On the Ramsey-Turán numbers of graphs and hypergraphs

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Abstract

Let t be an integer, $f(n)$ a function, and H a graph. Define the t-Ramsey-Turán number of H, $\mathbf{RT}_t(n, H, f(n))$, to be the maximum number of edges in an n-vertex, H-free graph G with $\alpha_t(G) \leq f(n)$, where $\alpha_t(G)$ is the maximum number of vertices in a K_t-free induced subgraph of G. Erdős, Hajnal, Simonovits, Sós, and Szemerédi [5] posed several open questions about $\mathbf{RT}_t(n, K_s, o(n))$, among them finding the minimum ℓ such that $\mathbf{RT}_t(n, K_{t+\ell}, o(n)) = \Omega(n^2)$, where it is easy to see that $\mathbf{RT}_t(n, K_{t+1}, o(n)) = o(n^2)$. In this paper, we answer this question by proving that $\mathbf{RT}_t(n, K_{t+2}, o(n)) = \Omega(n^2)$; our constructions also imply several results on the Ramsey-Turán numbers of hypergraphs.

1 Introduction

Let $\mathcal{H}$ be an r-uniform hypergraph and $f(n)$ a function. The **Ramsey-Turán number of** $\mathcal{H}$, $\mathbf{RT}(n, \mathcal{H}, f(n))$, is the maximum number of edges in an n-vertex, r-uniform, $\mathcal{H}$-free hypergraph with independence number at most $f(n)$. In 1970, Erdős and Sós [8] initiated the study of Ramsey-Turán numbers of graphs when they started investigating whether excluding large independent sets in K_s-free graphs implies an improvement in Turán's theorem. One of the main problems in Ramsey-Turán theory is to determine the threshold function for

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$\mathcal{H}$ (see [17] for a survey). The **threshold function** for $\mathcal{H}$ is a function $t(n)$ such that $\mathbf{RT}(n, \mathcal{H}, t(n)) = \Omega(n^r)$ and if $f(n) = o(t(n))$ then $\mathbf{RT}(n, \mathcal{H}, f(n)) = o(n^r)$. Define

$$\theta(\mathcal{H}) = \lim_{\epsilon \to 0} \lim_{n \to \infty} \frac{\mathbf{RT}(n, \mathcal{H}, \epsilon n)}{n^r}.$$

In an abuse of notation, we write $\mathbf{RT}(n, \mathcal{H}, o(n)) = \theta(\mathcal{H}) n^r + o(n^r)$. An easy diagonalization argument shows that $t(n) = n$ is a threshold function for $\mathcal{H}$ if and only if $\mathrm{ex}(n, \mathcal{H}) = \Omega(n^r)$ and $\theta(\mathcal{H}) = 0$. Very few threshold functions are known exactly; instead we study the easier problem of deciding whether $t(n) = n$ is a threshold function or not.

Erdős, Hajnal, Sós, and Szemerédi [6, p. 80] proposed a problem about an extension of the concept of the Ramsey-Turán numbers of graphs. Let G be a graph and define the **K_t-independence number** of G as

$$\alpha_t(G) := \max \left\{ |S| : S \subseteq V(G), G[S] \text{ is } K_t\text{-free} \right\}.$$

Define $\mathbf{RT}_t(n, H, f(n))$ to be the maximum number of edges in an H-free graph G on n vertices with $\alpha_t(G) \leq f(n)$ and define

$$\theta_t(\mathcal{H}) = \lim_{\epsilon \to 0} \lim_{n \to \infty} \frac{1}{n^2} \mathbf{RT}_t(n, H, \epsilon n). \tag{1}$$

We write $\mathbf{RT}_t(n, H, o(n)) = \theta_t(\mathcal{H}) n^2 + o(n^2)$. For $t = 2$, it is easy to show that the limit in (1) exists; for $t \geq 3$, the fact that these limits exist is not obvious, it was one of the main results in [5].

For complete graphs of odd order, Erdős and Sós [8] proved that

$$\theta(K_{2s+1}) = \frac{1}{2} \left(1 - \frac{1}{s} \right),$$

leaving open the question of determining $\theta(K_{2s})$ for $s \geq 2$. The first celebrated result in Ramsey-Turán theory was that $\theta(K_4) = \frac{1}{8}$. In one of the first applications of the Regularity Lemma[1] to graph theory, Szemerédi [19] proved that $\theta(K_4) \leq \frac{1}{8}$ in 1972. Four years later, Bollobás and Erdős [3] provided a surprising geometric construction using high dimensional spheres which proved that Szemerédi's upper bound was tight. It was not until 1983 that Erdős, Hajnal, Sós, and Szemerédi [6] extended this result to all complete graphs of even order, determining $\theta(K_{2s})$.

It was also proved in [6] that $\theta(H) \leq \theta(K_s)$ for $s \geq 5$, where s is the minimum integer for which $V(H)$ can be partitioned into $\lceil s/2 \rceil$ sets $V_1, \ldots, V_{\lceil s/2 \rceil}$ such that $V_1, \ldots, V_{\lfloor s/2 \rfloor}$ span forests in H and if s is odd then $V_{\lceil s/2 \rceil}$ spans an independent set. For odd s this bound is sharp. The 'simplest' major open question is to decide if $\theta(K_{2,2,2}) = 0$.

The exact threshold function for K_s for $s \geq 4$ is also still unknown, but Sudakov [18] showed, using the so called "dependent random choice method," that $\mathbf{RT}(n, K_4, n2^{-\omega\sqrt{\log n}}) =$

[1] This was an earlier version of the Szemerédi Regularity Lemma.

$o(n^2)$, where $\omega = \omega(n)$ is any function going to infinity arbitrarily slowly. Note that $n2^{-\omega\sqrt{\log n}}/n^{1-\delta} \to \infty$ as $n \to \infty$ for any fixed δ.

No results about $\mathbf{RT}_t(n, H, o(n))$ for $t \geq 3$ were known until Erdős, Hajnal, Simonovits, Sós, and Szemerédi [5] proved that the limit in (1) exists when H is a complete graph, $\theta_t(K_s) \leq \frac{1}{2}\left(1 - \frac{t}{s-1}\right)$, and this is sharp for all $s \equiv 1 \pmod{t}$. Note that for $t = 2$ this was already known by Erdős and Sós [8]. Additionally, for some special cases, for $\ell = 1, 2, 3, 4, 5$ and $\ell \leq t + 1$ they proved that

$$\theta_t(K_{t+\ell}) \leq \frac{\ell - 1}{4t}.$$

In [5] a construction was given proving that

$$\theta_t(n, K_{2t}, o(n)) \geq \frac{1}{8}. \tag{2}$$

Unfortunately, the proof that the constructed graph has small independence number relied on a theorem of Bollobás [2] which has been withdrawn as incorrect [1]. Therefore, until now, it was unknown if $\theta_t(K_s)$ is positive for $s \leq 2t$. Erdős, Hajnal, Simonovits, Sós, and Szemerédi [5] posed several open problems.

Problem 1. ([5, Problem 2.12]) Find the minimum ℓ such that $\theta_t(K_{t+\ell}) > 0$.

In [6], Erdős, Hajnal, Sós, and Szemerédi wrote that to solve Problem 2 below "an analogue of the Bollobás-Erdős graph would be needed which we think will be extremely hard to find."

Problem 2. ([5], [6], and [17, Problem 17]) Determine if $\theta_3(K_5) > 0$.

As we already mentioned, Problem 2 is motivated by the history of the analogous question for $t = 2$: Erdős and Sós [8] observed that $\theta(K_5) > 0$ and $\theta(K_3) = 0$, leaving open the hard problem deciding if $\theta(K_4) > 0$, which was solved by Bollobás and Erdős [3]. For $t = 3$, it is easy to observe that $\theta_3(K_4) = 0$ and $\theta_3(K_7) > 0$, motivating Problem 2.

2 Results

The main result of our paper is solving Problems 1 and 2 by constructing graphs showing that $\theta_t(K_{t+\ell}) > 0$ for $2 \leq \ell \leq t$. This is a breakthrough step in the area; in this part of extremal graph theory constructions usually do not come easily.

Theorem 3. *For $t \geq 2$ and $2 \leq \ell \leq t$, let $u = \lceil t/2 \rceil$. Then*

$$\theta_t(K_{t+\ell}) \geq \frac{1}{2}\left(1 - \frac{1}{\ell}\right) 2^{-u^2}.$$

For comparison, trivially, $\theta_t(K_{t+1}) = 0$. Note that for $t = \ell$ the bound in (2) is better than in Theorem 3. Theorem 3 can also be used to give a lower bound for $\theta_t(K_{qt+\ell})$ for all $q \geq 1$ and $2 \leq \ell \leq t$. Let G be a member of the graph sequence constructed to prove Theorem 3 and let T be a complete $(q-1)$-partite graph with almost equal class sizes. In each class of T, insert a K_{t+1}-free graph with small K_t-independence number. (Such a graph exists by the Erdős-Rogers Theorem [7].) Lastly, completely join G and T. Any copy of $K_{qt+\ell}$ which appears in this graph can have at most t vertices in each part of T. This forces G to contain $t + \ell$ vertices of the copy of $K_{qt+\ell}$, which is a contradiction. This graph also has a small K_t-independence number. Letting $|G| = an$ we instantly conclude the following.

