An extension of Dirac's Theorem

József Balogh*
Department of Mathematical Sciences
University of Illinois at Urbana-Champaign
Urbana, Illinois 61801, USA
jobal@math.uiuc.edu

Frank Mousset†
Institute of Theoretical Computer Science
ETH Zürich
8092 Zürich, Switzerland
frank.mousset@inf.ethz.ch

Jozef Skokan
Department of Mathematics
London School of Economics
Houghton Street, London, WC2A 2AE, England
Department of Mathematical Sciences
University of Illinois at Urbana-Champaign
Urbana, Illinois 61801, USA
jozef@member.ams.org

April 14, 2015
Mathematics Subject Classifications: 05C35, 05C38, 05C75

Abstract

The following extension of Dirac's Theorem was conjectured by Enomoto, Kaneko and Tuza: if G is an n-vertex graph with minimum degree at least n/k, then there are $k-1$ cycles in G covering the

*Research is partially supported by Simons Fellowship, NSF CAREER Grant DMS-0745185, Marie Curie FP7-PEOPLE-2012-IIF 327763.
†Supported by grant no. 6910960 of the Fonds National de la Recherche, Luxembourg.

vertex set $V(G)$. Kouider proved this conjecture under the extra condition that G is 2-connected, and pointed out that this condition is necessary when $k > \sqrt{n}$. Here, we show that, for a fixed integer k and n sufficiently large, if G is an n-vertex graph with minimum degree at least n/k, then there are $k-1$ cycles in G covering the vertex set $V(G)$. This bound is best possible since there exist graphs with minimum degree $n/k-1$ which do not have this property. In our proof we use modern methods, including using the Szemerédi Regularity Lemma and the technique of 'connected matchings'. These methods also allow us to give a simple description of the extremal structure of graphs with minimum degree almost n/k that cannot be covered with $k-1$ cycles.

1 Introduction

The theorem of Dirac [5] that any graph G on $n \geq 3$ vertices with minimum degree $\delta(G) \geq n/2$ contains a Hamilton cycle is one of the classical results of graph theory. There is a rich collection of extensions of this theorem in various directions. One possible direction is to replace the Hamilton cycle with another spanning subgraph and ask what minimum degree guarantees its existence.

Pósa [9] and Seymour [20] were first to conjecture that any n-vertex graph with minimum degree at least $kn/(k+1)$ contains the k-th power of a Hamilton cycle. This conjecture was proved for large n by Komlós, Sárközy, and Szemerédi [11], [13] using the regularity method. Later, Levitt, Sárközy, Szemerédi [17] and Chau, DeBiasio, and Kierstead [4] proved the same result with different methods, for smaller values of n. When we consider the square of a Hamilton path instead of the square of a Hamilton cycle, Fan and Kierstead [7] proved that $(2n-1)/3$ is the optimal minimum degree for every n.

In this paper we explore the other direction, when the minimum degree of a graph gets smaller. Already Dirac observed that either an n-vertex graph is not 2-connected or it must contain a cycle of length at least $\min\{n, 2\delta(G)\}$. From this one easily deduces that a graph with minimum degree $n/2-c$ is either 'close' to a graph consisting of two disjoint copies of $K_{n/2}$ or it contains an almost spanning cycle (of length at least $n-2c$).

We are interested in the following question of Enomoto, Kaneko and Tuza [6] from 1987: if an n-vertex graph has minimum degree cn, how many cycles do we need to cover its vertices? For $c \in [1/2, 1)$, we need one by

Dirac's theorem. For $c < 1/2$, the question is answered by the following theorem.

Theorem 1. *Let k be a positive integer and let G be a graph of sufficiently large order n, with minimum degree $\delta(G) \geq n/k$. Then the vertex set of G can be covered with $k-1$ edge-disjoint cycles.*

Kouider [15] proved the Theorem 1 for every k and every n, under the extra condition that G is 2-connected. Additionally, it was pointed out that this condition is needed when $k > \sqrt{n}$. Furthermore, it was mentioned in [15] that the case $k = 3$ was settled by Alon and by Enomoto, Kaneko and Tuza [6].

Extremal structure. The minimum degree condition is tight by at least two types of examples of graphs with km vertices and minimum degree $m-1$ that cannot be covered with $k-1$ cycles. The first type consists of graphs with minimum degree $m-1$ that contain an independent set of size $(k-1)m+1$, such as for example the complete bipartite graph $G = K_{m-1,(k-1)m+1}$, or the graph $G = K_{km} \setminus K_{(k-1)m+1}$. The other type consists of block graphs on km vertices that have k biconnected components, each of which is a clique of size at least m (one can see that at least one of those cliques must have size exactly m). Moreover, for $k \geq 3$, the two types of graphs could be mixed, and in general, one could imagine a graph which, after one deletes all articulation points, decomposes into the disjoint union of cliques and graphs with a large independent set.

It is natural to ask whether there are families of examples that differ substantially from the ones given above. In Theorem 2 below, we prove that this is not the case. It would be interesting to find the characterization of extremal families when $k < \sqrt{n}$ and $k = k(n) \to \infty$ as $n \to \infty$.

1.1 Notation

The notation used in this article is fairly standard. We write $[n]$ for the set $\{1, 2, \dots, n\}$ of the first n natural numbers. Also, in many cases, we assume that large numbers are integers, hoping that it will not lead to confusion. For numbers a, b, c, where $c > 0$, we write $a \in b \pm c$ to mean $b - c \leq a \leq b + c$.

For a graph $G = (V, E)$ and two disjoint sets $A, B \subset V$, let $G[A]$ be the subgraph of G induced by A, $E(A, B)$ be the set of edges with one endpoint in A and the other in B, and $G[A, B]$ be the bipartite subgraph of G induced by sets A and B. The *density* of the pair (A, B) is $d(A, B) := |E(A, B)|/|A||B|$. For a vertex v, we denote by $d_A(v)$ the number of its neighbors in A. If $A = V$, we simply write $d(v)$ instead of $d_V(v)$.

1.2 Outline of the proof

An important step in the proof of Theorem 1 is the following stability variant of Theorem 1. It states that if a graph *almost* satisfies the minimum-degree condition $\delta(G) \geq n/k$, but cannot be covered with $k-1$ cycles, then its structure must approximate the extremal structure explained above: the graph can be split into disjoint parts, each of which is either close to a clique (this is captured by setting $r = 1$ in the definition below) or contains a large independent set.

Definition 1. Let G be a graph of order n. Given a positive integer k and real numbers $r \in [1, k]$ and $\beta \in (0, 1)$, we say that a subset $X \subseteq V(G)$ is (r, k, β)-*stable* if there exists a partition of X into (possibly empty) disjoint subsets A and B such that the following holds:

(a) $|A| \in n/k \pm \beta n$ and $|B| \in (r-1)n/k \pm \beta n$;

(b) we have $\delta(G[X]) \geq n/k^4 - \beta n$ and all but at most βn vertices have degree at least $n/k - \beta n$ in $G[X]$;

(c) $d_B(a) \geq |B|/k^4 - \beta n$ for all $a \in A$, and all but at most βn vertices $a \in A$ satisfy $d_B(a) \geq |B| - \beta n$;

(d) $d_A(b) \geq |A|/k^4 - \beta n$ for all $b \in B$, and all but at most βn vertices $b \in B$ satisfy $d_A(b) \geq |A| - \beta n$;

(e) the set B is almost independent in the sense that $e(G[B]) \leq \beta n^2$.

Theorem 2. *Given a positive integer k and $\beta > 0$, there is $\alpha > 0$ such that the following holds for all graphs G of sufficiently large order n and minimum degree at least $(1-\alpha)n/k$.*

Assume that the vertices of G cannot be covered by $k-1$ edge-disjoint cycles. Then there exists $r \in [k]$, positive integers $k_1, k_2, \ldots, k_r$, and a partition $X_1 \cup \ldots \cup X_r = V(G)$ such that $k_1 + \ldots + k_r = k$ and such that for each $i \in [r]$, the set X_i is (k_i, k, β)-stable in G.

The paper consists of two parts: the first part describes how Theorem 2 implies Theorem 1 and, in the second part, we prove Theorem 2. The proof of the first part is elementary. For the second part, we use the method of connected matchings, invented by Łuczak [18], which is based on an application of Szemerédi's Regularity Lemma [21]. This method seems to be widely applicable, for different applications see [1, 2, 3, 8, 19, 16].

2 Properties of stable sets

In this section, we shall prove two slightly stronger versions of Theorem 1 that apply to graphs induced on (r, k, β)-stable sets. The first one, Lemma 6, states that if X is (r, k, β)-stable in G and many vertices of $G[X]$ have degree at least $|X|/r$, then one can cover $G[X]$ with $\lceil r \rceil - 1$ edge-disjoint cycles. Secondly, in Lemma 7, we prove that if one drops the degree condition from Lemma 6, then it is nevertheless true that $G[X]$ can be covered with $\lceil r \rceil$ cycles. Moreover, $G[X]$ is 'connected' by such coverings in a sense that, for any two vertices u, v, there exists a covering of the vertex set by $\lceil r \rceil - 1$ cycles and one (u, v)-path. Actually, we only need this statement for integer r, which is why the statement of Lemma 7 only applies to $r \in [k]$.

Before proving these lemmas, we need some intermediate statements. The first one collects some observations about stable sets that will be useful later on. They follow immediately from Definition 1.

Lemma 3. *For every positive integer k and every $\beta \in (0, 1)$, there exists $n_0 \in \mathbb{N}$ such that the following holds for all real numbers $r \in [1, k]$. Let G be a graph of order $n \geq n_0$ and let X be (r, k, β)-stable in G. Then*

(i) X is (r, k, β')-stable for every β', $\beta \leq \beta' < 1$;

(ii) if G' is any spanning subgraph of G obtained by removing at most $2k$ incident edges from each vertex, then X is $(r, k, 2\beta)$-stable in G'.

We will also use the following version of Dirac's theorem for bipartite graphs.

Lemma 4. *Let G be a bipartite graph with partite sets A and B, both of size m, such that $\delta(G) > (m + 1)/2$. Then. for every pair $(a, b) \in A \times B$, G contains a Hamilton path with endpoints a and b. In particular, G contains a Hamilton cycle.*

Proof. The second statement follows from the first one by taking ab to be any edge in G.

