarrow (K_3; \frac{s-3}{r}, \frac{s-3}{r}, \frac{s-3}{r})$.

Proof sketch. First $(K_2; \frac{s-3}{r}) \rightsquigarrow (K_3; \frac{s-3}{r})$ since every vertex has weight at least $(s-1)/2r$. Assume we have a triangle uvz with an edge uv with weight at least $(s-3)/r + \epsilon$.

We now show that H contains $(K_3; \frac{s-3}{r}, \frac{s-3}{r})$. By Lemma 15, there is no K_4 containing the triangle uvz so for any other x, one of the weights from xu, xv, xz is at most ϵ (meaning the edge is missing from H). One of the other two weights on xu, xv, and xz must be less than $(s-3)/r + \epsilon$, otherwise we would have the configuration $(K_3; \frac{s-3}{r}, \frac{s-3}{r})$. All edges have weight at most $(s-2)/r$. Adding weight, one of u, v, or z will violate the minimum weight degree condition in H. A similar argument shows $(K_3; \frac{s-3}{r}, \frac{s-3}{r}) \rightsquigarrow (K_3; \frac{s-3}{r}, \frac{s-3}{r}, \frac{s-3}{r})$. □

Lemma 18. *If* $3(s-2) > r$, *then* $(K_3; \frac{s-3}{r}, \frac{s-3}{r}, \frac{s-3}{r})$ *is a bad configuration.*

Proof. Let uvz be a triangle with all edges with weight at least $\frac{s-3}{r}$. By Lemma 15, there is no K_4 containing this triangle. By the minimum weight condition, there is some vertex x sending $\frac{3(s-1)}{2r}$ to uvz and since $uvzx$ does not form a K_4, $w(x,u) + w(x,v) \geq \frac{3(s-1)}{4r}$. Now apply Lemma 16 with $a = s-3$ and $b = c = \frac{3(s-1)}{4}$ to obtain a contradiction.

$$\frac{3(s-1)}{4} + \frac{3(s-2)(s-1)}{4r} > s - 1.$$

□

The combination of the above two lemmas shows that if $3(s-2) > r$ (which is true for $r = 10, s = 6$ and $r = 12, s = 7$) then $(K_2; \frac{s-3}{r})$ is a bad configuration. In other words, every edge in H has weight at most $(s-3)/r + \epsilon$.

Proposition 19. $\theta_{10}(K_{16}) \leq \frac{1}{8}$.

Proof. Fix $r = 10$ and $s = 6$. First, Lemma 15 shows H is K_7-free (with $a = 0$.) Lemmas 17 and 18 show all edges have weight at most $3/10 + \epsilon$. Then Turán's Theorem implies the total weight is at most

$$\frac{1}{2}\left(1 - \frac{1}{6}\right)\left(\frac{3}{10} + \epsilon\right) \leq \frac{1}{8} + \epsilon,$$

a contradiction. □

Proposition 20. $\theta_{12}(K_{19}) \leq \frac{1}{8}$.

Proof sketch. Fix $r = 12$ and $s = 7$. Lemma 17 and Lemma 18 show that all edges have weight at most $4/12 + \epsilon$. We claim without proof that several configurations are bad.

- $(K_3; \frac{3.75}{12}, \ldots, \frac{3.75}{12})$ is a bad configuration.
- $(K_4; \frac{3}{12}, \ldots, \frac{3}{12})$ is a bad configuration.
- $(K_5; \frac{2}{12}, \ldots, \frac{2}{12})$ is a bad configuration.
- H is K_6-free.

All of the above facts have very similar proofs. As an example, Lemma 15 shows $(K_5; \frac{3}{12})$ is a bad configuration. So consider any K_4 with all edges with weight at least 3/12. There is some vertex x with at least weight 1 towards the K_4. But x is not adjacent to all of the K_4, so the weight is distributed over three edges. But the maximum weight on an edge is 4/12, a contradiction.

So assume the four bullet points above are true and count up the total weight. The edges with weight more than 3.75/12 are triangle free, so their total weight is at most $\frac{n^2}{4}\frac{4}{12}$. The edges with weight more than 3/12 are K_4-free so there are at most $n^2/3$ of them, and there are at most $n^2/3 - n^2/4$ edges that are uncounted by the bound $\frac{n^2}{4}\frac{4}{12}$ and these have weight at most 3.75/12. Thus the total weight is at most

$$\frac{n^2}{4}\frac{4}{12} + \left(\frac{n^2}{3} - \frac{n^2}{4}\right)\frac{3.75}{12} + \left(\frac{3n^2}{8} - \frac{n^2}{3}\right)\frac{3}{12} + \left(\frac{2n^2}{5} - \frac{3n^2}{8}\right)\frac{2}{12} < \frac{n^2}{8}.$$

□