Corollary 4. *For $t, q \geq 2$ and $2 \leq \ell \leq t$, let $u = \lceil t/2 \rceil$. For any $0 < a < 1$,*

$$\theta_t(K_{qt+\ell}) \geq \frac{1}{2}\left(1 - \frac{1}{\ell}\right) 2^{-u^2} a^2 + \binom{q-1}{2} \left(\frac{1-a}{q-1}\right)^2 + (1-a)a. \tag{3}$$

The precise formula for a is cumbersome so it is not included here. Instead, we list the optimized value for some small values of s and t. In [5] the following problem was posed.

Problem 5. ([5], [6], and [17, Problem 19])

$$\theta_3(K_5) \leq \frac{1}{12}, \quad \theta_3(K_6) \leq \frac{1}{6}, \quad \theta_3(K_8) \leq \frac{3}{11}, \quad \theta_3(K_9) \leq \frac{3}{10}.$$

Are any of these bounds tight?

Optimizing a in Corollary 4, the relative size of G and T, we obtain the following lower bounds for the graphs considered in Problem 5.

$$\frac{1}{64} \leq \theta_3(K_5), \quad \frac{1}{48} \leq \theta_3(K_6), \quad \frac{16}{63} \leq \theta_3(K_8), \quad \frac{12}{47} \leq \theta_3(K_9).$$

Note that in [5] it was proved that $\mathbf{RT}_3(n, K_7, o(n)) = \frac{1}{4}n^2 + o(n^2)$.

Theorem 3 follows from a result about hypergraphs. For $s > r$ let $\mathrm{TK}^r(s)$ be the r-uniform hypergraph obtained from the complete graph K_s by replacing each graph edge uv with a hypergraph edge which besides u, v contains $r-2$ new vertices. The **core vertices** of $\mathrm{TK}^r(s)$ are the s vertices of degree larger than one. Let $\mathcal{TK}^r(s)$ be the family of r-uniform hypergraphs $\mathcal{H}$ such that there exists a set S of s vertices of $\mathcal{H}$ where each pair of vertices from S are contained in some hyperedge of $\mathcal{H}$. The set S is called the set of **core vertices** of $\mathcal{H}$.

Let $T_s^r(n)$ be the complete n-vertex, r-uniform, s-partite hypergraph with part sizes as equal as possible. Mubayi [14] showed for $s > r$ that $\mathrm{ex}(n, \mathcal{TK}^r(s+1)) = |T_s^r(n)|$ and $\mathrm{ex}(n, \mathrm{TK}^r(s+1)) = (1 + o(1))\, |T_s^r(n)|$. Recently, Pikhurko [15], improving on [14], has shown that for large n, $\mathrm{ex}(n, \mathrm{TK}^r(s+1)) = |T_s^r(n)|$ and that $T_s^r(n)$ is the unique extremal example. Since in this case the extremal hypergraphs have large independent sets, it is interesting to study the behavior of the function $\mathbf{RT}(n, \mathrm{TK}^r(s), f(n))$ for $f(n) = o(n)$. A simple observation is the following.

Proposition 6. *For $r \geq 2$,*

1. $\mathbf{RT}(n, \mathrm{TK}^r(r+1), o(n)) = o(n^r)$.
2. $\mathbf{RT}(n, \mathcal{TK}^r(2r-1), o(n)) = o(n^r)$.

Proof. We prove only the statement for 3-uniform hypergraphs; the proof can be easily extended to every $r \geq 3$.

Let $\mathcal{H}$ be a 3-uniform, n-vertex hypergraph with independence number at most ϵn and at least $9\epsilon n^3 + 72n^2$ edges. For simplicity, assume 3 divides n and let $\mathcal{H}'$ be a 3-partite subhypergraph of $\mathcal{H}$ with equal part sizes and with at least $\frac{1}{9}$ of the edges of $\mathcal{H}$. Recall that for a pair of vertices x and y, their **codegree** $d(x, y)$ is the number of vertices z such that $\{x, y, z\}$ is an edge. For each pair x, y of vertices in different classes, delete all edges containing x and y if their codegree is at most 16. We delete at most $8n^2$ hyperedges. Thus we have a 3-partite hypergraph $\mathcal{H}'$ with at least ϵn^3 edges and the codegree of any pair of vertices from different classes is zero or at least 16.

Since $\mathcal{H}'$ has at least ϵn^3 cross-edges, the maximum codegree of $\mathcal{H}'$ is at least ϵn. Let x, y be a pair of vertices from different classes with codegree at least ϵn, and let Z be the set of vertices z in the third class such that $\{x, y, z\}$ is an edge. Since the independence number of $\mathcal{H}$ is at most ϵn, there exists a hyperedge E of $\mathcal{H}$ contained in Z. The vertices in E together with x, y form a hypergraph in $\mathcal{TK}^3(5)$. (In the r-uniform case, the edge E together with $r-1$ vertices will form a copy of $\mathcal{TK}^r(2r-1)$.) Thus any 3-uniform, n-vertex, $\mathcal{TK}^3(5)$-free hypergraph with independence number at most ϵn can have at most $9\epsilon n^3 + 72n^2$ edges.

To find a copy of $\mathrm{TK}^3(4)$, let z_1 and z_2 be two vertices from E. The core vertices in a copy of $\mathrm{TK}^3(4)$ are x, y, z_1, and z_2. The vertices z_1 and z_2 are contained together in the edge E, and since x and z_1 are contained together in a hyperedge of $\mathcal{H}'$, the codegree of x and z_1 is at least 16. Thus we can find an edge of $\mathcal{H}$ containing x and z_1 where the third vertex avoids all previously used vertices. Similarly we can find edges containing x, z_2 and x, y and y, z_i where the third vertex has not yet been used. Thus we find a copy of $\mathrm{TK}^3(4)$ in $\mathcal{H}$. (In the r-uniform case, take as core vertices two vertices from E together with $r-1$ other vertices to find a copy of $\mathrm{TK}^r(r+1)$.) □

Using our construction, we prove the following lower bounds.

Theorem 7. *Let $r \geq 3$ and let $u = \lceil r/2 \rceil$.*

(i) $\mathbf{RT}(n, \mathrm{TK}^r(r+2), o(n)) \geq 2^{-\binom{ur}{2}+r\binom{u}{2}} \left(\frac{n}{r}\right)^r$.

(ii) $\mathbf{RT}(n, \mathcal{TK}^r(2r), o(n)) \geq 2^{-\binom{ur}{2}+r\binom{u}{2}} \left(\frac{n}{r}\right)^r$.

Note that unlike in the Turán-density extremal case, where for large n we have $\mathrm{ex}(n, \mathrm{TK}^r(s)) = \mathrm{ex}(n, \mathcal{TK}^r(s))$, the Ramsey-Turán numbers for $\mathrm{TK}^r(s)$ and $\mathcal{TK}^r(s)$ are different. Let $\mathcal{F}^r(s)$ be the subfamily of $\mathcal{TK}^r(s)$ containing those hypergraphs where each edge contains exactly two core vertices. We can prove similarly to Theorem 7 (i) that $\theta(\mathcal{F}^r(r+2)) \geq 2^{-\binom{ur}{2}+r\binom{u}{2}} \left(\frac{1}{r}\right)^r$. Based on this we conjecture that $\mathcal{F}^r(s)$ behaves like $\mathrm{TK}^r(s)$.

Conjecture 8. *$\theta(\mathcal{F}^r(s)) = \theta(\mathrm{TK}^r(s))$ for $s > r$.*

Theorems 3 and 7 are corollaries of the following theorem, which is our main tool.

