On perfect packings in dense graphs

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Abstract

We say that a graph G has a perfect H-packing if there exists a set of vertex-disjoint copies of H which cover all the vertices in G. We consider various problems concerning perfect H-packings: Given $n, r, D \in \mathbb{N}$, we characterise the edge density threshold that ensures a perfect K_r-packing in any graph G on n vertices and with minimum degree $\delta(G) \geq D$. We also give two conjectures concerning degree sequence conditions which force a graph to contain a perfect H-packing. Other related embedding problems are also considered. Indeed, we give a degree sequence condition which forces a graph to contain a copy of K_r, thereby strengthening the minimum degree version of Turán's theorem. We also characterise the edge density threshold that ensures a graph G contains k vertex-disjoint cycles.

1 Introduction

Given two graphs H and G, a *perfect H-packing* in G is a collection of vertex-disjoint copies of H which cover all the vertices in G. Perfect H-packings are also referred to as *H-factors* or *perfect H-tilings*. Hell and Kirkpatrick [7] showed that the decision problem whether a graph G has a perfect H-packing is NP-complete precisely when H has a component consisting of at least 3 vertices. So for such graphs H, it is unlikely that there is a complete characterisation of those graphs containing a perfect H-packing. Thus, there has been significant attention on obtaining sufficient conditions that ensure a graph G contains a perfect H-packing.

A seminal result in the area is the Hajnal-Szemerédi theorem [6] which states that a graph G whose order n is divisible by r has a perfect K_r-packing provided that $\delta(G) \geq (r-1)n/r$. Kühn and Osthus [10, 11] characterised, up to an additive constant, the minimum degree which ensures a graph G contains a perfect H-packing for an arbitrary graph H.

It is easy to see that the minimum degree condition in the Hajnal-Szemerédi theorem cannot be lowered. Of course, this does not mean that one cannot strengthen this result. *Ore-type* degree conditions consider the sum of the degrees of non-adjacent vertices in a graph. The following Ore-type result of Kierstead and Kostochka [8] implies the Hajnal-Szemerédi theorem.

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Theorem 1 (Kierstead and Kostochka [8]) *Let $n, r \in \mathbb{N}$ such that r divides n. Suppose that G is a graph on n vertices such that for all non-adjacent $x \neq y \in V(G)$,*

$$d(x) + d(y) \geq 2(1 - 1/r)n - 1.$$

Then G contains a perfect K_r-packing.

Kühn, Osthus and Treglown [12] characterised, asymptotically, the Ore-type degree condition which ensures a graph G contains a perfect H-packing for an arbitrary graph H.

1.1 Degree sequence conditions forcing a perfect K_r-packing

Chvátal [3] gave a condition on the degree sequence of a graph which ensures Hamiltonicity: Suppose that G is a graph on n vertices and that the degrees of the graph are $d_1 \leq \cdots \leq d_n$. If $n \geq 3$ and $d_i \geq i + 1$ or $d_{n-i} \geq n - i$ for all $i < n/2$ then G is Hamiltonian. So in the case when n is even, this degree sequence condition ensures that G has a perfect K_2-packing (i.e. a perfect matching). We propose the following conjecture on the degree sequence of a graph which forces a perfect K_r-packing.

Conjecture 2 *Let $n, r \in \mathbb{N}$ such that r divides n. Suppose that G is a graph on n vertices with degree sequence $d_1 \leq \cdots \leq d_n$ such that:*

(α) $d_i \geq (r-2)n/r + i$ for all $i < n/r$;

(β) $d_{n/r+1} \geq (r-1)n/r$.

Then G contains a perfect K_r-packing.

Note that Conjecture 2, if true, is much stronger than the Hajnal-Szemerédi theorem since the degree condition allows for n/r vertices to have degree less than $(r-1)n/r$. Further, Proposition 14 in Section 5 shows that the condition on the degree sequence in Conjecture 2 is essentially "best possible". Chvátal [3] proved Conjecture 2 in the case when $r = 2$. We prove the conjecture in the case when G is additionally K_{r+1}-free (see Section 6).

It is also of interest to establish degree sequence conditions which force a single copy of K_r in a graph G. In Section 7 we give such a result, which is a consequence of the following structural theorem.

Theorem 3 *Suppose that $n, r \in \mathbb{N}$ such that $n \geq r$ and so that r divides n. Let G be a K_{r+1}-free graph on n vertices whose degree sequence $d_1 \leq \cdots \leq d_n$ is such that $d_{n/r} \geq (r-1)n/r$. Then $G \subseteq T(n, r)$.*

(Here $T(n, r)$ denotes the complete r-partite *Turán graph* on n vertices; so each vertex class has size $\lceil n/r \rceil$ or $\lfloor n/r \rfloor$.)

1.2 Perfect packings in dense graphs of low minimum degree

In Section 3 we consider the following natural problem: Let $n, r \in \mathbb{N}$ such that r divides n. Given some $D \in \mathbb{N}$, what edge density condition ensures that any graph G on n vertices and of minimum degree $\delta(G) \geq D$ contains a perfect K_r-packing? In Section 4.1 we deal with the case when $r = 2$. The following result completely answers this question for $r \geq 3$.

Theorem 4 *Let $n, r \in \mathbb{N}$ such that $r \geq 3$ and r divides n. Given any $D \in \mathbb{N}$ such that $r-1 \leq D \leq (r-1)n/r - 1$ define*

$$g(n,r,D) := \max\left\{\binom{n}{2} - \binom{n/r+1}{2}, D(n-D) + \binom{n-1-D}{2} + e(T(D,r-2))\right\}.$$

Suppose that G is a graph on n vertices with $\delta(G) \geq D$ and $e(G) > g(n,r,D)$. Then G contains a perfect K_r-packing. Moreover, there exists a graph G' on n vertices with $\delta(G') \geq D$ and $e(G') = g(n,r,D)$ but such that G' does not contain a perfect K_r-packing.