For the first statement, fix an $a \in A, b \in B$. Let $G' = G - a - b$ and note that G' is a bipartite graph with partite sets $A' = A \setminus \{a\}$ and $B' = B \setminus \{b\}$ with minimum degree $\delta(G') > (m-1)/2$.

It is easy to see that every bipartite graph with classes A' and B' of size $m-1$ and with minimum degree larger than $(m-1)/2$ is Hamiltonian. Indeed, in every edge-maximal counterexample there are two non-adjacent vertices $x \in A'$ and $y \in B'$ that are connected by a Hamilton path P. Let us orient the vertices of P from x towards y and let X be the set of the immediate predecessors of neighbors of x on P. Since the graph is bipartite and since $x \in A'$, we have $X \subseteq A'$; moreover, by the minimum degree condition, $|X| > (m-1)/2$. Because y has at least $(m-1)/2$ neighbors, all of them in A', and since $|A'| = m-1$, there is at least one vertex $z \in X$ that is a neighbor of y. Write

$$P = x, p_1, \dots, p_{t-1}, z, p_{t+1}, \dots, p_{2m-4}, y$$

for the vertices of P; by definition, p_{t+1} is adjacent to x. Then

$$x, p_{t+1}, \dots, p_{2m-4}, y, z, p_{t-1}, \dots, p_1, x$$

is a Hamilton cycle.

To complete the proof, let C be a Hamilton cycle in G'. Then, using the fact that $d_C(a), d_C(b) > (m-1)/2$, there has to be a neighbor of a that is adjacent to a neighbor of b on C. Connecting these neighbors to a and b gives the desired path in G. □

The following technical lemma is a consequence of (a weak form of) Lemma 4. It is our main tool for the proofs of Lemmas 6 and 7. The constant 0.01 does not have any deeper significance and is certainly not optimal.

Lemma 5. *Let m be a positive integer and let $0 < \gamma < 0.01$. Let G be a graph with vertex set $A \cup B$, where $|A| = m$ and $m \le |B| \le m + \gamma m$, satisfying the following conditions:*

(i) every $a \in A$ has at least $\sqrt{\gamma} m$ neighbors in B, and all but at most $\gamma m/2$ vertices of A have at least $(1-\gamma)m$ neighbours in B;

(ii) every $b \in B$ has at least $\sqrt{\gamma} m$ neighbors in A, and all but at most $\gamma m/2$ vertices of B have at least $(1-\gamma)m$ neighbours in A;

(iii) B contains a matching of size $|B| - |A|$.

Then G is Hamiltonian.

Proof. Let U_B be the set of vertices $b \in B$ such that $d_A(b) < (1-\gamma)m$, and let U_A be the set of vertices $a \in A$ such that $d_B(a) < (1-\gamma)m$. By our assumptions, we have $|U_A| + |U_B| \leq \gamma m$. Let M be a matching in B of size $|B| - |A| \leq \gamma m$ in B and define $U'_B := U_B \setminus V(M)$.

We will start the proof by showing that G contains an A-B-path P that uses exactly $2|U_A| + 2|U'_B| + 2|M|$ vertices of A and $2|U_A| + 2|U'_B| + 3|M|$ vertices of B, and that, moreover, covers all vertices of $U_A \cup U_B$. After finding this path, we will show that it can be extended to a Hamilton cycle.

First, we build a short path that covers the vertices of U_A. By the minimum degree condition, for every vertex in U_A, we can choose two neighbors in $B \setminus (U_B \cup V(M))$ in such a way that no two vertices of U_A choose the same neighbor. Since any two vertices in $B \setminus (U_B \cup V(M))$ have at least $(1-3\gamma)m \geq |U_A|$ common neighbors in $A \setminus U_A$, we can obtain a path P_1 starting and ending in $B \setminus U_B$ that uses exactly $2|U_A| - 1$ vertices of A and $2|U_A|$ vertices of B. Moreover we have $U_A \subseteq V(P_1) \subseteq V(G) \setminus (U_B \cup V(M))$.

We now construct another path P_2 that covers U_B in a similar fashion. For each edge in M, let us choose, for each endpoint, a neighbor in $A \setminus V(P_1)$. Also, for each vertex in U_B that is not incident to an edge of M, we choose two neighbors in $A \setminus V(P_1)$. Again, by the minimum degree condition, it is possible to do this in such a way that no two vertices choose the same neighbor, and, again, we can connect these neighbors by common neighbors in $B \setminus (V(P_1) \cup V(M))$ to get a path P_2. Then P_2 starts and ends in $A \setminus U_A$, and uses $2|M| + 2|U'_B|$ vertices of A and $3|M| + 2|U'_B| - 1$ vertices of B. Moreover, $U_B \subseteq V(P_2) \subseteq V(G) \setminus V(P_1)$.

Importantly, P_1 and P_2 are disjoint paths with endpoints in $B \setminus U_B$ and $A \setminus U_A$, respectively. Moreover, both paths have length at most $9\gamma m$. Using the degree condition, we can now connect an endpoint of P_1 to an endpoint of P_2 by a path p_1, x, y, p_2 such that $x \in A \setminus (V(P_1) \cup V(P_2))$ and $b \in B \setminus (V(P_1) \cup V(P_2))$, thus obtaining an A-B-path P with the desired properties. In particular, P uses $2|U_A| + 2|M| + 2|U'_B|$ vertices of A and $2|U_A| + 3|M| + 2|U'_B|$ vertices of B.

To extend P to a Hamilton cycle, let us call a vertex *uncovered* if it does not appear as an interior vertex of P. Since $|B| - |A| = |M|$, we see that A and B contain the same number m' of uncovered vertices each. Moreover, all uncovered vertices have degree at least

$$(1-\gamma)m > (m'+1)/2 + |V(P)|,$$

so we can apply Lemma 4, to complete the path P to a Hamilton cycle in G. □

Lemma 6. *For every positive integer k, there exist $\beta > 0$ and $n_0 \in \mathbb{N}$ such that the following holds for all graphs G of order $n \geq n_0$.*

Let $r \in [2,k]$ be a real number and let X be an (r,k,β)-stable subset of $V(G)$ such that all but at most n/k^3 vertices $x \in X$ satisfy $d_X(x) \geq |X|/r$. Then $G[X]$ can be covered with $\lceil r \rceil - 1$ edge-disjoint cycles.

Proof. Let X be (r,k,β)-stable in G for some $r \in [2,k]$, where we assume that β is sufficiently small depending on k. Let A and B be the sets given by Definition 1. Without loss of generality, we may assume that $|A| \leq |B|$: if $|A| > |B|$, then simply exchange A and B—it is easy to check that conditions (a) to (d) from Definition 1 are still satisfied.

We now modify the sets A and B slightly by moving vertices $x \in B$ with $d_B(x) \geq \beta^{1/3}n$ to A until either $(\lceil r \rceil - 1)|A| \geq |B|$ holds for the first time, or until there are no such vertices left. By Definition 1 (a), we see that in doing this, we do not move more than $2\beta n$ vertices from B to A. Thus by Definition 1 (b)–(d), after moving the vertices, we have

(i) $d_A(b) \geq |A| - 3\beta n$ for all but at most βn vertices $b \in B$,

(ii) $d_B(a) \geq |B| - \beta n$ for all but at most $3\beta n$ vertices $a \in A$, and

(iii) for all $a \in A$ and $b \in B$, we have $d_A(b) \geq n/k^4 - \beta n$ and $d_B(a) \geq \min\{n/k^4 - 3\beta n, \beta^{1/3}n\} \geq \beta^{1/3}n$,

for sufficiently small β.

If $r > 2$, then it is clear that after moving these vertices, we still have $|A| \leq |B|$ (we stop moving vertices already if $(\lceil r \rceil - 1)|A| \geq |B|$, so well before $|A| = |B|$). If $r = 2$, then this can fail, but only if $|A| = |B| + 1$. In this case, the minimum degree condition implies that A contains at least one edge, and we can use Lemma 5 to see that $G[X]$ is Hamiltonian (using A for B and B for A, and setting for example $\gamma = 100k\beta$). Thus, from now on, assume that $|A| \leq |B|$.

We claim that B contains a matching M of size at least

$$\frac{|B| - (\lceil r \rceil - 1)|A|}{\beta^{1/4}}.$$

Indeed, if $(\lceil r \rceil - 1)|A| \geq |B|$, then there is nothing to show. Otherwise, we know that $G[B]$ has maximum degree less than $\beta^{1/3}n$. Since at least half of the vertices of B have degree at least $|X|/r$ (Definition 1 implies $|X| \geq 2n/k^3$ for small β), we see that $G[B]$ must contain at least $|B|(|X|/r - |A|)/4$ edges. By Vizing's theorem, $G[B]$ contains a matching of size at least

$$\frac{|B|(|X|/r - |A|)}{4(\beta^{1/3}n + 1)} \geq \lceil r \rceil \frac{|X|/\lceil r \rceil - |A|}{\beta^{1/4}} = \frac{|B| - (\lceil r \rceil - 1)|A|}{\beta^{1/4}},$$

provided β is sufficiently small, and using $|X| = |A| + |B|$.

Let us choose, generously, $\gamma := 100k\beta$, and let $B_1, \ldots, B_{\lceil r \rceil - 1}$ be an equitable partition of B chosen uniformly at random among all equitable partitions of B into $\lceil r \rceil - 1$ parts. Similarly, let $A_1, A_2, \ldots, A_{\lceil r \rceil - 1}$ be an equitable partition of A chosen uniformly at random among equitable partitions of A.

By a standard concentration inequality (the Chernoff bound for the hypergeometric distribution) and using that $1/\beta$ is much larger than r, with probability at least $2/3$, each B_i contains a matching of size at least $s := \lceil |B|/(\lceil r \rceil - 1) - |A| \rceil$ and each vertex of A has degree at least $\sqrt{\gamma}|B_i|$ into B_i. Similarly, with probability $2/3$, each vertex of B has degree at least $\sqrt{\gamma}|A_i|$ into each part A_i. Thus, there exists a choice of $B_1, \ldots, B_{\lceil r \rceil - 1}$ and $A_1, \ldots, A_{\lceil r \rceil - 1}$ such that all these properties hold simultaneously.