Theorem 9. *(Construction) For any integer $r \geq 2$ there exist positive constants c_1, c_2 such that the following holds. For arbitrary small constants $\alpha, \beta > 0$, and any integer N, there exists an $m \geq N$ such that there exists an rm-vertex, r-uniform hypergraph $\mathcal{G}$ with vertex partition $W_1, \dots, W_r$ with $|W_i| = m$ and the following properties:*

- *(i) No subhypergraph from $\mathcal{TK}^r(4)$ is embedded into $\mathcal{G}$ so that both W_i and W_j contain two core vertices for some $i \neq j$.*
- *(ii) $e(\mathcal{G}) \geq 2^{r\binom{u}{2} - \binom{ru}{2}} m^r - c_2 \alpha m^r$, where $u = \lceil r/2 \rceil$.*
- *(iii) For any i, $\mathcal{G}[W_i]$ contains no connected hypergraph $\mathcal{F}$ with $|V(\mathcal{F})| \leq r^3$ and*

$$|V(\mathcal{F})| < r + (r-1)(|\mathcal{F}| - 1).$$

- *(iv) The independence number of $\mathcal{G}$ is at most $c_1 \beta m$.*

We show that if the independence number of a $\mathrm{TK}^3(6)$-free 3-uniform hypergraph with n vertices is at most $n2^{-\omega(\log n)^{2/3}}$, then it has $o(n^3)$ edges. The proof of Theorem 10 for every $r \geq 3$ extends to $\mathrm{TK}^r(2r)$-free r-uniform hypergraphs with independence number at most $n2^{-\omega(\log n)^{(r-1)/r}}$, we omit the details.

Theorem 10. *Let $w = w(n)$ be any function tending to infinity arbitrarily slowly, and let $f(n) = n2^{-w\cdot(\log n)^{2/3}}$. Then $\mathbf{RT}(n, \mathrm{TK}^3(6), f(n)) = o(n^3)$.*

In Section 3 we state several properties of the k-dimensional unit sphere which will be used in the construction. In Section 4 we describe two earlier constructions by Bollobás and Erdős [3] and Rödl [16]. In Section 5 we describe our construction and prove several properties of it, and in Section 6 we show how the construction presented in Section 5 can be modified to prove Theorems 3, 7, and 9. In Section 7 we prove Theorem 10 and lastly we state some open problems in Section 8. Throughout the paper, we often omit the floor and ceiling signs for the sake of simplicity.

3 Properties of the unit sphere

Let μ be the Lebesgue measure on the k-dimensional unit sphere $\mathbb{S}^k \subseteq \mathbb{R}^{k+1}$ normalized so that $\mu(\mathbb{S}^k) = 1$. For $A \subseteq \mathbb{S}^k$, define $\mathrm{diam}(A) = \sup\{d(x,y) : x, y \in A\}$ where $d(x,y)$ is the Euclidean distance in $\mathbb{R}^{k+1}$. For $A, B \subseteq \mathbb{S}^k$, define

$$d_{\max}(A, B) = \sup\{d(a,b) : a \in A, b \in B\}.$$

For $A \subseteq \mathbb{S}^k$ and $t \geq 2$, define

$$d_t(A) = \sup \left\{ \min_{i \neq j} d(x_i, x_j) : x_1, \ldots, x_t \in A \right\}.$$

In particular, let $\delta = \delta_t$ be the edge length of the t-simplex, i.e., $\delta_t = \sqrt{\frac{2t}{t-1}}$. A **spherical cap** is the intersection of the unit sphere $\mathbb{S}^k$ with a halfspace. The **center** of a spherical cap is the point in the spherical cap at maximum distance from H, where H is the hyperplane bounding the halfspace. The **height** of a spherical cap is the minimum distance between the center and H and the **diameter** of a spherical cap is the diameter of the sphere formed by the intersection of the spherical cap with H. Note that if a is the maximum distance between the center and a point of the spherical cap and h is the height, then $2h = a^2$.

Given any $\alpha, \beta > 0$, it is possible to select $\epsilon > 0$ small enough and then k large enough so that Properties (P1), (P2), and (P3) below are satisfied.

(P1) Let C be a spherical cap in $\mathbb{S}^k$ with height h, where $2h = \left(\sqrt{2} - \epsilon/\sqrt{k}\right)^2$ (this means that all points of the spherical cap are within distance $\sqrt{2} - \epsilon/\sqrt{k}$ of the center). Then $\mu(C) \geq \frac{1}{2} - \alpha$.

(P2) Let $C_1, \ldots, C_t$ be spherical caps in $\mathbb{S}^k$ with height h, where $2h = \left(\sqrt{2} - \epsilon/\sqrt{k}\right)^2$. Let z_i be the center of C_i. Assume for all $1 \leq i < j \leq t$ that $d(z_i, z_j) \leq \sqrt{2}$. Then $\mu(C_1 \cap \ldots \cap C_t) \geq \frac{1}{2^t} - t\alpha$.

(P3) Let C be a spherical cap with diameter $2 - \epsilon/(2\sqrt{k})$. Then $\mu(C) \leq \beta$.

We also use the following properties of high dimensional spheres.

(P4) For any $0 < \gamma < \frac{1}{4}$, it is impossible to have $p_1, p_2, q_1, q_2 \in \mathbb{S}^k$ such that $d(p_1, p_2) \geq 2 - \gamma$, $d(q_1, q_2) \geq 2 - \gamma$, and $d(p_i, q_j) \leq \sqrt{2} - \gamma$ for all $1 \leq i, j \leq 2$.

(P5) Let $A \subseteq \mathbb{S}^k$ and let C be a spherical cap of the same measure. Then $\mathrm{diam}(A) \geq \mathrm{diam}(C)$.

(P6) Let $A, B \subseteq \mathbb{S}^k$ with equal measure and let C be a cap of the same measure. Then $d_{\max}(A, B) \geq \mathrm{diam}(C)$.

Properties (P1) and (P2) follow directly from the formula for the measure of a spherical cap, Properties (P3), (P5), and (P6) are all folklore results that are easy corollaries of the isoperimetric inequality on the sphere [13], and Property (P4) is from [3], see also [5].

Erdős, Hajnal, Simonovits, Sós, and Szemerédi [5] gave a construction which they claim proved $\mathbf{RT}_t(n, K_{2t}, o(n)) \geq \frac{1}{8}n^2 - o(n^2)$. Unfortunately, the proof that the construction has small independence number relies on a theorem of Bollobás [2] which has been withdrawn as incorrect [1]. In [2], the following question was considered. Is it true that if C is a spherical cap with $\mu(C) = \mu(A)$, then $d_t(A) \geq d_t(C)$? If this were true as claimed in [5], then $\theta_t(K_{2t}) \geq \frac{1}{8}$. In a private communication, Bollobás [1] provided the following counterexample. Take C to be a cap of the sphere in three dimensions with small but positive measure and let C' be another cap of the same measure which is far from C. Let $A = C \cup C'$. Then if $\mu(C)$ is small enough we can approximate C and C' by circles with radius r. Then $d_3(A) \approx 2r$ since we can take two points of C and one point of C'. But if D is a cap with the same measure as A then D has radius about $\sqrt{2}r$ so $d_t(D) \approx \sqrt{6}r > d_3(A)$. This counterexample can be extended to higher dimensions and more than three points, but only seems to work when C has small measure.

4 Former constructions

In this section, we describe two previous constructions; our construction will use ideas from both.

The Bollobás-Erdős Graph, [3]. In order to prove that $\mathbf{RT}(n, K_4, o(n)) \geq \frac{n^2}{8} - o(n^2)$, we need to construct, for every $\alpha, \beta > 0$, a K_4-free graph G with n vertices, independence number at most βn, and at least $\frac{n^2}{8}(1-\alpha)$ edges. Given $\alpha, \beta \geq 0$, take ϵ small enough and k large enough so that Properties (P1) and (P3) hold. Divide the k-dimensional unit sphere $\mathbb{S}^k$ into $n/2$ domains having equal measure and diameter at most $\frac{\epsilon}{10\sqrt{k}}$. Choose a point from each domain and let P be the set of these points. Let $\phi : P \to \mathcal{P}(\mathbb{S}^k)$ map points of P to the corresponding domain of the sphere. Take as vertex set of G the disjoint union of two sets V_1 and V_2 both isomorphic to P. For $x, y \in V_i$ we make xy an edge of G if $d(x, y) \geq 2 - \epsilon/\sqrt{k}$. For $x \in V_1, y \in V_2$ we make xy an edge of G if $d(x, y) \leq \sqrt{2} - \epsilon/\sqrt{k}$. Then Property (P1) shows that every vertex in V_1 has at least $\frac{1}{2}|V_2|(1-\alpha)$ neighbors in V_2 so the total number of edges is at least $\frac{1}{8}n^2(1-\alpha)$. If I is a set in V_1 with $|I| \geq \beta|V_1| = \beta\frac{n}{2}$, then $\mu(\phi(I)) = |I|/|P| \geq \beta$. Let C be a spherical cap of measure $\mu(\phi(I))$. Properties (P3) and (P5) show that $2 - \epsilon/(2\sqrt{k}) \leq \text{diam}(C) \leq \text{diam}(\phi(I))$. For $p \in I$, each $\phi(p)$ has diameter at most $\epsilon/(10\sqrt{k})$ so we can find two points $p_1, p_2 \in I$ with $d(p_1, p_2) \geq 2 - \epsilon/\sqrt{k}$, showing that I is not independent. Finally, Property (P4) shows this graph has no K_4 as a subgraph since any K_4 must take two vertices from V_1 and two vertices from V_2 (the graph spanned by V_i is triangle-free). To summarize, we have constructed a K_4-free graph G on n vertices with independence number at most βn and at least $\frac{1}{8}n^2(1-\alpha)$ edges. Since this construction holds for any $\alpha, \beta > 0$, we have proved that $\theta(K_4) \geq \frac{1}{8}$.