Clearly a graph G of minimum degree $\delta(G) < r-1$ cannot contain a perfect K_r-packing. Further, regardless of edge density, every graph G whose order n is divisible by r and with $\delta(G) \geq (r-1)n/r$ contains a perfect K_r-packing. Thus, Theorem 4 considers all values of D where our problem was not solved previously. We prove Theorem 4 in Section 3. In Section 2 we prove the 'moreover' part of Theorem 4. That is, we show that the edge density condition in Theorem 4 is best possible for all values of D such that $r-1 \leq D \leq (r-1)n/r - 1$.

An *equitable k-colouring* of a graph G is a proper k-colouring of G such that any two colour classes differ in size by at most one. Let $n, r \in \mathbb{N}$ such that r divides n. Notice that a graph G on n vertices has a perfect K_r-packing if and only if the complement $\overline{G}$ of G has an equitable n/r-colouring. So, for example, the Hajnal-Szemerédi theorem can be stated in terms of equitable colourings: Let G be a graph on n vertices such that r divides n. If $\Delta(G) \leq n/r - 1$ then G has an equitable n/r-colouring.

It is often easier to work in the language of equitable colourings compared to perfect packings. Indeed, rather than prove Theorem 1 directly, Kierstead and Kostochka proved the equivalent statement for equitable colourings. Here we also find it more convenient to work with equitable colourings. Thus, instead of proving Theorem 4 directly we prove the following equivalent result.

Theorem 5 *Let $n, r \in \mathbb{N}$ such that $r \geq 3$ and r divides n. Given any $D \in \mathbb{N}$ such that $n/r \leq D \leq n - r$ define*

$$f(n,r,D) := \min\left\{\binom{n/r+1}{2}, D + e(\overline{T}(n-D-1, r-2))\right\}.$$

Suppose that G is a graph on n vertices with $\Delta(G) \leq D$ and $e(G) < f(n,r,D)$. Then G has an equitable n/r-colouring. Moreover, there exists a graph G' on n vertices with $\Delta(G') \leq D$ and $e(G') = f(n,r,D)$ but such that G' does not have an equitable n/r-colouring..

(Note that here $\overline{T}(n,r)$ denotes the complement of the Turán graph $T(n,r)$.)

1.3 Vertex-disjoint cycles in dense graphs

Given $k \in \mathbb{N}$, Corrádi and Hajnal [5] proved that every graph G on $n \geq 3k$ vertices and of minimum degree $\delta(G) \geq 2k$ contains at least k vertex-disjoint cycles. So when $n = 3k$, the Corrádi-Hajnal theorem is precisely the Hajnal-Szemerédi theorem in the case when $r = 3$. Recently, Allen, Böttcher, Hladký and Piguet (see [1]) characterised the density threshold that ensures a sufficiently large n-vertex graph G contains at least k vertex-disjoint triangles where $0 \leq k \leq n/3$. As an application of Theorem 4 we characterise the density threshold that ensures an n-vertex graph G contains at least k vertex-disjoint *cycles* where $n \geq 7k/2$.

Theorem 6 *Let $n, k \in \mathbb{N}$ such that $n \geq 7k/2$. Suppose that G is a graph on n vertices so that*

$$e(G) > (2k-1)(n-k).$$

Then G contains k vertex-disjoint cycles. Moreover, there exists a graph G' on n vertices with $e(G') = (2k-1)(n-k)$ such that G' does not contain k vertex-disjoint cycles.

We prove Theorem 6 in Section 4.2. Notice that $G' := K_n - E(K_{n-2k+1})$ does not contain k vertex-disjoint cycles and $e(G') = (2k-1)(n-k)$.

2 The extremal examples for Theorems 4 and 5

In this section we will give the extremal examples for Theorem 5. Since Theorems 4 and 5 are equivalent, the complements of the extremal graphs for Theorem 5 are the extremal graphs for Theorem 4.

Proposition 7 *Suppose that $n, r \in \mathbb{N}$ such that $r \geq 3$ and r divides n. Then there exists a graph G_1 on n vertices such that $\Delta(G_1) = n/r$,*

$$e(G_1) = \binom{n/r+1}{2},$$

but such that G_1 does not have an equitable n/r-colouring.

Proof. Let G_1 denote the disjoint union of a clique V on $n/r+1$ vertices and an independent set W of $(1-1/r)n-1$ vertices. So every independent set in G_1 contains at most one vertex from V. But since $|V| = n/r+1$, G_1 does not have an equitable n/r-colouring. Further, $\Delta(G_1) = n/r$ and $e(G_1) = \binom{n/r+1}{2}$. □

Proposition 8 *Suppose that $n, r \in \mathbb{N}$ such that $r \geq 3$ and $n = kr$ for some $k \geq 2$. Further, let $D \in \mathbb{N}$ such that $n/(r-1) \leq D \leq n-r$. Then there exists a graph G_2 on n vertices such that $\Delta(G_2) = D$,*

$$e(G_2) = D + e(\overline{T}(n-D-1, r-2)),$$

but such that G_2 does not have an equitable n/r-colouring.