Since $|A| \leq |B|$, we can reorder the parts so that $|A_i| \leq |B_i|$ holds for each $i \in [\lceil r \rceil - 1]$. Using $|B| \leq (\lceil r \rceil - 1)(|A| + s)$, we have

$$|A_i| \leq |B_i| \leq |A| + s,$$

so we may add arbitrary elements of A to A_i until $0 \leq |B_i| - |A_i| \leq s$ holds. Since B_i contains a matching of size $s \geq |B_i| - |A_i|$, it is now easily verified that the graph $G[A_i \cup B_i]$ satisfies the conditions of Lemma 5. In particular, by Definition 1 (a), we have

$$s = \left\lceil \frac{|B|}{\lceil r \rceil - 1} \right\rceil - |A| \leq \left\lceil \frac{(r-1)n/k + \beta n}{\lceil r \rceil - 1} \right\rceil - n/k + \beta n \leq \gamma |A_i|.$$

Thus, each subgraph $G[A_i \cup B_i]$ is Hamiltonian, and, together, these $\lceil r \rceil - 1$ Hamilton cycles cover the vertices of $G[X]$. □

Lemma 7. *For every positive integer k, there exist $\beta > 0$ and $n_0 \in \mathbb{N}$ such that the following holds for all graphs G of order $n \geq n_0$.*

Let $1 \leq r \leq k$ be a positive integer and let X be an (r, k, β)-stable subset of $V(G)$. Then X can be covered with r edge-disjoint cycles in $G[X]$.

Moreover, given any two vertices $x, y \in X$, there exists an edge-disjoint covering of $V(G[X])$ by $r-1$ cycles and a single path with endpoints x and y.

Proof. Let us assume that β is sufficiently small as a function of k.

The case $r = 1$ requires a special argument. In this case, we need to show that $G[X]$ is Hamilton-connected, i.e., that for any two vertices $x, y \in X$, there is a Hamilton path going from x to y. Clearly, this will also imply that $G[X]$ is Hamiltonian. Let U be the set of vertices of A that do not have degree $|X| - 3\beta n$ in $G[X]$. By Definition 1, we easily see that $|U| \leq \beta n$ and that $|U \cup B| \leq 2\beta n$. We now choose two vertices x' and y' in $A \setminus U$ such that $xx' \in E(G)$ and $yy' \in E(G)$, and such that $x' \neq y'$. This is possible because the minimum degree of $G[X]$ is $n/k^4 - \beta n$ and $|U \cup B| \leq 2\beta n$. Our first goal is to show that $G[X \setminus \{x, x', y, y'\}]$ is Hamiltonian. For this, let us write $A' = A \setminus \{x, x', y, y'\}$, $B' = B \setminus \{x, x', y, y'\}$, and $U' = U \setminus \{x, x', y, y'\}$.

Since every vertex in $A' \setminus U'$ has degree at least $|X| - 6\beta n$ in $G[A' \setminus U']$, we see that any two vertices of $A' \setminus U'$ have at least $10\beta n$ common neighbors in A', if β is small enough. Moreover, every vertex of $U' \cup B'$ has at least $n/k^4 - 4\beta n \geq 2|U' \cup B'|$ neighbors in $A' \setminus U'$.

Therefore we can build a path of length at most $8\beta n$ with endpoints in $A \setminus U$ that contains all vertices of $U' \cup B'$: first, for every vertex of $U' \cup B'$, choose two neighbors $x_{2i}, x_{2i+1} \in A' \setminus U'$, in such a way that no two vertices choose the same neighbor; then, using the fact that any two vertices of $A' \setminus U'$ have many common neighbors, connect each x_{2i+1} to $x_{2(i+1)}$ via a path of length two. Let $\{a_P, b_P\}$ be the endpoints of the path P created in this way.

Since the conditions of Dirac's theorem are satisfied with room to spare, the graph $H = G[A' \setminus V(P)]$ must be Hamiltonian. Let C be a Hamilton cycle in H. Since $a_P, b_P \in A' \setminus U'$, they have degree at least $|C|/2 + 1$ into $V(C)$, so there exists an edge $uv \in E(C)$ such that u is a neighbor of a_P and v is a neighbor of b_P. By opening C at this edge and connecting u, v to a_P, b_P, we obtain a cycle C' in G that spans all vertices except for x, x', y, y'. Again, x' and y' are in $A \setminus U$, so they both have at least $|C'|/2 + 1$ neighbors on C'. Thus, we can open C' at some edge $uv \in E(C)$, and connect x' to u and y' to v by two edges. Since $xx', yy' \in E(G)$, this gives a Hamilton path starting in x and ending in y.

The case $r \geq 2$ is proved in a similar way as Lemma 6. Let $X \subseteq V(G)$ be (r, k, β)-stable and let A, B be the corresponding partition. Without loss of generality, we can assume that $|A| \leq |B|$. Since, by Definition 1, we have $|A| \in n/k \pm \beta n$ and $|B| \in (r-1)n/k \pm \beta n$, we see that if β is sufficiently small, then $|A| > \lceil |B|/r \rceil$, with room to spare.

Let $\gamma := 100k\beta$ and let $A_1, \ldots, A_r$ and $B_1, \ldots, B_r$ be uniformly random equitable partitions of A and B. Since $|A| \leq |B|$, we can reorder the sets in such a way that $|A_i| \leq |B_i|$ holds for each $1 \leq i \leq r$. With some positive constant probability, it is even possible to do this so that the vertices x and y are part of the same set $A_i \cup B_i$. Moreover, by Chernoff bounds, with high probability, each vertex of B has at least $2\sqrt{\gamma}n$ neighbors in each set A_i, and similarly each vertex of A has at least $2\sqrt{\gamma}n$ neighbors in each B_i. Let us now fix any choice of $(A_i)_{i=1}^r$ and $(B_i)_{i=1}^r$ such that all these things hold. Without loss of generality, assume that $x, y \in A_1 \cup B_1$.

Since $|A| \geq |B|/r$, we can add arbitrary vertices of A to each A_i until we have $|A_i| = |B_i|$ for each $2 \leq i \leq r$. For A_1 and B_1, the modifications that we do are more subtle. If x and y are in different parts (say, $x \in A_1$ and $y \in B_1$), then we simply add vertices to A_1 until $|A_1| = |B_1|$. Otherwise, if $x, y \in B_1$, then we add vertices of A to A_1 until $|A_1| \geq |B_1| + 1$; this is possible since $|A| > \lceil |B|/r \rceil$. Finally, if $x, y \in A_1$, then we first add vertices of A to A_1 until $|A_1| = |B_1|$, and then we add to B_1 a single vertex from B_2.

To finish the proof, we will show that $G[A_i, B_i]$ is Hamiltonian for $2 \leq i \leq k$, and that $G[A_1, B_1]$ contains a Hamilton path from x to y. To see the first part, simply observe that with our choice of γ, and since $|A_i| = |B_i|$, the conditions of Lemma 5 are satisfied, and so each bipartite subgraph $G[A_i, B_i]$ is Hamiltonian.

For the second part, to make sure that everything is edge-disjoint, let us first remove one Hamilton cycle from each graph $G[A_i, B_i]$ for $i \geq 2$. This does not decrease the degrees of vertices in $A_1 \cup A_2$ by more than $2r$. Let $m := \min\{|A_1|, |B_1|\}$, and let us call a vertex of $A_1 \cup B_1$ *good* if it has degree at least $(1-\gamma)m$ in the bipartite subgraph $G[A_1, B_1]$. First, since each vertex has degree at least $\sqrt{\gamma}n$ in $G[A_1, B_1]$, and since there are at most γn bad (i.e., non-good) vertices, we see that there are vertices x_1, x_2 and y_1, y_2 such that x_2 and y_2 are good and such that x, x_1, x_2 and y, y_1, y_2 are paths in $G[A_1, B_1]$. Let $A_1' := A_1 \setminus \{x, y, x_1, y_1, x_2, y_2\}$ and $A_1' := A_1 \setminus \{x, y, x_1, y_1, x_2, y_2\}$. One easily checks that $G[A_1', B_1']$ satisfies the conditions of Lemma 5, and is thus Hamiltonian. Here it is important to note that by our construction, $|A_1'| = |B_1'|$. If C is a Hamilton cycle in $G[A_1', B_1']$, then, since x_2 and y_2 are good, there must be an edge $uv \in E(C)$ such that u is a neighbor of x_2 and v is a neighbor of y_2. Opening the cycle at the edge uv and connecting u to x_2 and v to y_2 yields the desired Hamilton path from x to y.

Clearly, the Hamilton path in $A_1 \cup B_1$ is edge-disjoint from the Hamilton cycles that we embedded into the sets $A_i \cup B_i$ for $i \geq 2$. Similarly, the Hamilton cycles are pairwise edge-disjoint. This completes the proof of the lemma. □

3 Proof of Theorem 1

In this section, we derive Theorem 1 from Theorem 2.

Proof of Theorem 1. Let G be a graph of sufficiently large order n with minimum degree $\delta(G) \geq n/k$. The case $k = 2$ follows from Dirac's theorem, so assume that $k \geq 3$. Fix a sufficiently small constant $\beta > 0$. By Theorem 2, there are integers $k_1, \ldots, k_r$ and a partition of $V(G)$ into sets $X_1, \ldots, X_r$, such that each set X_i is (k_i, β)-stable in G and such that $k_1 + \ldots + k_r = k$. We may assume that there is no covering of G by $k - 1$ edge-disjoint cycles.

By Lemma 7, it is immediate that the vertices of G can be covered by $k_1 + \ldots + k_r = k$ edge-disjoint cycles. If there are $i \neq j$ such that G contains two independent edges between X_i and X_j, then by Lemma 7, we can cover the graph with $k-1$ cycles as the paths from X_i and X_j could be merged into one cycle. So assume that this is not the case. Moreover, also by Lemma 7 (or by any simpler argument), we see that each graph $G[X_i]$ is connected.