The Rödl Graph, [16]. We do not know if $\mathbf{RT}(n, K_{2,2,2}, o(n))$ is $\Omega(n^2)$ or not. Erdős suggested that perhaps some modified version of the Bollobás-Erdős graph could be used to show it is $\Omega(n^2)$. In this direction, Rödl showed how to modify the Bollobás-Erdős graph to exclude both K_4 and $K_{3,3,3}$, proving that $\theta(\{K_4, K_{3,3,3}\}) \geq \frac{1}{8}$. The Rödl Graph is formed by blowing up the Bollobás-Erdős Graph so that each vertex is blown up into an independent set of size t and then randomly delete edges from inside each V_i (see Theorem 16). By randomly deleting edges inside each V_i, we can destroy (almost) all short cycles while not changing the density between V_1 and V_2. Since the original graph does not contain K_4, blowing up the graph will not produce any K_4's. Also, after destroying all short cycles, any graph which is not the union of a bipartite graph with a forest, such as $K_{3,3,3}$, will not be a subgraph of the final graph. One can check that the independence number of the obtained graph has smaller order of magnitude than its number of vertices.

5 Construction

Erdős, Hajnal, Simonovits, Sós, and Szemerédi [5] conjectured (see [5, Conjecture 2.9] and [17, Conjecture 18]) that the asymptotically extremal graphs for $\mathbf{RT}_t(n, K_s, o(n))$ with $s = tq + \ell$ $(1 \leq \ell \leq t)$ have the following structure. Partition n vertices into $q+1$ classes $V_0, \dots, V_q$. For each pair $\{i,j\} \neq \{0,1\}$ we almost completely join V_i to V_j and between V_0 and V_1 we place a graph with density $(\ell-1)/t + o(1)$. Lastly, inside each V_i we insert $o(n^2)$ edges. By optimizing the sizes of the V_is, the number of edges in this graph will be approximately

$$\left(1 - \frac{2t-\ell+1}{q(2t-\ell+1)-\ell+1}\right)\binom{n}{2}.$$

It was suggested in [5] that some modified version of the Bollobás-Erdős graph should be used between V_0 and V_1, but it was not known how to reduce the density of the Bollobás-Erdős graph while still maintaining some useful properties. Our construction is a modified version of the Bollobás-Erdős graph where we are able to reduce the density to roughly 2^{-t^2}. Unfortunately this is too low to match the conjecture but still enough to give a $\Omega(n^2)$ lower bound on $\mathbf{RT}_t(n, K_s, o(n))$.

Our construction depends on four parameters: two integers r and z and two small constants $\alpha, \beta > 0$. Fix an integer $r \geq 3$. Given $\alpha, \beta > 0$, fix ϵ and k so that Properties (P1), (P2) and (P3) hold. Define $\theta = \epsilon/\sqrt{k}$ and $u = \lceil r/2 \rceil$.

For sufficiently large integer z, partition the k-dimensional unit sphere $\mathbb{S}^k$ into z domains having equal measures and diameter at most $\theta/4$. Choose a point from each set and let P be the set of these points. Let $\phi : P \to \mathcal{P}(\mathbb{S}^k)$ map points of P to the corresponding domain of the sphere. The vertices of our hypergraph will be r copies of ordered u-tuples of points from P. Define

$$V = \left\{ (p_1, \dots, p_u) : p_i \in P \text{ and } d(p_i, p_j) \leq \sqrt{2} - \theta \text{ for all } i \neq j \right\}.$$

We will denote vertices in V as $\vec{v}$ and $\left\langle v^{(1)}, \dots, v^{(u)} \right\rangle$ as the coordinates of $\vec{v}$. Let $V_1, \dots, V_r$ be distinct sets isomorphic to V. Let $\mathcal{H} = \mathcal{H}(r, z, \alpha, \beta)$ be the hypergraph with vertex set $V_1 \dot\cup \dots \dot\cup V_r$ and the following hyperedges.

For each $1 \leq i \leq r$, make $E = \{\vec{v}_1, \dots, \vec{v}_r\} \subseteq V_i$ a hyperedge if $|E| = r$ and for every pair $\vec{v}_\ell, \vec{v}_m$ (with $\ell \neq m$), there exists some coordinate $1 \leq j \leq u$ such that $d(\vec{v}_\ell^{(j)}, \vec{v}_m^{(j)}) \geq 2 - \theta$. For cross-hyperedges, make $\{\vec{v}_1, \dots, \vec{v}_r\} \subseteq V(\mathcal{H})$ a hyperedge if $\vec{v}_1 \in V_1, \dots, \vec{v}_r \in V_r$ and $d(v_i^{(j)}, v_\ell^{(m)}) \leq \sqrt{2} - \theta$ for all $1 \leq i, \ell \leq r$ and all $1 \leq j, m \leq u$.

First, we claim some properties of $\mathcal{H}$.

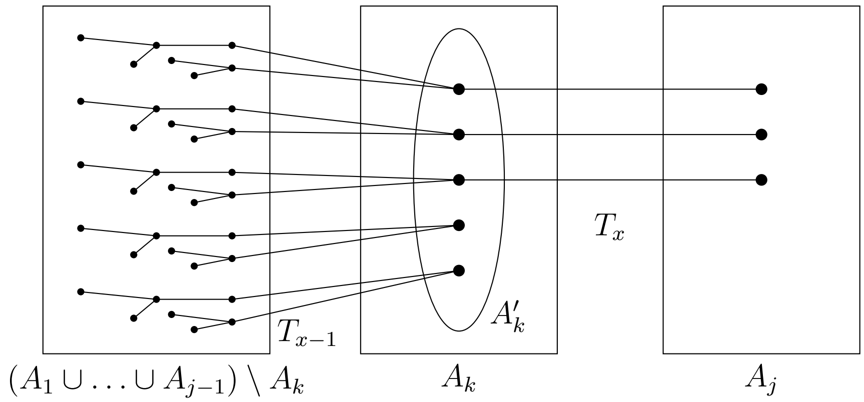

Figure 1: Embedding T during Lemma 13.

Lemma 11. *$\mathcal{H}$ contains no hypergraph in $\mathcal{TK}^r(4)$ embedded so that V_i contains two core vertices and V_j contains two core vertices with some $i \neq j$.*

Proof. Assume without loss of generality that there exist $\vec{v}_1, \vec{v}_2 \in V_1$ and $\vec{v}_3, \vec{v}_4 \in V_2$ which are all core vertices. We will find four points violating Property (P4). By the definition of hyperedges inside V_1, there is some coordinate i such that $d(v_1^{(i)}, v_2^{(i)}) \geq 2-\theta$. Similarly, there is some coordinate j such that $d(v_3^{(j)}, v_4^{(j)}) \geq 2 - \theta$. By the definition of cross-hyperedges, we know that all cross distances are at most $\sqrt{2} - \theta$. We therefore obtain four points $v_1^{(i)}, v_2^{(i)}, v_3^{(j)}, v_4^{(j)}$ which contradict Property (P4). □

Lemma 12. *Let $A_1, A_2 \subseteq P$ with $|A_1| = |A_2| \geq 2\beta z$ and let $t = |A_1|/2$. Then there exist t distinct points $p_1, \ldots, p_t \in A_1$ and t distinct points $q_1, \ldots, q_t \in A_2$ such that $d(p_i, q_i) \geq 2 - \theta$.*

Proof. Let G be the auxiliary bipartite graph on vertex set $A_1 \dot\cup A_2$ where $p \in A_1$ and $q \in A_2$ are adjacent if $d(p,q) \geq 2 - \theta$. We would like to find a matching of size at least t in G. Let M be a maximum matching in G, and assume $|E(M)| < t$. Let $G' = G - V(M)$ with $A'_1 = A_1 - V(M)$ and $A'_2 = A_2 - V(M)$. We will show that G' does not span an independent set, contradicting that M is a maximum matching.