Proof. Let G_2 denote the disjoint union of a copy K of $K_{1,D}$ and a copy of $\overline{T}(n-D-1, r-2)$. So $|G| = n$. Let v denote the vertex of degree D in K. The largest independent set in G_2 that contains v is of size $r-1$. Thus, G_2 does not have an equitable n/r-colouring. Further, $e(G_2) = D + e(\overline{T}(n-D-1, r-2))$.

Since $n/(r-1) \leq D$ we have that $n-1 \leq (r-1)D$. Thus,

$$\left\lceil \frac{n-D-1}{r-2} \right\rceil - 1 \leq \frac{n-D-1}{r-2} \leq D.$$

This implies that $\Delta(G_2) = D$. □

Clearly Propositions 7 and 8 show that one cannot lower the edge density condition in Theorem 5 in the case when $n/(r-1) \leq D \leq n-r$. The following result, together with Proposition 7, shows that Theorem 5 is best possible in the case when $n/r \leq D \leq n/(r-1)$.

Proposition 9 *Let $n, r \in \mathbb{N}$ such that $r \geq 3$ and r divides $n \geq 2r$. Suppose that $D \in \mathbb{N}$ such that $n/r \leq D \leq n/(r-1)$. Then*

$$f(n,r,D) = \binom{n/r+1}{2}.$$

The following simple consequence of Turán's theorem will be used in the proof of Theorem 5.

Fact 10 *Let $n, r \in \mathbb{N}$ such that $r \leq n$. Then*

$$e(T(n,r)) \leq \left(1 - \frac{1}{r}\right)\frac{n^2}{2} \quad \textit{and thus} \quad e(\overline{T}(n,r)) \geq \frac{n^2}{2r} - \frac{n}{2}.$$

We will also require the following easy result.

Lemma 11 *Let $n, r \in \mathbb{N}$ such that $r \geq 4$ and r divides $n \geq 3r$. Suppose that $D \in \mathbb{N}$ such that $n/r \leq D < (n+r)/(r-1)$. Then*

$$f(n,r,D) = \binom{n/r+1}{2}.$$

3 Proof of Theorem 5

3.1 Preliminaries

Suppose for a contradiction that the result is false. Let G be a counterexample with the fewest vertices. That is, $n = |V(G)| = rk$ for some $k \in \mathbb{N}$, $\Delta(G) \leq D$ for some $D \in \mathbb{N}$ such that $n/r \leq D \leq n-r$, $e(G) < f(n,r,D)$ and G has no equitable n/r-colouring. By the Hajnal-Szemerédi theorem, $\Delta(G) \geq n/r$. Notice that given fixed n and r, $f(n,r,D)$ is non-increasing with respect to D. Thus, we may assume that $\Delta(G) = D$.

We first show that $k \geq 4$. Indeed, if $n = 2r$ then $f(n,r,D) \leq \binom{3}{2} = 3$. But it is easy to see that every graph G_1 on $2r$ vertices and with $e(G_1) \leq 2$ has an equitable 2-colouring. If $n = 3r$ then $f(n,r,D) \leq \binom{4}{2} = 6$. Consider any graph G_1 on $3r$ vertices with $e(G_1) \leq 5$ and $3 \leq \Delta(G_1) \leq 5$. Let x denote the vertex in G_1 where $d_{G_1}(x) = \Delta(G_1)$. Since $3 \leq d_{G_1}(x) \leq 5$, x lies in an independent set I in G_1 of size r. But then $G_1 - I$ contains $2r$ vertices and at most 2 edges. So $G_1 - I$ has an equitable 2-colouring and hence G_1 has an equitable 3-colouring.

Let $v \in V(G)$ such that $d_G(v) = D$. Set $G^* := G - (N_G(v) \cup \{v\})$. Since $f(n,r,D) \leq D + e(\overline{T}(n-D-1, r-2))$ we have that $e(G^*) < e(\overline{T}(n-D-1, r-2))$. Thus, by Turán's theorem, G^* contains an independent set of size $r-1$. Hence, v lies in an independent set in G of size r. Amongst all such independent sets of size r that contain v, choose a set $I = \{v, x_1, \dots, x_{r-1}\}$ such that $d_G(x_1) + \dots + d_G(x_{r-1})$ is maximised.

Set $G' := G - I$, $n' := |V(G')| = n - r$ and $D' := \Delta(G') \leq D$. Notice that $D' \geq n'/r$. (Indeed, if not, then by the Hajnal-Szemerédi theorem G' contains an equitable n'/r-colouring. Thus, as I is an independent set in G this gives us an equitable n/r-colouring of G, a contradiction.) Furthermore, $D' \leq n' - r$. If not then

$$e(G) \geq D + D' \geq 2(n' - r + 1) = 2n - 4r + 2.$$

Since $D \geq D' \geq n' - r + 1 = n - 2r + 1$ we have that

$$e(G) < f(n, r, D) \leq f(n, r, n - 2r + 1) \leq (n - 2r + 1) + e(\overline{T}(2r - 2, r - 2))$$
$$\leq (n - 2r + 1) + (r + 3) = n - r + 4.$$

Therefore, $2n - 4r + 2 < n - r + 4$ and so $n < 3r + 2$ a contradiction since $n = kr \geq 4r$.

Since $n'/r \leq D' \leq n' - r$, if $e(G') < f(n', r, D')$ then the minimality of G implies that G' has an equitable n'/r-colouring. This then implies that G has an equitable n/r-colouring, a contradiction. Thus,

$$e(G') \geq f(n', r, D'). \tag{1}$$

We now split our argument into three cases.