Let D be the directed graph on the vertex set $\{X_1, \ldots, X_r\}$ that has a directed edge from X_i to X_j if and only if some vertex of X_j has at least n/k^4 neighbors in X_i. If this graph contains a (not necessarily directed) cycle, then we can cover G with $k - 1$ cycles. Indeed, let $a \neq b$ be such that X_a and X_b both have at least one out-edge on this cycle; such indices a and b must necessarily exist. Then, using the fact that each set X_i is connected, there are vertices $u_1, v_1 \in X_a$ and $u_2, v_2 \in X_b$ such that u_1, u_2 and v_1, v_2 can be connected by disjoint paths in G using no additonal vertices of either X_a or X_b. By Lemma 7, $G[X_a]$ can be covered $k_a - 1$ cycles and a path joining u_1 to v_1. Similarly, $G[X_b]$ can be covered $k_b - 1$ cycles and a path joining u_2 to v_2. Combining these with the disjoint paths joining u_1 to u_2 and v_1 to v_2, there exist $k_a + k_b - 1$ cycles in G that together cover the vertices in $X_a \cup X_b$ (and also some additional vertices). Remove these cycles from the graph. By

Lemma 3, every set X_i is still $(k_i, 2\beta)$-stable in the rest of the graph, so by Lemma 7, it can be covered by k_i cycles. From this it follows that the vertices of G can be covered with $k_1 + \cdots + k_r - 1 = k - 1$ cycles. Therefore, we may assume that D does not contain any cycle.

Let $d_D^+(X_i)$ denote the out-degree of X_i in D. Since D does not contain any cycles, we have $\sum_{i=1}^{r} d_D^+(X_i) \le r-1$. Then there must be some $i \in [r]$ such that $|X_i| \le k_i\delta(G) - d_D^+(X_i)$. Sure enough, assuming the opposite results in the contradiction

$$n = \sum_{i=1}^{r} |X_i| \ge \sum_{i=1}^{r} (k_i\delta(G) - d_D^+(X_i) + 1) = n - \sum_{i=1}^{r} d_D^+(X_i) + r \ge n + 1.$$

Let F be the set of all vertices $u \notin X_i$ that have n/k^4 neighbors in X_i. Note that for each $j \ne i$, there can be at most one vertex from X_j in F. By the definition of D, there are $d_D^+(X_i)$ vertices in F. Let $X = X_i \cup F$. It is easy to see that X is $(k_i, k, 2\beta)$-stable in G. Moreover, there cannot be more than n/k^3 vertices $x \in X_i$ that do not have at least $\delta(G)$ neighbors in X: otherwise, there would be some $j \ne i$ such that at least n/k^4 vertices of X_i have a neighbor in $X_j \setminus F$. Since they cannot all have the same neighbor (or the neighbor would be in F), there must be two independent edges between X_i and X_j. But we assumed that this does not happen.

In particular, since

$$k_i\delta(G) \ge |X| \ge \delta(G) + 1,$$

we have $k_i \ge 2$. By Lemma 6, we can cover $G[X]$ with $k_i - 1$ edge-disjoint cycles. Since each $G[X_j]$ can be covered with k_j cycles, there is a covering of the vertices of G with $\sum_{i=1}^{r} k_i - 1 = k - 1$ cycles. □

4 Regularity

In our proof of Theorem 2, we shall use Szemerédi's Regularity Lemma [21] and some related results.

For $\epsilon \in (0, 1)$, we say that a pair (A, B) of disjoint sets of vertices of a graph $G = (V, E)$ is ϵ-regular if, for all subsets $X \subseteq A$ and $Y \subseteq B$ such that

$|X| \geq \epsilon|A|$ and $|Y| \geq \epsilon|B|$, we have

$$\big|d(X,Y) - d(A,B)\big| \leq \epsilon.$$

A pair (A,B) is called (ϵ,δ)-*super-regular* if it is ϵ-regular and

$$d_B(a) \geq \delta|B| \text{ for } \forall a \in A \qquad \text{and} \qquad d_A(b) \geq \delta|A| \text{ for } \forall b \in B.$$

Theorem 8 (Degree form of the Regularity Lemma, [14])**.** *For every $\epsilon > 0$ and positive integer t_0, there is an $M = M(\epsilon, t_0)$ such that if $G = (V,E)$ is any graph of order $n \geq M$ and $d \in [0,1]$ is any real number, then there is $t_0 \leq t \leq M$, a partition $(V_i)_{i=0}^t$ of the vertex set V, and a subgraph $G' \subseteq G$ with the following properties:*

(R1) $|V_0| \leq \epsilon|V|$,

(R2) all clusters V_i, $i \in [t]$, are of the same size $m \in ((1-\epsilon)n/tn, n/t)$,

(R3) $d_{G'}(v) > d_G(v) - (d+\epsilon)n$ for all $v \in V \setminus V_0$,

(R4) $e(G'[V_i]) = 0$ for all $1 \leq i \leq t$,

(R5) for all $1 \leq i < j \leq t$, the pair (V_i, V_j) is ϵ-regular, with a density either 0 or greater than d.

Given a partition $(V_i)_{i=0}^t$ of the vertex set V and a subgraph $G' \subseteq G$ satisfying conditions (R1)–(R5), we define the (ϵ,d)-*reduced graph* R with vertex set $[t]$ and edge set corresponding to those pairs ij for which (V_i, V_j) is ϵ-regular and with density at least d.

We shall also use the following special case of the famous blow-up lemma of Komlós, Sárközy and Szemerédi [12]. (See also Lemma 24 and the first remark after Lemma 25 in [10].)

Lemma 9. *Given $\delta > 0$ there exists an $\epsilon > 0$ such that the following holds. Let $G = (A,B,E)$ be an (ϵ,δ)-super-regular pair with $|A| = |B|$ and $x \in A, y \in B$. Then G contains a Hamilton path with endpoints x and y.*

5 Proof of Theorem 2

Let $k \geq 3$ be a fixed integer. Without loss of generality, we may assume that $\beta > 0$ is sufficiently small, depending on k. Let us choose constants $\epsilon, d, \alpha \in (0, 1)$ and $t_0 \in \mathbb{N}$ such that

$$\frac{1}{t_0} \ll \alpha \ll \epsilon \ll d \ll \beta \ll \frac{1}{k},$$

where by $a \ll b$ we mean that a is chosen to be sufficiently smaller than b.

We apply the degree form of the Regularity Lemma with parameters ϵ/k^3 (for ϵ), t_0, and $d + \epsilon/k^3$ (for d) to obtain M. Let G be a graph of order n with $\delta(G) \geq (1 - \alpha)n/k$, where $n \geq M$ is sufficiently large. By Theorem 8, there exists an $(\epsilon/k^3, d + \epsilon/k^3)$-reduced graph R on t vertices, $t_0 \leq t \leq M$. The reason for choosing ϵ/k^3 instead of ϵ is that, during the proof, we will modify the regular partition slightly, moving at most $\epsilon|V_i|/2$ vertices from some clusters to the exceptional set V_0 (but without changing the structure of the reduced graph R). After doing this, we are left with an exceptional set of size at most ϵn, and each edge of R corresponds to an ϵ-regular pair in G with density at least d. Although it is not clear at this point which vertices will be moved to the exceptional set, everything that follows will be in terms of this larger set, which we also denote by V_0. Thus, conceptually, we have applied the Regularity Lemma with parameters ϵ, t_0, d, except that we have some control over which vertices should go to the exceptional set.

Properties of the reduced graph. The graph R is a graph on t vertices, and one can show that

$$\delta(R) \geq \frac{(1 - 2dk)t}{k} > \frac{t}{k+1}. \tag{1}$$

Indeed, the vertices of every cluster in R with degree less than $(1 - 2dk)t/k$ would have degree at most $(1 - 2dk)(t/k) \cdot (n/t) + \epsilon n \leq (1 - \alpha)n/k - (d + \epsilon)n$ in G', contradicting property (R3) of Theorem 8.

Let us denote by r the number of connected components of R and by $R_1, \ldots, R_r$ the components themselves. Since each component has size at least $\delta(R) > t/(k + 1)$, there can be at most k components altogether.

For each component R_i, define

$$s_i := \frac{kv(R_i)}{(1 - 2dk)t}. \tag{2}$$

Using (1), this means that

$$\delta(R_i) \geq \delta(R) \geq (1-2dk)t/k = v(R_i)/s_i. \tag{3}$$

In particular, we have $s_i > 1$. Since $v(R_1) + \ldots + v(R_r) = t$, we have

$$\sum_{i=1}^{r} s_i = \frac{k}{1-2dk} \leq (1+3dk)k. \tag{4}$$

Creating the partition of $V(G)$. For each $i \in [r]$, let G_i be the preimage of R_i (i.e., the subgraph of G induced by the vertices in clusters in R_i). Note that

$$\bigcup_{i=1}^{r} V(G_i) = V(G) \setminus V_0.$$

By the definition of s_i, we have

$$v(G_i) \leq \frac{nv(R_i)}{t} = \frac{(1-2dk)ns_i}{k}.$$

Using inequalities

$$(1+dk)(1-2dk) = 1 - dk - 2d^2k^2 \leq 1 - \alpha - (d+\epsilon)k$$

and $\delta(G) \geq (1-\alpha)n/k$ and properties (R3)–(R5), we obtain that

$$\delta(G_i) \geq \frac{(1-\alpha)n}{k} - (d+\epsilon)n \geq (1+dk)\frac{v(G_i)}{s_i}. \tag{5}$$

It is also useful to remember that every G_i has linear size in n:

$$v(G_i) \geq \delta(G_i) \geq \frac{(1-\alpha)n}{k} - (d+\epsilon)n \geq \frac{n}{k^2}. \tag{6}$$

Since the minimum degree of G is at least $(1-\alpha)n/k$, and since there are at most k components, it is clear that for every vertex $v \in V(G)$, there exists at least one $i \in [r]$ such that the degree of v into $V(G_i)$ is at least $(\delta(G) - \epsilon n)/k$. Thus, we can partition the exceptional set V_0 into sets $U_1, \ldots, U_r$, where U_i contains only vertices with at least

$$(1-\alpha-k\epsilon)n/k^2 \geq n/k^3$$

neighbors in $V(G_i)$. The sets $Y_i := V(G_i) \cup U_i$ form a partition of the vertex set of G. This is *almost* the partition that Theorem 2 asks for; as we will see, the sets Y_i for which $s_i \in (2, 2+4dk^2]$ might have to be partitioned further.