Since $|E(M)| < t$, $|A'_i| > t \geq \beta z$. Let $B_i = \phi(A'_i)$ so that $\mu(B_i) = |A'_i|/z \geq \beta$. Let C be a spherical cap of measure β so $\mu(B_i) \geq \mu(C)$. Properties (P3) and (P6) show that $2-\theta/2 \leq \operatorname{diam}(C) \leq d_{\max}(B_1, B_2)$. Since each $\phi(p)$ has diameter at most $\theta/4$, we must have some $p \in A'_1$ and $q \in A'_2$ with $d(p,q) \geq 2 - \theta$. In other words, pq is an edge of G' which contradicts that M was a maximum matching. □

Lemma 13. *If $A_1, \ldots, A_r \subseteq P$ with $|A_i| \geq 2^r \beta z$ and T is a tree on vertex set $[r]$, then there exist $p_1 \in A_1, \ldots, p_r \in A_r$ such that if $ij \in E(T)$ then $d(p_i, p_j) \geq 2 - \theta$.*

Proof. Assume $|A_i| = 2^r \beta z$. Let G be the auxiliary graph on vertex set $A_1 \dot\cup \ldots \dot\cup A_r$ where $p \in A_i$ and $q \in A_j$ are adjacent if $i \neq j$ and $d(p,q) \geq 2 - \theta$. We would like to find an embedding of T into G such that $i \in V(T)$ is embedded into A_i.

Let $T_1 \subseteq T_2 \subseteq \cdots \subseteq T_{r-1} = T$ be subtrees of T where T_x is formed by deleting a leaf of T_{x+1}. We prove by induction on x that we can find $2^{r-x}\beta z$ vertex disjoint embeddings of T_x into G where $i \in V(T_x)$ is embedded into A_i. Since T_1 is just a single edge, Lemma 12 shows that we can find $|A_i|/2 = 2^{r-1}\beta z$ vertex disjoint embeddings of T_1.

Assume $x \geq 2$. By induction, we can find at least $2^{r-x+1}\beta z$ vertex disjoint embeddings of T_{x-1} into G. Let $j \in V(T_x)$ be the leaf of T_x deleted to form T_{x-1} and let k be the neighbor of j in T_x. Let A'_k be the set of vertices in A_k used by the embeddings of T_{x-1}, so that $|A'_k| \geq 2^{r-x+1}\beta z$. We now apply Lemma 12 to A'_k and A_j to find a matching between A'_k and A_j using at least $|A'_k|/2 \geq 2^{r-x}\beta z$ edges. Since the vertices of this matching are distinct, at least $2^{r-x}\beta z$ of the embeddings of T_{x-1} extend to embeddings of T_x. □

Lemma 14. *For every s, $\alpha(\mathcal{H}[V_s]) \leq r^u 2^{u+r}\beta z^u$.*

Proof. Fix an arbitrary set $X \subseteq V_s$ with $|X| = r^u 2^{u+r}\beta z^u$. Let $T_1, \ldots, T_u$ be trees for which $V(T_i) = [r]$ and $\cup T_i$ is the complete graph on vertex set $[r]$. Observe that the only property that we use about our trees is that they cover the edge set of a K_r. Note that if $\vec{v}_1, \ldots, \vec{v}_r \in X$ such that $d(v_i^{(j)}, v_i^{(\ell)}) \geq 2 - \theta$ when $j\ell \in E(T_i)$, then $\{\vec{v}_1, \ldots, \vec{v}_r\}$ forms a hyperedge inside X. We will find these vertices by repeatedly applying Lemma 13.

Let $0 \leq j < u$. Assume we have already selected $v_1^{(1)}, \ldots, v_1^{(j)}, v_2^{(1)}, \ldots, v_2^{(j)}, \ldots, v_r^{(1)}, \ldots, v_r^{(j)}$, that is coordinates 1 through j for all r vertices to be found. For each i, define the set of candidates to continue the future vertex $\vec{v}_i$ as

$$C_i^{(j)} = \left\{ \left\langle v_i^{(1)}, \ldots, v_i^{(j)}, q_{j+1}, \ldots, q_u \right\rangle \in X : q_{j+1}, \ldots, q_u \in P \right\}.$$

Initially, $C_i^{(0)} = X$. Throughout the selection process we maintain that the size of $\left|C_i^{(j)}\right|$ is at least $r^{u-j}2^{u-j+r}\beta z^{u-j}$.

We now show how to select $v_1^{(j+1)}, \ldots, v_r^{(j+1)}$. For $1 \leq i \leq r$, call a $(j+1)$-tuple $(v_i^{(1)}, \ldots, v_i^{(j)}, p)$ **bad** if

$$\left| \left\{ \left\langle v_i^{(1)}, \ldots, v_i^{(j)}, p, q_{j+2}, \ldots, q_u \right\rangle \in C_i^{(j)} : q_{j+2}, \ldots, q_u \in P \right\} \right| < r^{u-j-1}2^{u-j-1+r}\beta z^{u-j-1}.$$

Form D_i by deleting all vertices $\vec{w}$ from $C_i^{(j)}$ where the first $j+1$ coordinates of $\vec{w}$ form a bad tuple. Counting the number of vertices we delete, there are r choices for i, there are at most z choices for p, and there are at most $r^{u-j-1}2^{u-j-1+r}\beta z^{u-j-1}$ choices for the rest of the coordinates. Thus the number of vertices we delete is at most $r^{u-j}2^{u-j-1+r}\beta z^{u-j}$ so $|D_i| \geq r^{u-j}2^{u-j-1+r}\beta z^{u-j}$.

Now define

$$A_i = \left\{ p \in P : \exists q_{j+2}, \ldots, q_u \in P \text{ where } \left\langle v_i^{(1)}, \ldots, v_i^{(j)}, p, q_{j+2}, \ldots, q_u \right\rangle \in D_i \right\}.$$

If $|A_i| < 2^r\beta z$, then $|D_i| < 2^r\beta z^{u-j} \leq 2^{u-j+r}\beta z^{u-j}$ which is a contradiction. Now apply Lemma 13 to $A_1, \ldots, A_r$ and T_{j+1} to obtain $v_1^{(j+1)} \in A_1, \ldots, v_r^{(j+1)} \in A_r$. Since none of the tuples $(v_i^{(1)}, \dots, v_i^{(j+1)})$ are bad,

$$\left| C_i^{(j+1)} \right| \geq r^{u-j-1} 2^{u-j-1+r} \beta z^{u-j-1}$$

for every i. ☐

Lemma 15. *Let $\mathcal{E} = \{\{\vec{v}_1, \dots, \vec{v}_r\} \in \mathcal{H} : \vec{v}_i \in V_i\}$. Then there exists a constant c depending only on r such that*

$$|V(\mathcal{H})| \leq r 2^{-\binom{u}{2}} z^u$$

and

$$|\mathcal{E}| \geq 2^{-\binom{ru}{2}} z^{ru} - c\alpha z^{ru} \geq 2^{r\binom{u}{2} - \binom{ru}{2}} \left(\frac{|V(\mathcal{H})|}{r} \right)^r - c\alpha \left| V(\mathcal{H}) \right|^r.$$

Proof. By Property (P2), each V_i has size at most $z \prod_{i=1}^{u-1} \left(\frac{z}{2^i} - i\alpha z \right)$ so the number of vertices is at most $r 2^{-\binom{u}{2}} z^u$. Using Property (P2) there are at least

$$z \prod_{i=1}^{ru-1} \left(\frac{z}{2^i} - i\alpha z \right)$$

choices of ru points on the sphere with pairwise distance at most $\sqrt{2} - \theta$. Each of these ru-sets of points form a cross-hyperedge. ☐

6 Proofs of Theorems 3, 7, and 9

We now turn our attention to proving Theorems 3, 7, and 9. Consider the construction $\mathcal{H}$ from Section 5 and assume $\mathrm{TK}^r(r+2)$ is a subhypergraph. Lemma 11 tells us it is impossible to have two core vertices in two different classes, so we must have three core vertices in some part. $\mathcal{H}$ itself may contain a copy of $\mathrm{TK}^r(3)$ inside one part, but by using an idea of Rödl [16] we are able to eliminate this possibility by blowing up the hypergraph $\mathcal{H}$. In [16], Rödl proved a variant of the following theorem for graphs and the special case when $\mathcal{F}$ is a cycle.