3.2 Case 1: $\mathbf{f(n', r, D') = \binom{n'/r+1}{2}}$.

By (1), $e(G') \geq \binom{n'/r+1}{2} = \binom{n/r}{2}$. Since $d_G(v) = D \geq n/r$,

$$e(G) \geq \frac{n}{r} + \binom{n/r}{2} = \binom{n/r + 1}{2} \geq f(n, r, D),$$

a contradiction, as desired.

3.3 Case 2: $\mathbf{D' \leq D - 1}$ and $\mathbf{f(n', r, D') = D' + e(\overline{T}(n' - D' - 1, r - 2))}$.

The following claim will be useful.

Claim 12 $D' < \frac{r-1}{2r-3}n - \frac{(r^2-r+1)}{2r-3}$.

Proof. Note that

$$D + D' + e(\overline{T}(n' - D' - 1, r - 2)) \overset{(1)}{\leq} e(G) < f(n, r, D) \leq D + e(\overline{T}(n - D - 1, r - 2)). \tag{2}$$

Since $D' \leq D - 1$, clearly $e(\overline{T}(n' - D, r - 2)) \leq e(\overline{T}(n' - D' - 1, r - 2))$. Thus, (2) implies that

$$D' + e(\overline{T}(n' - D, r - 2)) < e(\overline{T}(n - D - 1, r - 2)). \tag{3}$$

One can obtain $\overline{T}(n - D - 1, r - 2)$ from $\overline{T}(n' - D, r - 2)$ by adding $r - 1$ vertices and at most

$$(n' - D) + \frac{n - D - 2}{r - 2} \quad \text{edges.} \tag{4}$$

Hence (3) and (4) give

$$D' < n' - D + \frac{n - D - 2}{r - 2}.$$

Rearranging, and using that $D' \leq D - 1$ and $n' = n - r$ we get that

$$\left(2 + \frac{1}{r - 2}\right) D' < \left(1 + \frac{1}{r - 2}\right) n - \frac{(r^2 - r + 1)}{r - 2}.$$

Thus,

$$D' < \frac{r - 1}{2r - 3}n - \frac{(r^2 - r + 1)}{2r - 3},$$

as desired. □

Since we are in Case 2 we have that

$$D' + e(\overline{T}(n - r - D' - 1, r - 2)) \leq \binom{n'/r + 1}{2} = \binom{n/r}{2}. \tag{5}$$

Notice that for fixed n and r, $D' + e(\overline{T}(n - r - D' - 1, r - 2))$ is non-increasing as D' increases. Hence, (5) and Claim 12 imply that

$$D'' + e(\overline{T}(n - r - D'' - 1, r - 2)) \leq \frac{n^2}{2r^2} - \frac{n}{2r} \tag{6}$$

where $D'' := \lfloor (r-1)n/(2r-3) - (r^2 - r + 1)/(2r-3) \rfloor$. Note that

$$n - r - \frac{r-1}{2r-3}n + \frac{(r^2 - r + 1)}{2r-3} - 1 = \frac{r-2}{2r-3}n + \frac{4-r^2}{2r-3}.$$

So Fact 10 and (6) imply that

$$\left(\frac{r-1}{2r-3}n - \frac{(r^2-r+1)}{2r-3} - \frac{(2r-4)}{2r-3}\right) + \frac{1}{2(r-2)}\left(\frac{r-2}{2r-3}n + \frac{4-r^2}{2r-3}\right)^2 - \frac{1}{2}\left(\frac{r-2}{2r-3}n + \frac{4-r^2}{2r-3}\right)$$
$$\leq \frac{n^2}{2r^2} - \frac{n}{2r}.$$

Next we will move all terms from the previous equation to the left hand side and simplify. The coefficient of n^2 is

$$\frac{r-2}{2(2r-3)^2} - \frac{1}{2r^2} = \frac{r^3 - 6r^2 + 12r - 9}{2r^2(2r-3)^2}. \tag{7}$$

The coefficient of n is

$$\frac{r-1}{2r-3} - \frac{(r-2)}{2(2r-3)} + \frac{1}{2r} + \frac{(4-r^2)}{(2r-3)^2} = \frac{r^2 - 4r + 9}{2r(2r-3)^2}. \tag{8}$$

The constant term is

$$-\frac{(r^2 + r - 3)}{2r-3} + \frac{(r^2-4)^2}{2(r-2)(2r-3)^2} + \frac{(r^2-4)}{2(2r-3)} = \frac{-r^4 + 3r^3 + 4r^2 - 26r + 28}{2(r-2)(2r-3)^2}. \tag{9}$$

Since $n \geq 4r$, (7)–(9) imply that

$$\frac{8(r^3 - 6r^2 + 12r - 9)}{(2r-3)^2} + \frac{2(r^2 - 4r + 9)}{(2r-3)^2} + \frac{-r^4 + 3r^3 + 4r^2 - 26r + 28}{2(r-2)(2r-3)^2} \leq 0. \tag{10}$$

Multiplying (10) by $2(r-2)(2r-3)^2$ we get

$$15r^4 - 121r^3 + 364r^2 - 486r + 244 \leq 0$$

This yields a contradiction, since it is easy to check that

$$15r^4 - 121r^3 + 364r^2 - 486r + 244 > 0$$

for all $r \in \mathbb{N}$ such that $r \geq 3$.