Stability. We distinguish three cases; each case is handled by a different lemma.

Lemma 10. *If $1 \leq s_i \leq 2$, then $G[Y_i]$ is Hamiltonian. Moreover, if $s_i < 1 + 4dk^2$, then Y_i is actually $(1, k, \beta)$-stable in G.*

Lemma 11. *If $2 < s_i \leq 2 + 4dk^2$, then one of the following holds:*

(i) $G[Y_i]$ is Hamiltonian,

(ii) Y_i is $(2, k, \beta)$-stable in G, or

(iii) there is a partition of Y_i into two $(1, k, \beta)$-stable sets in G.

Lemma 12. *If $2 + 4dk^2 < s_i$, then either Y_i can be covered with $\lfloor s_i - 4dk^2 \rfloor$ edge-disjoint cycles in $G[Y_i]$, or Y_i is $(\lfloor s_i \rfloor, k, \beta)$-stable in G.*

Given these three lemmas, it is straightforward to complete the proof of Theorem 2.

Proof of Theorem 2. First of all, by Lemma 7, we see that Lemmas 10, 11 and 12 imply that each graph $G[Y_i]$ can be covered with $\lfloor s_i \rfloor$ cycles. By (4), we have $\sum_{i=1}^{r} \lfloor s_i \rfloor \leq k$. Since we may assume that G cannot be covered with $k-1$ cycles, this inequality is really an equality, i.e., $\sum_{i=1}^{r} \lfloor s_i \rfloor = k$.

This implies that for every $i \in [r]$, we have $\lfloor s_i \rfloor \leq s_i < \lfloor s_i \rfloor + 4dk^2$: otherwise, using (4) we obtain the contradiction

$$k = \sum_{i=1}^{r} \lfloor s_i \rfloor = \sum_{i=1}^{r} s_i - \sum_{i=1}^{r} (s_i - \lfloor s_i \rfloor) \leq (1 + 3dk)k - 4dk^2 < k.$$

But then, unless we can cover the vertices of G with $k-1$ edge-disjoint cycles, Lemmas 10, 11 and 12 tell us that the situation is as follows:

- if $1 \leq s_i \leq 2$, then Y_i is $(1, k, \beta)$-stable;

- if $2 < s_i \leq 2 + 4dk^2$, then Y_i is either $(2, k, \beta)$-stable or the union of two disjoint $(1, k, \beta)$-stable sets;
- if $s_i > 2 + 4dk^2$, then Y_i is $(\lfloor s_i \rfloor, k, \beta)$-stable.

This completes the proof of Theorem 2. $\square$

Thus, it only remains to prove Lemmas 10, 11 and 12. The first two of them are relatively easy to prove without using the regularity method, however the proof of Lemma 12 is more involved.

Proof of Lemma 10. We start by proving the first part. Let $U_i := Y_i \setminus V(G_i)$ and recall that by (5), we have $\delta(G_i) \geq (1 + dk)v(G_i)/s_i \geq (1 + dk)v(G_i)/2$. In particular, any two vertices $u, v \in V(G_i)$ are connected by at least $dkv(G_i)$ disjoint paths of length two. Moreover, by the definition of Y_i, each vertex of U_i has degree at least n/k^3 into $V(G_i)$. Since $|U_i| \leq \epsilon n$, this means that we can greedily construct a path P of length $4|U_i| - 2$ in $G[Y_i]$ such that P starts and ends in vertices of G_i and contains all vertices of U_i.

The graph $G_i - V(P)$ still satisfies Dirac's condition. Let C be a Hamilton cycle in $G_i - V(P)$ and let $u, v \in G_i$ be the endpoints of P. Then, by the minimum degree condition, there are vertices u', v' that are adjacent on C and such that $uu', vv' \in E(G_i)$. By opening the cycle C on the edge $u'v'$ and connecting u' to u and v' to v, we obtain a Hamilon cycle in $G[Y_i]$.

To see the second statement of the lemma, just let $A = Y_i$ and $B = \emptyset$. Since d is very small compared to β, the conditions of Definition 1 are easily verified. $\square$

Proof of Lemma 11. Assume that $2 < s_i \leq 2 + 4dk^2$. Let $U_i := Y_i \setminus V(G_i)$. Recall that by (5), we have

$$\delta(G_i) \geq (1 + dk)v(G_i)/s_i \geq (1 - 3dk^2)v(G_i)/2.$$

Moreover, by the definition of Y_i, we know that each vertex of U_i has at least n/k^3 neighbors in $V(G_i)$.

We will show that at least one of the followings holds:

(i) $G[Y_i]$ is Hamiltonian,

(ii) $G[Y_i]$ contains an independent set of size at least $(1 - 10dk^2)|Y_i|/2$, or

(iii) Y_i contains two disjoint sets A, B of size at least $(1 - 5dk^2)|Y_i|/2$ such that $e(A, B) = 0$.

We claim that if we are in case (ii), then Y_i is $(2, k, \beta)$-stable in G. Indeed, let I be the independent set of size at least $(1 - 10dk^2)|Y_i|/2$. Since we have $\delta(G_i) \geq (1 - 3dk^2)v(G_i)/2$, we also have $|I| \leq (1 + 3dk^2)v(G_i)/2$. Therefore, there are at most

$$|I|(v(G_i) - |I| - \delta(G_i)) \leq 14dk^2 v(G_i)^2/4$$

non-edges between I and $V(G_i) \setminus I$, and so there can be at most $14dk^2 v(G_i)/\beta$ vertices in $V(G_i) \setminus I$ that do not have at least $(1 - \beta/2)v(G_i)/2$ neighbors in I. Let A be the set of the vertices in $V(G_i) \setminus I$ that have at least $(1 - \beta/2)v(G_i)/2$ neighbors in I, and let $B := I$. Let $U_i' := Y_i \setminus (A \cup B)$. Since each vertex of U_i' has degree at least $n/k^3 - 14dk^2 v(G_i)/\beta \geq n/k^4$ into $A \cup B$, we can assign the vertices of U_i' to A or B in such a way that each vertex in B has $|A|/k^4$ neighbors in $|A|$, and each vertex in A has $|B|/k^4$ neighbors in B. The other conditions of Definition 1 are now easy to check. Similarly, if we are in case (iii), then one easily proves that Y_i can be partitioned into two $(1, k, \beta)$-stable sets (i.e., two almost-cliques).

Thus, from now on, we shall assume that neither (ii) nor (iii) holds. Then for any two vertices $u, v \in V(G_i)$ and every subset $A \subseteq V(G_i)$ of size at least $v(G_i) - dn$, the graph $G[A \cup \{u, v\}]$ contains a path of length at most three that goes from u to v. To see this, observe that by (6), both u and v have at least

$$(1 - 3dk^2)v(G_i)/2 - dn \geq (1 - 5dk^2)v(G_i)/2$$

neigbors in A. If they have a common neighbor in A, or if there is an edge from a neighbor of u in A to a neighbor of v in A, then we are done. Otherwise, the neighborhoods of u and v are disjoint subsets of size at least $(1-5dk^2)v(G_i)/2$ with no edges between them, and we are in case (iii).

Recall that $|U_i| \leq \epsilon n$ and that every vertex of U_i has degree at least n/k^3 into Y_i. From the previous observation, it is now easy to see that $G[Y_i]$ must contain a path P of length $5|U_i| - 3$ that contains all vertices of U_i and whose endpoints are in Y_i. Let us write a_P and b_P for the endpoints of P.

Let G_i' be the subgraph of G_i induced by $\{a_P, b_P\} \cup (V(G_i) - P)$. Note that $v(G_i') \geq v(G_i) - 5\epsilon n$, and that, consequently, G_i' has minimum degree at least $(1 - 4dk^2)|Y_i|/2$. We may also assume that $G_i' - \{a_P, b_P\}$ is at least two-connected, since otherwise, by the minimum degree of G_i', the graph would contain two sets X, Y of size at least

$$\delta(G_i') - 2 \geq (1 - 4dk^2)|Y_i|/2 - 2$$

that intersect only in an articulation point. But then, we would be in case (iii), contradicting our assumption.

We will show that G'_i contains a Hamilton path joining a_P to b_P. Clearly, this path will combine with P to yield a Hamilton cycle in $G[Y_i]$. Our strategy is the following. First, we will prove that $G'_i - \{a_P, b_P\}$ must contain a nearly spanning cycle. Then, we will connect a_P and b_P with this cycle to form a nearly spanning path from a_P to b_P in G'_i. Finally, we will absorb the few remaining vertices of G'_i into the path to get a Hamilton path.

To obtain the first part, we use the well-known fact (also due to Dirac [5]) that every two-connected graph with minimum degree δ contains a cycle of length at least 2δ. In our case, this means that $G'_i - \{a_P, b_P\}$ contains a cycle C of length

$$|C| \geq 2\delta(G'_i - \{a_P, b_P\}) \geq (1 - 5dk^2)|Y_i|.$$

Since both a_P and b_P have degree larger than $|C|/3$ into C, there must be a neighbor of a_P on C that is within distance at most two to a neighbor of b_P on C, the distance being measured along the cycle C (and making sure that $a_P \neq b_P$). Therefore there is a path P' in G'_i with endpoints a_P and b_P that has length at least $(1 - 6dk^2)|Y_i|$.

To complete the proof, we show how to handle the at most $6dk^2|Y_i|$ vertices of G'_i that do not belong to P'. Consider any such vertex $v \in V(G'_i)$ and let X be the set of all neighbors of v on P' that are not within distance less than two of either a_P or b_P (again, the distance being measured on P'). There must be at least $(1 - 10dk^2)|Y_i|/2$ such vertices. If any two neighbours u and w are consecutive on P', then we can easily absorb v to P' by following P' from a_P to u, using uv and vw, and following P' from w to b_P. So, assume this is not the case.

Orient P' from a_P to b_P, and let Y be the set of the immediate successors of vertices in X on the path. Since this is a set of size at least $(1-10dk^2)|Y_i|/2$, it must contain at least one edge uw, or else we would be in case (ii). However, using this edge, one can rotate the path P' to obtain a path going from a_P to b_P that contains all vertices of P', as well as the additional vertex v. Indeed: let u' be the predecessor of u and w' be the predecessor of w on P'. We absorb v to P' by following P' from a_P to u', using $u'v$ and vw', following P' from w' to u , using uw, and following P' from w to b_P.