Theorem 16. *Let $\mathcal{H}$ be an r-uniform hypergraph on n vertices. Let $0 < \gamma < 1$ and let ℓ be a positive integer. Then there exists a $t = t(\mathcal{H}, \ell, \gamma, r)$ and an r-uniform hypergraph $\mathcal{G}$ with vertex set $V(\mathcal{H}) \times [t]$ with the following properties.*

- *For all $\{a_1, \dots, a_r\} \in \mathcal{H}$ and all sets $U_i \subseteq \{a_i\} \times [t]$ with $|U_i| \geq \gamma t$ for each $1 \leq i \leq r$, there exists at least one hyperedge of $\mathcal{G}$ with one vertex in each U_i.*
- *$\mathcal{G}$ does not contain as a subhypergraph any v-vertex hypergraph $\mathcal{F}$ with m edges where $v \leq \ell$ and $v + (1 + \gamma - r)(m-1) < r$.*

Proof. Let $\mathcal{H}'$ be the t-blowup of $\mathcal{H}$. That is, $V(\mathcal{H}') = V(\mathcal{H}) \times [t]$ and the hyperedges are $\{\{(a_1, i_1), (a_2, i_2), \ldots, (a_r, i_r)\} : \{a_1, \ldots, a_r\} \in \mathcal{H}, 1 \leq i_1, \ldots, i_r \leq t\}$. Let $\mathcal{H}''$ be a random subhypergraph of $\mathcal{H}'$ where each hyperedge is chosen independently with probability $p = t^{1+\gamma-r}$ (note that $r \geq 2$ and γ is small so that $p < 1$). Let $\mathcal{F}$ be a v-vertex hypergraph with $|\mathcal{F}| = m$ and where $v + (1+\gamma-r)(m-1) < r$. The expected number of copies of $\mathcal{F}$ in $\mathcal{H}''$ is bounded by $c_1 t^v p^m = o(pt^r)$ where c_1 is some constant depending only on $\mathcal{H}$ and ℓ. We now delete one hyperedge from each copy of $\mathcal{F}$ in $\mathcal{H}''$. There are at most 2^{ℓ^r} such hypergraphs $\mathcal{F}$ so we can make t sufficiently large so that we delete fewer than $\frac{\gamma^r}{2} pt^r$ hyperedges. $\mathcal{G}$ is the resulting graph which now satisfies the second property.

Now fix a hyperedge $E = \{a_1, \ldots, a_r\} \in \mathcal{H}$ and $U_i \subseteq \{a_i\} \times [t]$ with $|U_i| = \gamma t$ for $1 \leq i \leq r$. We now show that the probability that all blowups of the hyperedge E intersecting all V_i are deleted is exponentially small. Before deletion, the expected number of blowups of E where the copy of a_i appears in U_i for each i is $p(\gamma t)^r$. By Chernoff's Inequality, the probability that there are at most $\frac{1}{2} p(\gamma t)^r$ such blowups of E is bounded by $e^{-c_2 pt^r}$ where c_2 is some constant depending only on γ. Since we delete only $\frac{1}{2} p(\gamma t)^r$ hyperedges in total, the probability that we delete all blowups of E where the copy of a_i appears in U_i for each i is at most $e^{-c_2 pt^r}$.

We now use the union bound to bound the probability that there is some hyperedge $E = \{a_1, \ldots, a_r\} \in \mathcal{H}$ and some $U_i \subseteq \{a_i\} \times [t]$ with $|U_i| = \gamma t$ for $1 \leq i \leq r$ where we deleted all blowups of the edge E where the copy of a_i appears in U_i for each i. This probability is bounded by

$$|\mathcal{H}| \binom{t}{\gamma t}^r e^{-c_2 pt^r} \leq |\mathcal{H}| \left(\frac{e}{\gamma}\right)^{\gamma r t} e^{-c_2 pt^r} \leq |\mathcal{H}| \, e^{c_3 t} e^{-c_4 t^{1+\gamma}} = o(e^{-t})$$

where c_3 and c_4 are constants depending only on γ and r. □

By combining the construction from Section 5 and the previous theorem, we prove Theorem 9.

Proof of Theorem 9. Let $z = N$ and let $\mathcal{H} = \mathcal{H}(r, z, \alpha, \beta)$ be the hypergraph constructed in Section 5 and $V_1, \ldots, V_r$ the partition of the vertex set of $\mathcal{H}$. Let $\mathcal{E}_1$ be the set of cross-hyperedges, that is $\mathcal{E}_1 = \{\{\vec{v}_1, \ldots, \vec{v}_r\} \in \mathcal{H} : \vec{v}_i \in V_i\}$ and let $\mathcal{E}_2 = \mathcal{H} - \mathcal{E}_1$ so $\mathcal{E}_2$ is the set of hyperedges which are inside some V_i. Let $\gamma = \beta$ and $\ell = r^3$ and apply Theorem 16 to $\mathcal{E}_2$ to obtain $\mathcal{E}_2'$ where $V(\mathcal{E}_2') = V(\mathcal{H}) \times [t]$. Let $\mathcal{G}$ be $\mathcal{E}_2'$ together with all the hyperedges

$$\{\{(\vec{v}_1, a_1), \ldots, (\vec{v}_r, a_r)\} : \{\vec{v}_1, \ldots, \vec{v}_r\} \in \mathcal{E}_1, 1 \leq a_i \leq t\}.$$

Let $m = |V_i| \, t \approx 2^{-u(u-1)/2} z^u t$ so that $\mathcal{G}$ has rm vertices, and let $W_i = V_i \times [t]$. Now we verify the claimed properties of $\mathcal{G}$.

(i) By Lemma 11, $\mathcal{H}$ contains no hypergraph in $\mathcal{TK}^r(4)$ embedded so that V_i has two core vertices and V_j has two core vertices. Since the blow up preserves this, the same holds for $\mathcal{G}$, W_i, and W_j.

(ii) By Lemma 15, $|\mathcal{E}_1| \geq 2^{-\binom{ru}{2}} z^{ru} - c_2 \alpha z^{ru}$. Because during blow up we keep all cross hyperedges,

$$e(\mathcal{G}) \geq 2^{-\binom{ru}{2}} z^{ru} t^r - c_2 \alpha z^{ru} t^r = \left(2^{r\binom{u}{2} - \binom{ru}{2}} - c_2 \alpha 2^{u(u-1)r/2}\right) m^r$$

where c_2 is some constant depending only on r.

(iii) Theorem 16 shows that $\mathcal{G}[W_i]$ does not contain as a subhypergraph any hypergraph $\mathcal{F}$ with $|V(\mathcal{F})| \leq r^3 = \ell$ and $|V(\mathcal{F})| + (1-r)(|\mathcal{F}|-1) < r$.

(iv) Let I be a vertex set in $\mathcal{G}[W_1]$ with $|I| = r^u 2^{u+r+1}\beta z^u t$. For $\vec{v} \in V_1$, call $\vec{v}$ γ**-bad** if there are fewer than γt indices $1 \leq i \leq t$ such that $(\vec{v}, i) \in I$. Form I' by deleting all pairs $(\vec{v}, i)$ from I where $\vec{v}$ is γ-bad. We deleted at most $z^u \gamma t = z^u \beta t$ pairs so $|I'| \geq r^u 2^{u+r}\beta z^u t$. Define $A = \{\vec{v} \in V_1 : (\vec{v}, i) \in I' \text{ for some } 1 \leq i \leq t\}$. Then $|A| \geq r^u 2^{u+r}\beta z^u$, so by Lemma 14 we must have a hyperedge $\{\vec{v}_1, \ldots, \vec{v}_r\}$ contained in $\mathcal{H}[A] \subseteq \mathcal{E}_2$. Define $B_i = (\{\vec{v}_i\} \times [t]) \cap I'$. Since no $\vec{v}_i$ is γ-bad we have $|B_i| \geq \gamma t$ for every $1 \leq i \leq r$. By Theorem 16 there exists a hyperedge of $\mathcal{E}_2'[W_1] \subseteq \mathcal{G}[W_1]$ with one vertex in each B_i, which is a hyperedge contained in I. This shows that the independence number of $\mathcal{G}[W_i]$ is at most $r^u 2^{u+r+1}\beta z^u t$ for each $1 \leq i \leq r$, which implies that $\mathcal{G}$ has independence number at most $r^{u+1} 2^{u+r+1}\beta z^u t \leq c_1 \beta m$, where c_1 is a constant depending only on r. □

Proof of Theorem 7 (i). Let $\mathcal{G}$ be the construction from Theorem 9 and assume that $\mathrm{TK}^r(r+2)$ is a subhypergraph. Since we cannot have two core vertices in two different parts, the copy of $\mathrm{TK}^r(r+2)$ must have three core vertices in one part. Let $\mathcal{F} = \mathrm{TK}^r(3)$. Then $|\mathcal{F}| = 3$ and $|V(\mathcal{F})| = 3 + 3(r-2) = 3r - 3 < r + 2(r-1) = r + (r-1)(|\mathcal{F}| - 1) = 3r - 2$ which contradicts Theorem 9 (iii).