3.4 Case 3: $\mathbf{D'} = \mathbf{D}$ and $\mathbf{f(n', r, D') = D' + e(\overline{T}(n' - D' - 1, r - 2))}$.

By (1) we have that

$$e(G') \geq f(n', r, D') = D' + e(\overline{T}(n' - D' - 1, r - 2)). \tag{11}$$

Consider any vertex $x \in V(G')$ such that $d_{G'}(x) = D' = D$. Since $\Delta(G) = D$, x is not adjacent to any vertex in $I = \{v, x_1, \dots, x_{r-1}\}$. Further, I was chosen such that $d_G(x_1) + \cdots + d_G(x_{r-1})$ is maximised. Thus, $d_G(x_1) = \cdots = d_G(x_{r-1}) = D$. Together with (11) this implies that

$$e(G) \geq (r+1)D + e(\overline{T}(n' - D - 1, r - 2)). \tag{12}$$

Since $e(G) < f(n, r, D) \leq D + e(\overline{T}(n - D - 1, r - 2))$, (12) implies that

$$rD + e(\overline{T}(n' - D - 1, r - 2)) < e(\overline{T}(n - D - 1, r - 2)). \tag{13}$$

One can obtain $\overline{T}(n - D - 1, r - 2)$ from $\overline{T}(n' - D - 1, r - 2)$ by adding r vertices and at most

$$(n' - D - 1) + \frac{2(n - D - 3)}{r - 2} + 1 \ \text{ edges.} \tag{14}$$

Thus, (13) and (14) imply that

$$rD < n - r - D + \frac{2(n - D - 3)}{r - 2}$$

and so

$$\left(r + 1 + \frac{2}{r-2}\right) D < \left(1 + \frac{2}{r-2}\right) n + \frac{(-r^2 + 2r - 6)}{r - 2} < \left(1 + \frac{2}{r-2}\right) n. \tag{15}$$

If $r = 3$ then (15) implies that

$$D < \frac{n}{2}.$$

Since $f(n', 3, D) = \min\{\binom{n'/3+1}{2}, D + \binom{n'-D-1}{2}\}$ it is easy to see that if $f(n', 3, D) = D + \binom{n'-D-1}{2}$ then $D \geq 2n'/3 + 1 = 2n/3 - 1$. Thus, $2n/3 - 1 \leq D < n/2$, a contradiction since $n \geq 4r = 12$.

If $r \geq 4$ then (15) implies that

$$D < \frac{n}{r-1} = \frac{n'}{r-1} + \frac{r}{r-1}.$$

Since $n' \geq 3r$, Lemma 11 implies that $f(n', r, D') = \binom{n'/r+1}{2}$ and so we are in Case 1, which we have already dealt with.

4 Perfect matchings and cycles in dense graphs

4.1 Perfect matchings in dense graphs

In this section we establish the density threshold that ensures every graph G on an even number n of vertices and of minimum degree $\delta(G) \geq d$ contains a perfect matching. Note that we only consider values of d such that $1 \le d < n/2$, since if $\delta(G) \ge n/2$ then G has a perfect matching, regardless of the edge density.

For a positive even n and an integer $0 \le d < n/2$, let A, B and C be disjoint sets with $|A| = d+1$, $|B| = d$, $|C| = n - 2d - 1$. Let $H = H(n,d)$ be the graph with the vertex set $A \cup B \cup C$ such that $H[B \cup C] = K_{n-d-1}$, and each vertex in A is adjacent to each vertex in B and to no vertex in C. So H does not contain a perfect matching. Let

$$h(n,d) := |E(H(n,d))| = \binom{n-d-1}{2} + d(d+1). \tag{16}$$

Note that for a fixed even n, $h(n,d)$ decreases with d in the interval $[0, n/3 - 5/6]$ and increases with d in $[n/3 - 5/6, 0.5n - 1]$.

Proposition 13 *For an even positive n and integer $1 \le d < n/2$, let $f(2,n,d)$ denote the maximum integer c such that some n-vertex graph with minimum degree at least d and at least c edges has no perfect matching. Then*

$$f(2,n,d) = \max\{h(n,d), h(n, 0.5n-1)\}. \tag{17}$$

Proof. The examples of $H(n,d)$ show that $f(2,n,d) \ge \max\{h(n,d), h(n,0.5n-1)\}$. If G is an n-vertex graph with $\delta(G) \ge n/2$, then G has a perfect matching. Thus, it is enough to prove that if an n-vertex graph G with $d \le \delta(G) < n/2$ has no perfect matching, then

$$e(G) \le h(n,d') \text{ for some } d \le d' < 0.5n. \tag{18}$$

So, let G be an n-vertex graph with $\delta(G) \ge d$ and no perfect matching such that the number of edges in G is maximised. By Tutte's Theorem, there is $S \subset V(G)$ such that $G - S$ has more than $|S|$ components of odd order. We may choose such S as large as possible; then every component of $G - S$ has an odd order. Let $s := |S|$ and $W_1, \ldots, W_t$ be the vertex sets of the components of $G - S$. We may order them so that $|W_1| \le |W_2| \le \ldots \le |W_t|$.

If $s = 0$ then as n is even, G consists of at least 2 components. But then since $\delta(G) \ge d$ we have that $e(G) \le e(K_{d+1}) + e(K_{n-d-1}) \le h(n,d)$ and so (18) holds. So we may assume that $s \ge 1$. Since n is even and $s \ge 1$ we have that $t \ge s + 2 \ge 3$. By the maximality of $e(G)$, each of $G[W_1], \ldots, G[W_t], G[S]$ is a complete graph, and every $v \in S$ is adjacent to every $w \in V(G) - S$. Let G' be obtained from G by replacing $W_1, \ldots, W_t$ with $t-1$ single vertices and a copy of $K_{n-s-(t-1)}$, each of which is completely joined to S. Then
(a) G' has no perfect matching, since $G' - S$ has t odd components;
(b) $e(G') \ge e(G)$, since $\binom{n-s-(t-1)}{2} \ge \sum_{i=1}^{t} \binom{|W_i|}{2}$;
(c) $\delta(G') \ge s$;
(d) if $t = s + 2$, then $G' = H(n,s)$, otherwise $e(G') < e(H(n,s))$.