In this way, it is possible to absorb all left-over vertices until the path spans the whole of G'_i. □

It remains to handle the case $s_i > 2 + 4dk^2$. Unfortunately, the proof is quite involved. The first step is the following lemma, which is a statement about R_i, the component of the reduced graph that corresponds to Y_i.

Lemma 13. *Assume that $s_i > 2 + 4dk^2$ and let $m_i = \lfloor s_i - 4dk^2 \rfloor$. Let $t_i = |V(R_i)|$. Then at least one of the followings is the case:*

(i) The graph R_i contains a subset $I \subseteq V(R_i)$ of size $(s_i - 1)t_i/s_i - 6dk^2 s_i t_i$ that is almost independent in the sense that $e(I) \leq 4dk^2 s_i t_i^2$.

(ii) The graph R_i contains matchings $M_1, \ldots, M_{m_i}$ and disjoint subsets of vertices D_1, D_2 with the following properties:

(a) $D_1 \cap V(M_1) = \emptyset$, and for $j > 1$, $D_2 \cap V(M_j) = \emptyset$;

(b) each vertex of R_i has at least $dt_i/(3s_i)$ neighbors in each set D_1, D_2;

(c) the matchings $M_1, \ldots, M_{m_i}$ cover the vertex set of R_i.

Proof. Recall that, by (1), we can assume that t_i is very large compared to $1/d$.

First, we show that it is possible to choose disjoint subsets $D_1, D_2 \subseteq V(R_i)$, each of size at most $2dt_i$, in such a way that every vertex in $V(R_i)$ has at least $dt_i/(3s_i)$ neighbors in D_j, for $j \in \{1, 2\}$. Indeed, let D be a d-random subset of t_i and let us split it randomly into disjoint sets D_1 and D_2. The expected size of D_1 and D_2 is $dt_i/2$. Thus, by Markov's inequality, with probability at least $1/2$, we have $|D_1|, |D_2| \leq 2dt_i$. Fix some vertex $v \in V(R_i)$. The expected number of neighbors of v that are in D_1 is at least $d\delta(R_i)/2 \geq dt_i/(2s_i)$. Using the Chernoff bound, the probability that the neighborhood of v does not contain at least $dt_i/(3s_i)$ elements of D_1 is smaller than $1/(4t_i)$, provided that t_i is large enough. Similarly, the probability that the neighborhood of v does not contain at least $\beta t_i/(3s_i)$ elements of D_2 is smaller than $1/(4t_i)$. The union bound shows that there exists a good choice for D_1 and D_2. From now on, fix such a choice.

Let $m_i := \lfloor s_i - 4dk^2 \rfloor$ and observe that by assumption, we have $m_i \geq 2$. We want to cover the set $V(R_i)$ by m matchings $M_1, \ldots, M_{m_i}$, so that M_1 is disjoint from D_1 and $M_2, \ldots M_{m_i}$ are disjoint from D_2. To do this, we first let M_1 be a maximal matching that is disjoint from D_1. Now, to choose the matchings M_j for $j \geq 2$, we partition the set $V(R_i) \setminus V(M_1)$ equitably into sets $A_2, \ldots, A_{m_i}$. Then we let M_j be a matching that is disjoint from D_2 and that covers the maximum number of vertices of A_j (among all matchings that are disjoint from D_2); moreover, we assume that M_j has maximum size among all such matchings. There are now two cases.

Non-extremal case. If $|M_1| \geq t_i/s_i + 2dk^2 s_i t_i$, then we claim that we are in case (ii) of the lemma. The only thing to check is whether the matchings cover t_i. The set $V(R_i) \setminus V(M_1)$ has size

$$
\begin{aligned}
t_i - 2|M_1| &\leq t_i - \frac{2t_i}{s_i} - 4dk^2 s_i t_i = \frac{t_i(s_i - 2 - 4dk^2 s_i^2)}{s_i} \\
&\leq \frac{t_i(s_i - 4dk^2 - 2)(1 - 2ds_i)}{s_i} \leq \frac{t_i(\lfloor s_i - 4dk^2 \rfloor - 1)(1 - 2ds_i)}{s_i},
\end{aligned}
$$

so for each $2 \leq j \leq m_i$, we have

$$
|A_j| \leq \left\lceil \frac{t_i - 2|M_1|}{\lfloor s_i - 4dk^2 \rfloor - 1} \right\rceil \leq \lceil t_i/s_i - 2dt_i \rceil \leq \delta(R_i) - |D_2|.
$$

Thus, there exists a matching disjoint from D_2 that covers A_j completely, and since M_j was chosen to cover the most vertices of A_j among all matchings disjoint from D_2, the matchings cover every vertex of R_i.

Extremal case. If $|M_1| < t_i/s_i + 2dk^2 s_i t_i$, then we will see that the graph must have a special structure.

We will first show that $|M_1| \geq t_i/s_i - 2dt_i$. Write U for the set $V(R_i) \setminus (D_1 \cup V(M_1))$ of *uncovered* vertices that are not in D_1. Note that U is an independent set in R_i (or the matching M_1 would not be maximal). If $|U| \leq 1$, then, since $s_i > 2 + 4dk^2$ and $|D_1| \leq 2dt_i$, we have

$$
2|M_1| \geq t_i - |D_1| - 1 \geq t_i - 2dt_i - 1 \geq 2t_i/s_i,
$$

and we are done. Otherwise, there are at least two vertices $u, v \in U$. Since M_1 is maximal, we know that every neighbor of u is either in D_1 or is covered by an edge of M_1, and similarly for v. Moreover, there are *no* edges of M_1 between a neighbor of u and a neighbor of v. Therefore

$$
2t_i/s_i \leq 2\delta(R_i) \leq d(u) + d(v) \leq 2|D_1| + 2|M_1|,
$$

which implies that

$$
|M_1| \geq t_i/s_i - |D_1| \geq t_i/s_i - 2dt_i. \tag{7}
$$

Now, since $|M_1| < t_i/s_i + 2dk^2 s_i t_i$, we have

$$|U| = t_i - |D_1| - 2|M_1| \geq (s_i - 2)t_i/s_i - 5dk^2 s_i t_i. \tag{8}$$

We claim that there exists a set of size $|U| + |M_1|$ which contains very few edges. For this, observe that by the maximality of M_1, for every edge $xy \in M_1$ at least one of the vertices x, y has at most one neighbor in U. Thus, we may split $V(M_1)$ into two disjoint sets A and B of size $|M_1|$ by placing, for each edge of M_1, an endpoint with at most one neighbor in U into A, and the other endpoint into B. Then we have $e(A, U) \leq |A|$; the 'nearly independent set' that we are looking for will be $U \cup A$.

To show that $U \cup A$ contains few edges, we will first show that most vertices in B have at least two neighbors in U. Indeed, let $X := \{v \in B \mid d_U(v) < 2\}$. Since U is an independent set and since $V(R_i) = A \cup B \cup U \cup D_2$, we have

$$\begin{aligned} |X| + |U|(|B| - |X|) &\geq e(B, U) \geq |U|\delta(R_i) - e(U, V(R_i) \setminus B) \\ &\geq |U|\delta(R_i) - e(U, A) - e(U, D_2)) \geq |U|\delta(R_i) - |B| - |U||D_2|. \end{aligned}$$

Rewriting this, and using that $|B| - \delta(R_i) \leq 2dk^2 s_i t_i$ and $|D_2| \leq 2dt_i$, as well as the fact that $|U| = \Omega(t_i)$ is sufficiently large, we get

$$|X| \leq \frac{|B| + |U||B| - |U|\delta(R_i) + |U||D_2|}{|U| - 1} \leq 3dk^2 s_i t_i.$$

Let us now estimate the number of edges inside of $U \cup A$. We know that $e(U) = 0$ and $e(U, A) \leq |A|$. To bound $e(A)$, consider some edge $xy \in E(A)$ and denote by x' and y' the vertices matched to x and y in M_1, respectively. Then, by the maximality of M_1, we can see that at least one of x' and y' has at most one neighbor in U. Thus

$$e(U \cup A) \leq e(A) + e(U, A) \leq |A||X| + |A| \leq 4dk^2 s_i t_i^2$$

and, using (7) and (8),

$$|U \cup A| = |U| + |M_1| \geq (s_i - 1)t_i/s_i - 6dk^2 s_i t_i.$$

So we are in case (i) of the lemma. □

Case (i) of Lemma 13 will be relatively easy to handle (see the proof of Lemma 12 below). It remains to show that if R_i is in case (ii) of Lemma 13, then $G[Y_i]$ can be covered with m_i edge-disjoint cycles. Our main tool is going to be Lemma 9.

Lemma 14. *Assume that R_i is in case (ii) of Lemma 13, i.e., that there are matchings $M_1, \ldots, M_{m_i}$ that cover the vertex set of R_i, and sets $D_1, D_2 \subseteq V(R_i)$ with the following properties:*

(a) $D_1 \cap V(M_1) = \emptyset$, and for $j > 1$, $D_2 \cap V(M_j) = \emptyset$, and

(b) each vertex of R_i has at least $dv(R_i)/(3s_i)$ neighbors in each set D_1, D_2.

Then $G[Y_i]$ can be covered with m_i edge-disjoint cycles.

Proof. We may assume that ϵ is small enough so that Lemma 9 applies with $\delta = d/3$. Before continuing, we do some pre-processing. Recall that, at the beginning, we applied the Regularity Lemma with parameter ϵ/k^3 instead of ϵ. Using the fact that $M_i \leq k$, we can move a $(k\epsilon/k^3)$-fraction of each cluster to the exceptional set V_0 to achieve that the pairs of the matchings $M_1, \ldots, M_{m_i}$ are actually $(\epsilon, d/2)$-super-regular. Since we might do this for each component R_i, in total, we move at most a (ϵ/k)-fraction of the vertices of each cluster to the exceptional set. Moreover, this can be done in such a way that the clusters are still all of equal size. This is the only place where we modify the regular partition. Since we already accounted for this when applying the Regularity Lemma, we shall forget that we ever moved these vertices, and just remember that edges of the matchings correspond to super-regular pairs.