Let $n = |V(\mathcal{G})|$. From Theorem 9, we know that $n = rm$ and that

$$|\mathcal{G}| \geq 2^{r\binom{u}{2} - \binom{ru}{2}} m^r - c_1 \alpha m^r = 2^{r\binom{u}{2} - \binom{ru}{2}} \left(\frac{n}{r}\right)^r - c_2 \alpha n^r,$$

where c_1 and c_2 are constants depending only on r. Thus for any $\alpha > 0$, we know that

$$\begin{aligned}\lim_{\beta \to 0} \lim_{n \to \infty} \frac{\mathbf{RT}(n, \mathrm{TK}^r(r+2), \beta n)}{n^r} &\geq \lim_{\beta \to 0} \lim_{n \to \infty} \frac{2^{r\binom{u}{2} - \binom{ru}{2}} \left(\frac{n}{r}\right)^r - c_2 \alpha n^r}{n^r} \\ &\geq 2^{r\binom{u}{2} - \binom{ru}{2}} \left(\frac{1}{r}\right)^r - c_2 \alpha,\end{aligned}$$

yielding $\theta(\mathrm{TK}^r(r+2)) \geq 2^{r\binom{u}{2} - \binom{ru}{2}} (1/r)^r$. □

Proof of Theorem 7 (ii). The proof is similar to the proof of Theorem 7 (i). No copy of a hypergraph in $\mathcal{TK}^r(2r)$ can have two core vertices in two different parts so it must have at least $r+1$ core vertices in a single part. To complete the proof, we just need to show that every minimal hypergraph $\mathcal{F}$ in $\mathcal{TK}^r(r+1)$ satisfies $|V(\mathcal{F})| \leq r + (r-1)(|\mathcal{F}| - 1)$. Let $v_1, \ldots, v_{r+1}$ be the core vertices of $\mathcal{F}$. For $1 \leq a < b \leq r+1$, since $\mathcal{F} \in \mathcal{TK}^r(r+1)$ there exists some hyperedge containing both $\vec{v}_a$ and $\vec{v}_b$. Let $E_{a,b}$ be a hyperedge of containing both $\vec{v}_a$ and $\vec{v}_b$ (if there are more than one such hyperedges, pick one arbitrarily.) Now consider the ordering

$$E_{1,2}, E_{1,3}, \ldots, E_{1,r+1}, E_{2,3}, \ldots, E_{2,r+1}, E_{3,4}, \ldots, E_{r,r+1}.$$

Since $\mathcal{F}$ is minimal, all hyperedges of $\mathcal{F}$ appear in the ordering somewhere. Now let $F_1, \ldots, F_m$ be a list of the hyperedges of $\mathcal{F}$ where for each hyperedge $D \in \mathcal{F}$, we keep the first copy of D in the ordering and remove all other copies. By the choice of ordering, each F_i must use at least one vertex from the previous hyperedges. Therefore, $|V(\mathcal{F})| \leq r+(r-1)(m-1)$. In fact, the last hyperedge must use at least two previous vertices so we can reduce the bound by one to $|V(\mathcal{F})| \leq r+(r-1)(m-1)-1$. □

The **shadow graph** of a hypergraph $\mathcal{H}$ is a graph G with $V(G) = V(\mathcal{H})$ and $xy \in E(G)$ if there exists some hyperedge E of $\mathcal{H}$ with $x, y \in E$. We will now show that Theorem 3 follows by looking at the shadow graph of the hypergraph from Theorem 9.

Proof of Theorem 3. Let $r = t$ and let $\mathcal{G}$ be the hypergraph constructed in Theorem 9 with parts $W_1, \ldots, W_r$. Let G be the shadow graph of $\mathcal{G}[W_1 \cup W_2 \cup \ldots \cup W_\ell]$, so we take only the shadow graph of the first ℓ parts. Then $\alpha_t(G)$ is small because any hyperedge inside $\mathcal{G}[W_i]$ turns into a copy of K_t in $G[W_i]$ for $1 \leq i \leq \ell$. Assume G contains $K_{t+\ell}$. It is not possible to have two of the vertices in W_i and two of the vertices in W_j with $i \neq j$ because then $\mathcal{G}$ would contain a $\mathcal{TK}^r(4)$ arranged so that two core vertices are in W_i and two core vertices are in W_j. Thus we can assume without loss of generality that $G[W_1]$ contains K_{t+1}. This implies that $\mathcal{G}[W_1]$ contains a hypergraph in $\mathcal{TK}^r(r+1)$ which was excluded by the proof of Theorem 7 (ii).

To compute the number of edges of G, we must use Property (P1). Edges between W_i and W_j are chosen by picking $2u$ points within distance $\sqrt{2} - \theta$ on the sphere and then blowing each vertex up into size t. Therefore, we have at least $t^2 z \prod_{i=1}^{2u-1} \left(\frac{z}{2^i} - i\alpha z\right)$ edges between W_i and W_j. Thus

$$|E(G)| \geq \binom{\ell}{2} 2^{-\binom{2u}{2}} z^{2u} t^2 - c_1 \alpha z^{2u} t^2, \tag{4}$$

where c_1 is some constant depending only on r. Each W_i has size at most $2^{-\binom{u}{2}} z^u t$ so G has at most $\ell 2^{-\binom{u}{2}} z^u t$ vertices. Thus

$$\frac{2^{\binom{u}{2}}}{\ell} |V(G)| \leq z^u t. \tag{5}$$

Combining (4) with (5), we obtain

$$\begin{aligned} |E(G)| &\geq \binom{\ell}{2} 2^{-\binom{2u}{2}} \left(\frac{2^{\binom{u}{2}}}{\ell} |V(G)| \right)^2 - c_2 \alpha |V(G)|^2 \\ &\geq \frac{1}{2} \frac{\ell(\ell-1)}{\ell^2} 2^{u(u-1)-u(2u-1)} |V(G)|^2 - c_2 \alpha |V(G)|^2 \\ &\geq \frac{1}{2} \left(1 - \frac{1}{\ell}\right) 2^{-u^2} |V(G)|^2 - c_2 \alpha |V(G)|^2 \end{aligned}$$

for some constant c_2 depending only on r. □

7 Lower bounds on the Ramsey-Turán threshold functions

The main tool to prove Theorem 10 is the method of dependent random choice. It is a simple yet surprisingly powerful technique which has found applications in Extremal Graph Theory, Ramsey Theory, Additive Combinatorics, and Combinatorial Geometry. Early versions of this technique were proved and applied by several researchers, starting with Gowers, Kostochka, Rödl, and Sudakov. Gowers [11] used a variant of dependent random choice in an alternate proof of Szemerédi's Theorem [20] for four-term arithmetic progressions, Kostochka and Rödl [12] used it to investigate bipartite Ramsey numbers, and Sudakov [18] used it to prove $\mathbf{RT}(n, K_4, 2^{-w(n)\sqrt{\log n}}) = o(n^2)$, where $w(n)$ is arbitrary function tending to infinity. Since then, many other applications of the dependent random choice method have been found (see [10] for a survey).

Lemma 17. *(Dependent Random Choice, Lemma 2.1 in [10]). Let a, m, n, r, t be positive integers. Let G be an n-vertex graph with average degree $d := 2\,|E(G)|\,/n$. If*

$$\frac{d^t}{n^{t-1}} - \binom{n}{r}\left(\frac{m}{n}\right)^t \geq a,$$

then G contains a subset U of at least a vertices such that any r vertices in U have at least m common neighbors.

Conlon, Fox, and Sudakov [4], investigating the Ramsey numbers of sparse hypergraphs, extended Lemma 17 to hypergraphs. The **weight** $w(S)$ of a set S of hyperedges in a hypergraph is the number of vertices in the union of these edges.

Lemma 18. *(Hypergraph Dependent Random Choice, Lemma 1 in [4]). Suppose s, Δ are positive integers, $\epsilon, \beta > 0$, and G_r is an r-uniform, r-partite hypergraph with parts $V_1, \ldots, V_r$, each part having size N. Suppose G_r has at least ϵN^r edges. Then there exists an $(r-1)$-uniform, $(r-1)$-partite hypergraph G_{r-1} on the vertex set $V_2 \cup \ldots \cup V_r$ which has at least $\frac{1}{2}\epsilon^s N^{r-1}$ edges and such that for each nonnegative integer $w \leq (r-1)\Delta$, there are at most $4r\Delta\epsilon^{-s}\beta^s w^{r\Delta} r^w N^w$ dangerous sets of edges of G_{r-1} with weight w, where a set S of edges of G_{r-1} is dangerous if $|S| \leq \Delta$ and the number of vertices $v \in V_1$ such that for every edge $e \in S$, $e + v \in G_r$ is less than βN.*

The main idea of the proof of Theorem 10 is to first apply Lemma 18 to obtain a graph G and then apply Lemma 17 to G. Lemma 17 guarantees a set U large enough so that we can find a hyperedge E_3 contained inside U. The vertices of E_3 have a large number of common neighborhood in G, sufficient to find a hyperedge E_2 among the common neighbors. Then the hypergraph dependent random choice lemma shows that we can extend the edges of G spanned by $E_2 \cup E_3$ to hyperedges. We thus find the following hypergraph. Let F be the 3-uniform hypergraph with vertices $\{x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2, z_3\}$ and edges $\{x_1x_2x_3, y_1y_2y_3, z_1z_2z_3\} \cup \{x_iy_jz_k : 1 \leq i, j, k \leq 3\}$. Note that $F \in \mathcal{TK}^3(9)$. For a 3-uniform hypergraph, the **codegree** $d(x, y)$ of a pair of vertices x, y is the number of edges E with $x, y \in E$.