Thus if $s \ge d$, then (18) is proved. So, suppose $d > s$. Since $\delta(G) \ge d$, we have $d \le s + (|W_1| - 1) \le \ldots \le s + (|W_t| - 1)$. Construct G'' from G as follows:
(i) delete all edges in $G[W_1]$;
(ii) move a set W'' of $|W_1| - 1$ vertices from W_2 to S and connect them by edges to every vertex in G. Denote $S'' := S \cup W''$.

Observe that G'' has no perfect matching, since $|S''| = s + |W_1| - 1$ and $G'' - S''$ has $|W_1| + (t-1)$ components of odd order. Furthermore $d_{G''}(v) \ge d_G(v)$ for every $v \in V(G) = V(G'')$ and $d_{G''}(w) > d_G(w)$ for every $w \in W_3$. It follows that $e(G'') > e(G)$ and $\delta(G'') \ge d$, a contradiction to the choice of G. □

4.2 Proof of Theorem 6

Suppose for a contradiction that the result is false. Then there is a graph G on $n \geq 7k/2$ vertices with

$$e(G) > (2k-1)(n-k) \tag{19}$$

but such that G does not contain k vertex-disjoint cycles.

Let $v_1 \in V(G)$ such that $d_G(v_1) = \delta(G)$. If $\delta(G) \geq 2k$ then the Corrádi-Hajnal theorem implies that G contains k vertex-disjoint cycles, a contradiction. So $d_G(v_1) \leq 2k-1$. Let $v_2 \in V(G - v_1)$ such that $d_{G-v_1}(v_2) = \delta(G - v_1)$. Again we may assume that $d_{G-v_1}(v_2) \leq 2k-1$. Repeating this argument we obtain distinct vertices $v_1, \dots, v_{n-3k}$ so that $G' := G - \{v_1, \dots, v_{n-3k}\}$ is a graph on $3k$ vertices with $\delta(G') \leq 2k-1$. The choice of $v_1, \dots, v_{n-3k}$ and (19) implies that

$$e(G') > (2k-1)(n-k) - (2k-1)(n-3k) = 2k(2k-1). \tag{20}$$

If $k = 1$ this implies that $|G'| = 3$ and $e(G') > 2$, a contradiction. When $k = 2$ we have that $|G'| = 6$ and $e(G') > 12$. But then G' contains two vertex-disjoint triangles, a contradiction. Thus, $k \geq 3$.

Consider the case when $\delta(G') \geq k-1 \geq 2$. It is easy to check that $g(3k, 3, k-1) = \binom{3k}{2} - \binom{k+1}{2} = 2k(2k-1)$. Since G' does not contain a perfect K_3-packing, Theorem 4 implies that

$$e(G') \leq 2k(2k-1),$$

a contradiction to (20), as desired.

Now consider the case when $s := \delta(G') \leq k-2$. For $2 \leq s \leq k-2$, $g(3k, 3, s) = \binom{3k}{2} - \binom{s}{2} - (3k-1-s)$. Since G' does not contain a perfect K_3-packing, Theorem 4 implies that

$$e(G') \leq \binom{3k}{2} - \binom{s}{2} - (3k-1-s). \tag{21}$$

If $s = 0, 1$ then it is easy to see that (21) also holds. (In this case, $\binom{s}{2} := s(s-1)/2 = 0$.)

If k is even then, since $\delta(G') = s$, v_{n-3k} must have at most $s+1$ neighbours in $V(G')$, v_{n-3k-1} has at most $s+2$ neighbours in $V(G') \cup \{v_{n-3k}\}$ and so on until $v_{n-7k/2+1}$ has at most $s + k/2$ neighbours in $V(G') \cup \{v_{n-3k}, \dots, v_{n-7k/2+2}\}$. Hence, (21) implies that

$$e(G) \leq \binom{3k}{2} - \binom{s}{2} - (3k-1-s) + (s+1) + \dots + (s+k/2) + (n-7k/2)(2k-1).$$

Comparing with (19), after rearranging and simplifying we get

$$\frac{5k}{2}(2k-1) < \frac{3k(3k-1)}{2} - \frac{s(s-1)}{2} - 3k + 1 + s + \frac{sk}{2} + \frac{k^2}{8} + \frac{k}{4}.$$

This implies that

$$\frac{3k^2}{4} + \frac{7k}{2} < -s^2 + s(3+k) + 2. \tag{22}$$

Note that $-s^2 + s(3+k) + 2$ is maximised when $s = (3+k)/2$. So (22) implies that

$$\frac{3k^2}{4} + \frac{7k}{2} < -\frac{(3+k)^2}{4} + \frac{(3+k)^2}{2} + 2,$$

and therefore

$$2k^2 + 8k < 17,$$

a contradiction as $k \geq 3$. The case when k is odd is similar. □

5 The extremal examples for Conjecture 2

Proposition 14 *Suppose that $n, r, k \in \mathbb{N}$ such that $r \geq 2$ divides n and $1 \leq k < n/r$. Then there exists a graph G on n vertices whose degree sequence $d_1 \leq \cdots \leq d_n$ satisfies*

- $d_i = (r-2)n/r + k - 1$ *for all* $1 \leq i \leq k$;
- $d_i = (r-1)n/r$ *for all* $k+1 \leq i \leq (r-2)n/r + k$;
- $d_i = n - k - 1$ *for all* $(r-2)n/r + k + 1 \leq i \leq n - k + 1$;
- $d_i = n - 1$ *for all* $n - k + 2 \leq i \leq n$,

but such that G does not contain a perfect K_r-packing.