The general idea is to use Lemma 9 to cover the preimage of each matching by a single cycle in $G[Y_i]$, and to do this in such a way that all the vertices in U_i are absorbed. We start by assigning the exceptional vertices to clusters C into which they have large degree.

Assigning the exceptional vertices. For each matching M_j, let us write $V_{M_j} \subseteq V(G_i)$ for the union of all clusters in M_j. As the matchings M_j cover the vertices of R_i, we have $\bigcup_{j=1}^{m_i} V(M_j) = V(G_i)$. Since each vertex of U_i has degree at least n/k^3 into $V(G_i)$, and since $m_i \leq k$, we see that for every $u \in U_i$ there exists a $j_u \in [m]$ such that u has n/k^4 neighbours in $V_{M_{j_u}}$. Let us write $U_i^{(j)} := \{u \in U_i \mid j_u = j\}$ for the exceptional vertices assigned to the matching M_j in this way.

Since $|V(M_j)| \leq |V(R)| \leq t$ and since each cluster has size at most $2n/t$, it follows that for each vertex $u \in U_i^{(j)}$, there are at least $t/(4k^4)$ clusters $C \in V(M_j)$ such that $d_C(u) \geq n/(2k^4 t)$. Indeed, if this were not true, then the degree of u into $V(M_j)$ would be strictly below

$$t \cdot \frac{n}{2k^4 t} + \frac{t}{4k^4} \cdot \frac{2n}{t} = \frac{n}{k^4},$$

a contradiction with the definition of $U_i^{(j)}$.

We now assign the vertices of $U_i^{(j)}$ to clusters in $V(M_j)$ in such a way that

(i) if u is assigned to the cluster C, then $d_C(u) \geq n/(2k^4t)$, and

(ii) at most $4k^4\epsilon n/t$ vertices are assigned to each cluster.

Since $|U_i^{(j)}| \leq \epsilon n$ and since each vertex has $t/(4k^4)$ candidates, such an assignment exists. Take any such assignment and write U_C for the exceptional vertices assigned to the cluster $C \in V(M_j)$.

Covering the matchings. From all the above, it is clear that the sets

$$V_{M_1} \cup U_i^{(1)}, V_{M_2} \cup U_i^{(2)}, \ldots, V_{M_{m_i}} \cup U_i^{(m_i)}$$

cover the set Y_i. In the following, we will cover each set $V_{M_j} \cup U_i^{(j)}$ by a single cycle in H_i (however, this cycle might use vertices outside of $V_{M_j} \cup U_i^{(j)}$). Moreover, we will do this in such a way that the cycles are *edge-disjoint*.

Thus, fix some $j \in [m_i]$, and assume $\ell \in \{1, 2\}$ is such that D_ℓ is disjoint from the matching M_j. The embedding proceeds in two steps: first, for each cluster $C \in V(M_j)$, we connect the vertices of U_C by a short path using only vertices from D_ℓ, C, and U_C (and making sure that the paths for different clusters are vertex-disjoint); second, we use Lemma 9 to connect these short paths into a cycle spanning the whole of $V_{M_j} \cup U_i^{(j)}$.

In the first step, it is important to make sure that each path uses exactly the right number of vertices in the cluster C, as otherwise the second step might fail. Because we do not want to make this completely precise at this point, we assign to each cluster C an integer

$$s_C \in [8k^4\epsilon n/t, 100k^4\epsilon n/t],$$

and we will make sure that after creating the short path for C, the number of vertices of C not used by the path is exactly $|C| - s_C$. This allows us enough control over the number of remaining vertices per cluster, without hurting the super-regularity of the pairs corresponding to edges of M_j in a significant way.

We will also assume that for each $j' < j$, we already covered the vertices in $V_{M_{j'}} \cup U_i^{(j')}$ with a cycle. Since we want the new cycle to be edge-disjoint from these other cycles, we remove the corresponding cycles from the graph (thus removing at most $2k$ edges in each vertex). It is easy to see that the remaining graph still has all necessary regularity properties.

Step 1: creating the small paths. First, we assign each $C \in V(M_j)$ to a neighbor D_C of C in D_ℓ in such a way that we assign at most $3s_i/d$ clusters of $V(M_j)$ to each cluster in D_ℓ. This is possible because each vertex of R_i has $dt_i/(3s_i)$ neighbors in D_ℓ and because there are at most t_i clusters in $V(M_j)$.

During the construction of the paths, for every $D \in D_\ell$ and $C \in V(M_j)$, we maintain sets $A(D) \subseteq D$ and $A(C) \subseteq C$ of *available* vertices; initially $A(D) = D$ and $A(C) = C$ for all D and C, i.e., all vertices are available. The sets $A(D)$ and $A(C)$ will shrink during the construction of the paths, however, it will be true throughout that for each $C \in V(M_j)$ and $D \in D_\ell$, we have $|A(C)| \geq |C| - K\epsilon|C|/d$ and $|A(D)| \geq |D| - K\epsilon|D|/d$, where $K = K(k)$ is a sufficiently large constant. Since ϵ is very small compared to d, this means that almost all vertices are available throughout the process.

For each cluster $C \in V(M_j)$, we shall first build a path P'_C covering the vertices of U_C. The path will have the form

$$P'_C = x_1 u_1 y_1 \ z_2 \ x_2 u_2 y_2 \ z_3 \ x_3 u_3 y_3 \ \cdots \ z_{|U_C|} \ x_{|U_C|} u_{|U_C|} y_{|U_C|},$$

where $x_p, y_p \in C$, $u_p \in U_C$ and $z_p \in D_C$. After doing this, we will extend this path to a path P_C that uses exactly s_C vertices of C, completing the first step in the outline given above.

We now describe how to construct P'_C. Recall that every $u \in U_C$ has $n/(2k^4 t) \geq 2K\epsilon|C|/d$ neighbors in C. Order the vertices of U_C arbitrarily. For the first vertex $u_1 \in U_C$, let x_1 be an arbitrary neighbor of u_1 in $A(C)$, and let y_1 be a vertex in $A(C)$ that has at least $dn/(3t)$ neighbors in $A(D_C)$. Assuming that $|A(C)| \geq |C| - K\epsilon|C|/d$ and $|A(D_C)| \geq |D_C| - K\epsilon|D_C|/d$, such neighbors exists by regularity. Remove x_1, y_1 from $A(C)$.

At every subsequent step, consider the current $u_p \in U_C$. Provided that $|A(C)| \geq |C| - K\epsilon|C|/d$, there is a neighbor x_p of u_p in $A(C)$ that has a neighbor z_p in the neighborhood of y_{p-1} in $A(D_C)$, which we may assume (by induction) to be of size at least $dn/(3t) \geq \epsilon|D_C|$. Similarly, there is a neighbor $y_p \in A(C)$ that has at least $dn/(3t)$ neighbors in $A(D_C) \setminus \{z_p\}$, again provided that $A(C)$ and $A(D_C)$ are large. Remove z_p from $A(D_C)$ and remove x_p, y_p from $A(C)$.

We can continue in this way as long as $A(C)$ and $A(D_C)$ are sufficiently large, which they will be, since for every vertex in U_C we remove at most one vertex from $A(D_C)$ and two from $A(C)$, and $|U_C| \leq 4k^4\epsilon n/t \leq K\epsilon|C|/(12s_i)$. Moreover, only $3s_i/d$ clusters have chosen D_C, so it follows that each $A(C)$ and $A(D)$ lose at most $K\epsilon|C|/(2d)$ (resp. $K\epsilon|D|/(2d)$) vertices altogether.

Note that the path P'_C uses exactly $2|U_C| \leq 8k^4\epsilon n/t$ vertices from C. However, we would like to have a path that uses exactly $s_C \in [8k^4\epsilon n/t, 100k^4\epsilon n/t]$ vertices of C. For this reason, we will extend the path in the following way.

By construction, $y_{|U_C|}$ has $dn/(3t) \geq \epsilon|D_C|$ neighbors in $A(D_C)$. The typical vertex in $A(C)$ has a neighbor in this neighborhood, as well as $dn/(3t)$ additional neighbors in $A(D_C)$. Thus we may take such a vertex $x_{|U_C|+1}$ and a common neighbor $z_{|U_C|+1} \in A(D_C)$ of $x_{|U_C|+1}$ and $y_{|U_C|}$, and create a longer path $P'_C\ z_{|U_C|+1}\ x_{|U_C+1|}$. Then, we remove $z_{|U_C|+1}$ from $A(D_C)$ and $x_{|U_C|+1}$ from $A(C)$. As before, this process will not fail while $|A(C)| \geq |C| - K\epsilon|C|/d$ and $|A(D)| \geq |D| - K\epsilon|D|/d$ hold for all $C \in V(M_j)$ and $D \in D_\ell$. If K is large enough, then this means that we can continue for at least $100(k+1)k^3\epsilon n/t$ steps, and we do so until the path contains exactly s_C vertices of C.

Call the resulting path P_C. Observe that for different $C, C' \in V(M_j)$, the paths P_C and $P_{C'}$ are vertex-disjoint. Moreover, each path P_C has its endpoints in C, uses s_C vertices of C (and no vertices of other clusters in $V(M_j)$), and visits all vertices in U_C.

Step 2: finishing the embedding.

Let T_j be a minimal tree in R_i containing the matching M_j as a subgraph (such a tree exists because R_i is connected), and let $m = |T_j| - 1$ be the number of edges of T_j. For each $C \in V(M_j)$, choose $s_C \in [8(k+1)k^3\epsilon n/t, 100(k+1)k^3\epsilon n/t]$ such that

$$|C| - s_C = \lfloor n/t \rfloor - \lfloor 20k^4\epsilon n/t \rfloor + d_{T_j}(C).$$

This is possible since $n/t \geq |C| \geq (1-\epsilon)n/t$ and since $d_{T_j}(C) \leq t$.