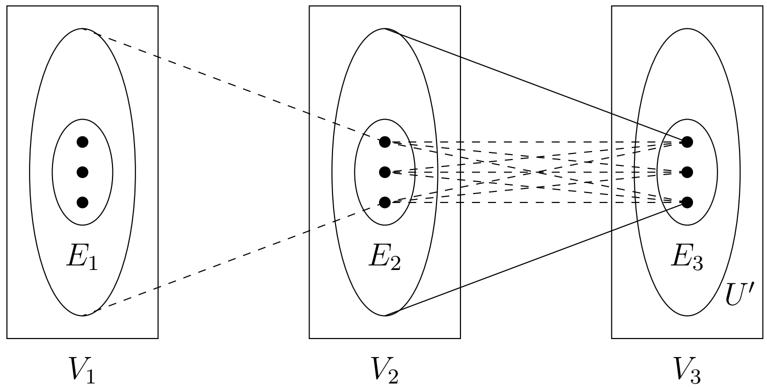

Figure 2: Embedding F in Theorem 19.

Theorem 19. *Let $\gamma = \gamma(n)$ be any function going to infinity arbitrarily slowly. Let $\beta = \beta(n) = 2^{-\gamma(\log n)^{2/3}}$. There exists a constant b such that if $\mathcal{H}$ is an n-vertex, 3-uniform hypergraph with independence number at most $\frac{1}{3}\beta n$ and at least $bn^3 2^{-\gamma^3/28} + 144n^2 = o(n^3)$ edges, then $\mathcal{H}$ contains F and $\mathrm{TK}^3(6)$.*

Proof. Let $N = \frac{n}{3}, \Delta = 9, w = 6, r = 3, c = 4r\Delta w^{r\Delta} r^w$, $s = \frac{w+1}{\gamma}\sqrt[3]{\log n}$, and $\epsilon = 2^{-\gamma^2 \sqrt[3]{\log n}/4}$. Let $b = 9c$, so $\mathcal{H}$ has at least $9c\epsilon^{1/s}N^3 + 144n^2$ edges and independence number at most $\frac{1}{3}\beta n$.

For simplicity, assume 3 divides n and let $\mathcal{H}'$ be a 3-partite subhypergraph of $\mathcal{H}$ with equal part sizes. We can always find such a hypergraph $\mathcal{H}'$ with at least $\frac{1}{9}$ of the edges of $\mathcal{H}$. For each pair x, y of vertices in different parts, delete all edges containing both x and y if the codegree $d(x, y)$ is at most 16. We delete at most $16n^2$ hyperedges. Thus we have a 3-partite hypergraph $\mathcal{H}'$ with at least $c\epsilon^{1/s}N^3$ edges and the codegree of any pair of vertices from different parts is zero or at least 16. Let V_1, V_2, V_3 be the parts of $\mathcal{H}'$, each part having size N.

We now apply Lemma 18 to $\mathcal{H}'$ to obtain a graph G on $V_2 \cup V_3$ with at least $\frac{1}{2}\left(c\epsilon^{1/s}\right)^s N^2 \geq 2\epsilon N^2$ edges and at most

$$4r\Delta\left(c\epsilon^{1/s}\right)^{-s}\beta^s w^{r\Delta} r^w N^w \leq \epsilon^{-1}\beta^s N^6 \leq 2^{\gamma^2\sqrt[3]{\log n}/4 - 7\log n} N^6$$

dangerous sets of edges of weight 6. When n is large, the number of dangerous sets is at most $1/2$ so we can assume G has no dangerous sets of weight 6.

We now apply Lemma 17 to G with $t = \frac{4}{\gamma}\sqrt[3]{\log n}$, $d = 4\epsilon N$, and $a = m = 2\beta N$. Let $n_1 = |V(G)| = 2N$. We check

$$\begin{aligned}\frac{d^t}{n_1^{t-1}} - \binom{n_1}{3}\left(\frac{m}{n_1}\right)^t &\geq 4\epsilon^t N - 2N^3\beta^t \geq 2^{2-\gamma(\log n)^{2/3}} N - 2^{1-4\log n} N^3 \\ &\geq 4\beta N - \frac{1}{2} \geq a = m.\end{aligned}$$

Therefore we have a subset U of $V(G)$ with $|U| = m = 2\beta N$ such that every three vertices of U have at least βN common neighbors in G. Either V_2 or V_3 contains at least half of the vertices of U, so assume by symmetry that $U' = U \cap V_3$ has at least βN vertices.

The set U' contains a hyperedge E_3 of $\mathcal{H}$ since the size of U' is larger than the independence number of $\mathcal{H}$. The vertices of E_3 have at least βN common neighbors in G, so the common neighbors contain a hyperedge E_2. By Lemma 18 G is bipartite, so $E_3 \subseteq V_3$ implies that $E_2 \subseteq V_2$. If we take S to be the nine edges of G spanned by the vertices $E_2 \cup E_3$, then S has weight 6 so it is not dangerous. Therefore, we find at least βN vertices v in V_1 such that vxy is a hyperedge for all $x \in E_2$ and all $y \in E_3$. These βN vertices contain a hyperedge E_1 of $\mathcal{H}$.

Let $E_1 = \{x_1, x_2, x_3\}$, $E_2 = \{y_1, y_2, y_3\}$, and $E_3 = \{z_1, z_2, z_3\}$. These vertices form a copy of F within $\mathcal{H}$. We also find a copy of $\mathrm{TK}^3(6)$ with core vertices $x_1, x_2, y_1, y_2, z_1, z_2$. Vertices x_1 and x_2 are contained together in the hyperedge $x_1x_2x_3$. Since x_i and y_j are contained together in at least one hyperedge of $\mathcal{H}'$, the codegree of x_i and y_j in $\mathcal{H}$ is at least 16. We can therefore find a distinct vertex in V_3 which is contained in a hyperedge together with x_i and y_j. The pairs x_i, z_j and y_i, z_j are handled similarly. $\square$

8 Open problems

There are many open problems remaining in Ramsey-Turán theory.

- The exact value of $\mathbf{RT}_3(n, K_s, o(n))$ for small values of s are mostly still unknown. Erdős, Hajnal, Simonovits, Sós, and Szemerédi [5] proved that $\theta_3(K_s) = \frac{1}{2}\left(1 - \frac{3}{s-1}\right)$ when $s \equiv 1 \pmod 3$. The best bound for $s = 5$ is our lower bound of $\frac{1}{64}$ and an upper bound of $\frac{1}{12}n^2$ by [5]. For $s = 6$, $\frac{1}{48} \leq \theta_3(K_6) \leq \frac{1}{6}$. In [5] a construction is given which is conjectured to show $\theta_3(K_6) \geq 1/8$; most likely the construction is correct. Based on these bounds, the following question is natural. Is there a construction determining $\theta(K_s)$ where the density of edges between classes is not $2^{-\ell}$ for some integer ℓ?

- In the area of the Ramsey-Turán theory, one of the major open problems is to prove a generalization of the Erdős-Stone Theorem [9] by proving that $\theta(H) = \theta(K_s)$ where s is equal to some parameter depending only on H. Let s be the minimum number such that $V(H)$ can be partitioned into $\lceil s/2 \rceil$ sets $V_1, \ldots, V_{\lceil s/2 \rceil}$ such that $V_1, \ldots, V_{\lfloor s/2 \rfloor}$ span forests and if s is odd $V_{\lceil s/2 \rceil}$ spans an independent set. In [5] it was proved that $\theta(H) \leq \theta(K_s)$, where the inequality is sharp for odd s. In several papers, Erdős mentioned the simplest open case when $H = K_{2,2,2}$, where one would like to know at least if $\theta(K_{2,2,2}) = 0$ (see [17, Problem 4], [6, p. 72], [18, Problem 1.3] among others).

- Can the theorem of Bollobás [2] can be (partially) saved? We think that the following version of the Erdős Conjecture could be true. Recall that for $A \subseteq \mathbb{S}^k$ and $t \geq 2$,

$$d_t(A) = \sup \left\{ \min_{i \neq j} d(x_i, x_j) : x_1, \ldots, x_t \in A \right\}.$$

Conjecture 20. *For every t positive integer and $\epsilon > 0$ there is a k_0 such that the following holds: For every $k > k_0$ and measurable $A \subseteq \mathbb{S}^k$ if $C \subseteq \mathbb{S}^k$ is a spherical cap with $\mu(C) = \mu(A) > \epsilon$, then $d_t(A) \geq d_t(C)$.*

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