Proof. Let G' denote the complete $(r-2)$-partite graph whose vertex classes $V_1, \dots, V_{r-2}$ each have size n/r. Obtain G from G' by adding the following vertices and edges: Add a set V_{r-1} of $2n/r - 2k + 1$ vertices to G', a set V_r of $k-1$ vertices and a set V_0 of k vertices. Add all edges from $V_0 \cup V_{r-1} \cup V_r$ to $V_1 \cup \cdots \cup V_{r-2}$. Further, add all edges with both endpoints in $V_{r-1} \cup V_r$. Add all possible edges between V_0 and V_r.

So V_0 is an independent set, and there are no edges between V_0 and V_{r-1}. This implies that any copy of K_r in G containing a vertex from V_0 must also contain at least one vertex from V_r. But since $|V_0| > |V_r|$ this implies that G does not contain a perfect K_r-packing. Furthermore, G has our desired degree sequence. □

Notice that the graphs G considered in Proposition 14 satisfy (β) from Conjecture 2 and only fail to satisfy (α) in the case when $i = k$ (and in this case $d_k = (r-2)n/r + k - 1$).

Let $n, r \in \mathbb{N}$ such that r divides n. Denote by $T^*(n, r)$ the complete r-partite graph on n vertices with $r-2$ vertex classes of size n/r, one vertex class of size $n/r - 1$ and one vertex class of size $n/r + 1$. Then $T^*(n, r)$ does not contain a perfect K_r-packing. Furthermore, $T^*(n, r)$ satisfies (α) but condition (β) fails; we have that $d_{n/r+1} = (r-1)n/r - 1$ here. Thus, together $T^*(n, r)$ and Proposition 14 show that, if true, Conjecture 2 is essentially best possible.

6 Some special cases of Conjecture 2

The following is a simple consequence of Chvátal's theorem.

Theorem 15 (Chvátal [3]) *Suppose that G is a graph on $n \geq 2$ vertices and the degrees of the graph are $d_1 \leq \cdots \leq d_n$. If*

$$d_i \geq i \quad \text{or} \quad d_{n-i+1} \geq n - i \quad \text{for all} \quad 1 \leq i \leq n/2$$

then G contains a Hamilton path.

It is easy to see that Theorem 15 implies Conjecture 2 in the case when $r = 2$. We now give a simple proof of Conjecture 2 in the case when G is K_{r+1}-free.

Theorem 16 *Let $n, r \in \mathbb{N}$ such that $r \geq 2$ divides n. Suppose that G is a graph on n vertices with degree sequence $d_1 \leq \cdots \leq d_n$ such that:*

- $d_i \geq (r-2)n/r + i$ *for all* $i < n/r$;
- $d_{n/r+1} \geq (r-1)n/r$.

Further suppose that no vertex $x \in V(G)$ *of degree less than* $(r-1)n/r$ *lies in a copy of* K_{r+1}. *Then* G *contains a perfect* K_r*-packing.*

Proof. We prove the theorem by induction on n. In the case when $n = r$ then $d_{n/r+1} = d_2 \geq (r-1)r/r = r-1$. This implies that every vertex in G has degree $r-1$. Hence $G = K_r$ as desired. So suppose that $n > r$ and the result holds for smaller values of n. Let $x_1 \in V(G)$ such that $d_G(x_1) = d_1 \geq (r-2)n/r + 1$. If $d_G(x_1) \geq (r-1)n/r$ then $\delta(G) \geq (r-1)n/r$. Thus G contains a perfect K_r-packing by the Hajnal-Szemerédi theorem. So we may assume that $(r-2)n/r+1 \leq d_G(x_1) < (r-1)n/r$. In particular, x_1 does not lie in a copy of K_{r+1}. We first find a copy of K_r containing x_1. If $r = 2$, x_1 has a neighbour and so we have our desired copy of K_2. So assume that $r \geq 3$. Certainly $N_G(x_1)$ contains a vertex x_2 such that $d_G(x_2) \geq (r-1)n/r$. Thus, $|N_G(x_1) \cap N_G(x_2)| \geq (r-3)n/r + 1 > 0$. So if $r = 3$ we obtain our desired copy of K_r. Otherwise, we can find a vertex $x_3 \in N_G(x_1) \cap N_G(x_2)$ such that $d_G(x_3) \geq (r-1)n/r$. We can repeat this argument until we have obtained vertices $x_1, \dots, x_r$ that together form a copy K'_r of K_r.

Let $G' := G - V(K'_r)$ and set $n' := n - r = |V(G')|$. Since G does not contain a copy of K_{r+1} containing x_1, every vertex $x \in V(G) \backslash V(K'_r)$ sends at most $r-1$ edges to K'_r in G. Thus, $d_{G'}(x) \geq d_G(x) - (r-1)$ for all $x \in V(G')$. So if $d_G(x) \geq (r-1)n/r$ then $d_{G'}(x) \geq (r-1)n/r - (r-1) = (r-1)n'/r$ for all $x \in V(G')$. If a vertex $y \in V(G')$ does not lie in a copy of K_{r+1} in G then clearly y does not lie in a copy of K_{r+1} in G'. This means that no vertex $y \in V(G')$ of degree less than $(r-1)n'/r$ lies in a copy of K_{r+1}.