By doubling the edges of T_j and considering an Euler tour in the resulting graph, one can see that there exists a surjective homomorphism $\pi\colon C_{2m} \to T_j$ that covers each edge of T_j exactly twice, i.e., for each edge $e \in T_j$, there are exactly two edges $e_1, e_2 \in E(C_{2m})$ such that $\pi(e_1) = \pi(e_2) = e$. For each edge $e \in M_j$, we (arbitrarily) color the edge e_1 red. Let us, for the moment, remove all red edges from C_{2m}, resulting in the graph C'_{2m}, which is just a system of disjoint paths. We now choose any embedding

$$\iota\colon C_{2m'} \to G$$

with the property that every $x \in V(C'_{2m})$ is mapped to a vertex in $\pi(x)$, and whose image is disjoint from the vertices of the paths P_C. Such an embedding exists by regularity: for every path $x_1, \dots, x_r$ in C'_{2m}, we may first embed x_1 to a vertex in $\pi(x_1)$ that has at least $d|\pi(x_2)|$ neighbors in $\pi(x_2)$. Of these neighbors, at least half will have at least $d|\pi(x_3)|$ neighbors in $\pi(x_3)$, so we may embed x_2 to any such neighbor. Continuing in this way, we can completely embed $x_1, \dots, x_r$ in G in the desired way, and we can do this for every path in C'_{2m}. Note that some vertices might be embedded into the same cluster of R_i; however, as $m \le t$ is a constant and as each cluster has linear size, this does not pose any difficulty.

At this point, we have merely embedded some disjoint paths into G. For each red edge $xy \in E(C_{2m})$, we will now embed into G a path with endpoints $\iota(x)$ and $\iota(y)$ that contains all vertices of $P_{\pi(x)} \cup P_{\pi(y)}$, and that, moreover, contains all vertices of $\pi(x) \cup \pi(y)$ that are not in the image of ι. Thus, we will extend ι to an embedding of a subdivision of C_{2m} into G whose image contains the set $V_{M_j} \cup U_i^{(j)}$, as required. Since for each red edge xy, the pair $(\pi(x), \pi(y))$ is $(\epsilon, d/2)$-super-regular, this is relatively easy to achieve: first, we connect an endpoint of $P_{\pi(x)}$ to $\iota(x)$ by a path of length four; similarly, we connect an endpoint of $P_{\pi(y)}$ to $\iota(y)$ by a path of length four; finally, we use the Lemma 9 to connect the other endpoint of $P_{\pi(x)}$ to the other endpoint of $P_{\pi(y)}$ by a Hamilton path in the bipartite subgraph of $G[\pi(x), \pi(y)]$ induced by the remaining vertices. The only thing to check is that this subgraph is balanced. However, this follows from our choice of s_C and the fact that the image of ι intersects each cluster C in exactly $d_{T_j}(C)$ vertices. □

Proof of Lemma 12. Let $m_i = \lfloor s_i - 4dk^2 \rfloor$. Combining Lemma 13 with Lemma 14, we obtain that either the graph can be covered with $\lfloor s_i - 4dk^2 \rfloor$ cycles, or the component R_i of the reduced graph contains a set I of size $(s_i - 1)t_i/s_i - 6dk^2 s_i t_i$ that spans at most $4dk^2 s_i t_i^2$ edges, where $t_i = v(R_i)$.

In the former case, we are done. We claim that in the latter case, the set Y_i is $(s_i, k, \beta/2)$-stable in G. To avoid a technical proof of a simple statement, we use standard asymptotic notation for the following argument; e.g., $O(dn)$ designates a quantity bounded from above by Cdn, where C is some costant depending only on k.

We first need to show how to choose the sets A and B. For this, let I be an almost independent set as above, and let B' be the union of the clusters in I. Note that since by (3), we have $\delta(R_i) \ge t_i/s_i$, we obtain an upper bound on the size of I: if I was larger than, say, $(s_i - 1)t_i/s_i + 16dk^2 s_i t_i$, then $R_i[I]$

would contain more than

$$|I| \cdot 8dk^2 s_i t_i \geq 4dk^2 s_i t_i^2$$

edges, a contradiction. Thus, we have

$$|I| \in (s_i - 1)t_i/s_i \pm O(dt_i). \tag{9}$$

Using (9), the definition (2) of s_i, and the fact that d and ϵ are much smaller than β, we easily obtain that

$$|B'| \in (s_i - 1)v(G_i)/s_i \pm O(dn). \tag{10}$$

Since I contains only $O(dt_i^2)$ edges, using (6), we see that B' contains at most $O(dn^2)$ edges. edges. Thus, all but at most $O(\sqrt{d}n)$ vertices of B' have at most $\sqrt{d}n$ neighbors in B'. Let B be the subset of B' consisting of the vertices that have at most $\sqrt{d}n$ neighbors in B', and so (by (5)) at least $v(G_i)/s_i - O(\sqrt{d}n)$ neighbors in $|V(G_i) \setminus B|$. Since $|V(G_i) \setminus B| = v(G_i)/s_i \pm O(dn)$, all but $O(\sqrt{d}n/\beta)$ vertices of $V(G_i) \setminus B$ have at least $(1 - \beta/3)$ neighbors in B. Let A be the set of all such vertices in $V(G_i) \setminus B$.

With this construction, each vertex of B has at least $(1-\beta/3)|A|$ neighbors in A, and each vertex of A has at least $(1-\beta/3)|B|$ neighbors in B. Moreover, $|A \cup B| = v(G_i) - O(\sqrt{d}n/\beta)$, and $|B| \in (s_i - 1)n/k \pm \beta n/3$ and $|A| \in n/k \pm \beta n/3$.

By the definition of Y_i, each vertex of $Y_i \setminus (A \cup B)$ has at least n/k^4 neighbors in $A \cup B$. Thus, it is possible to distribute these vertices over $A \cup B$ in such a way that each vertex of B has at least $|A|/k^4$ neighbors in A, and every vertex of A has at least $|B|/k^4$ neighbors in B. Then all conditions of Definition 1 are satisfied, i.e., Y_i is $(s_i, k, \beta/2)$-stable in G.

Given this, it is now easy to complete the proof of Lemma 12. By (5), most vertices of Y_i have degree at least $|Y_i|/s_i$ in $G[Y_i]$. Therefore, by Lemma 6, we can cover $G[Y_i]$ with $\lceil s_i \rceil - 1$ edge-disjoint cycles. Thus, either we can cover it with $\lfloor s_i - 4dk^2 \rfloor$ cycles, or $s_i \leq \lfloor s_i \rfloor + 4dk^2$. But in the latter case, one easily checks that Y_i is actually a $(\lfloor s_i \rfloor, k, \beta)$-stable subset of G. This completes the proof of Lemma 12. □

References

[1] J. Balogh, J. Barát, D. Gerbner, A. Gyárfás and G. Sárközy, Partitioning 2-edge-colored graphs by monochromatic paths and cycles, *Combinatorica*, **34** (2014), 507–526.

[2] F. S. Benevides, J. Skokan, The 3-colored Ramsey number of even cycles, *J. Combin. Theory Ser. B*, **99**, (2009), 690–708.

[3] J. Böttcher, J. Hladký, D. Piguet, The tripartite Ramsey number for trees, *J. Graph Theory*, **69** (2012), 264–300.

[4] Ph. Chau, L. DeBiasio, H. A. Kierstead, Pósa's conjecture for graphs of order at least $2 \cdot 10^8$, *Random Structures Algorithms*, **39** (2011), 507–525.

[5] G. A. Dirac, Some theorems on abstract graphs, *Proc. Lond. Math. Soc.*, **2** (1952), 69–81.

[6] H. Enomoto, A. Kaneko, and Zs. Tuza, *Graphs of order n and minimum degree n/4*, Combinatorics (Proc. Coll. Math. Soc. János Bolyai, Eger, 1987) North-Holland, New York, 1988, pp. 213–220.

[7] G. Fan, H.A. Kierstead, Hamiltonian square-paths, *J. Comb. Theory Ser. B*, **67** (1996), 167–182.

[8] A. Figaj, T. Łuczak, The Ramsey number for a triple of long even cycles. *J. Combin. Theory Ser. B*, **97** (2007), 584–596.

[9] P. Erdős, *Problem 9*, Theory of graphs and its applications, M. Fiedler (Editor), Czech Acad Sci Publ, Prague, 1964, pp. 159.

[10] J. Komlós, The Blow-up Lemma. *Combin. Probab. Comput.*, **8** (1999), 161–176.

[11] J. Komlós, G. N. Sárközy, E. Szemerédi, On the square of a Hamiltonian cycle in dense graphs. *Random Structures Algorithms*, **9** (1996), 193–211.

[12] J. Komlós, G. N. Sárközy, E. Szemerédi, The Blow-up Lemma. *Combinatorica*, **17** (1996), 109–123.

[13] J. Komlós, G. N. Sárközy, E. Szemerédi, Proof of the Seymour conjecture for large graphs, *Ann. Comb.*, **2** (1998), 43–60.

[14] J. Komlós, M. Simonovits, *Szemerédi's regularity lemma and its applications in graph theory*, Combinatorics, Paul Erdős is eighty, Vol. 2 (Keszthely, 1993), Bolyai Soc. Math. Stud., vol. 2, János Bolyai Math. Soc., Budapest, 1996, pp. 295–352.

[15] M. Kouider, Covering vertices by cycles, *J. Graph Theory* **18** (1994) 757–776.

[16] R. Lang, M. J. Stein, Local colourings and monochromatic partitions in complete bipartite graphs, submitted.

[17] I. Levitt, G. N. Sárközy, E. Szemerédi, How to avoid using the Regularity Lemma: Pósa's conjecture revisited, *Discrete Math.*, **310** (2010), 630–641.

[18] T. Łuczak, $R(C_n, C_n, C_n) \leq (4 + o(1))n$, *J. Combin. Theory Ser. B*, **75** (1999), 174–187.

[19] G. N. Sárközy, S. M. Selkow, F. Song, An improved bound for vertex partitions by connected monochromatic k-regular graphs, *J. Graph Theory*, **73** (2013), 127–145.

[20] P. Seymour, *Problem section*, Combinatorics: Proceedings of the British Combinatorial Conference 1973, T. P. McDonough and V. C. Mavron (Editors), Cambridge University Press, Cambridge, 1974, pp. 201–202.

[21] E. Szemerédi, Regular partitions of graphs. In: *Problèmes Combinatoires et Théorie des Graphes.* Colloques Internationaux CNRS 260. Orsay, 1976, pp. 399–401.