Let $d'_1 \leq \dots \leq d'_{n'}$ denote the degree sequence of G'. It is easy to check that $d'_i \geq (r-2)n'/r + i$ for all $i < n'/r$ and that $d'_{n'/r+1} \geq (r-1)n'/r$. Indeed, since $x_1 \in V(K'_r)$ where $d_G(x_1) = d_1$, we have that $d'_i \geq d_{i+1} - (r-1)$ for all $1 \leq i \leq n'$. Thus, for all $1 \leq i < n'/r = n/r - 1$, $d'_i \geq d_{i+1} - (r-1) \geq (r-2)n/r + (i+1) - (r-1) = (r-2)n'/r + i$. Similarly, $d'_{n'/r+1} = d'_{n/r} \geq d_{n/r+1} - (r-1) \geq (r-1)n/r - (r-1) = (r-1)n'/r$. Hence, by induction G' contains a perfect K_r-packing. Together with K'_r this gives us our desired perfect K_r-packing in G. □

7 Degree sequences forcing a copy of K_r in a graph

Proof of Theorem 3. Consider any $x_1 \in V(G)$ such that $d_G(x_1) \geq (r-1)n/r$. Since $d_{n/r} \geq (r-1)n/r$ we can greedily select vertices $x_2, \dots, x_{r-1}$ such that

- $x_1, \dots, x_{r-1}$ induce a copy of K_{r-1} in G;
- $d_G(x_i) \geq (r-1)n/r$ for all $1 \leq i \leq r-1$.

Note that since G is K_{r+1}-free, $\cap_{i=1}^{r-1} N_G(x_i)$ is an independent set. The choice of $x_1, \dots, x_{r-1}$ implies that $|\cap_{i=1}^{r-1} N_G(x_i)| \geq n/r$. Let V_1 denote a subset of $\cap_{i=1}^{r-1} N_G(x_i)$ of size n/r. Thus V_1 contains a vertex x_1^1 of degree at least $(r-1)n/r$.

As before we can find vertices $x_2^1, \dots, x_{r-1}^1$ such that

- $x_1^1, \dots, x_{r-1}^1$ induce a copy of K_{r-1} in G;
- $d_G(x_i^1) \geq (r-1)n/r$ for all $1 \leq i \leq r-1$.

So $\cap_{i=1}^{r-1} N_G(x_i^1)$ is an independent set of size at least n/r. Let V_2 denote a subset of $\cap_{i=1}^{r-1} N_G(x_i^1)$ of size n/r. Note that $N_G(x_1^1) \cap V_1 = \emptyset$ since $x_1^1 \in V_1$ and V_1 is an independent set. Thus as $V_2 \subseteq N_G(x_1^1)$, $V_1 \cap V_2 = \emptyset$.

Our aim is to find disjoint sets $V_1, \dots, V_r \subseteq V(G)$ of size n/r and vertices $x_1^1, \dots, x_{r-1}^1, x_1^2, \dots, x_{r-1}^2, \dots, x_1^{r-1}, \dots, x_{r-1}^{r-1}$ with the following properties:

- $G[V_j]$ is an independent set for all $1 \leq j \leq r$;
- Given any $1 \leq j \leq r-1$, $x_k^j \in V_k$ for each $1 \leq k \leq j$;
- $d_G(x_k^j) \geq (r-1)n/r$ for all $1 \leq j \leq r-1$ and $1 \leq k \leq r-1$;
- $x_1^j, \dots, x_{r-1}^j$ induce a copy of K_{r-1} in G for all $1 \leq j \leq r-1$.

Clearly finding such a partition $V_1, \dots, V_r$ of $V(G)$ implies that $G \subseteq T(n,r)$.

Suppose that for some $1 < j < r$ we have defined sets $V_1, \dots, V_j$ and vertices $x_1^1, \dots, x_{r-1}^1, \dots, x_1^{j-1}, \dots, x_{r-1}^{j-1}$ with our desired properties. Since $d_{n/r} \geq (r-1)n/r$ and $V_1, \dots, V_j$ are independent sets of size n/r we can choose vertices $x_1^j, \dots, x_j^j$ such that for all $1 \leq k \leq j$,

- $x_k^j \in V_k$ and $d_G(x_k^j) \geq (r-1)n/r$.

This degree condition, together with the fact that $x_1^j, \dots, x_j^j$ lie in different vertex classes, implies that these vertices form a copy of K_j in G. We now greedily select further vertices $x_{j+1}^j, \dots, x_{r-1}^j$ such that

- $x_1^j, \dots, x_{r-1}^j$ induce a copy of K_{r-1} in G;
- $d_G(x_k^j) \geq (r-1)n/r$ for all $j+1 \leq k \leq r-1$.

So $\cap_{i=1}^{r-1} N_G(x_i^j)$ is an independent set of size at least n/r. Let V_{j+1} denote a subset of $\cap_{i=1}^{r-1} N_G(x_i^j)$ of size n/r. Note that, for each $1 \leq k \leq j$, $N_G(x_k^j) \cap V_k = \emptyset$ since $x_k^j \in V_k$ and V_k is an independent set. Thus as $V_{j+1} \subseteq N_G(x_k^j)$ for each $1 \leq k \leq j$, V_{j+1} is disjoint from $V_1 \cup \dots \cup V_j$.

Repeating this argument we obtain our desired sets $V_1, \dots, V_r \subseteq V(G)$ and vertices $x_1^1, \dots, x_{r-1}^1, x_1^2, \dots, x_{r-1}^2, \dots, x_1^{r-1}, \dots, x_{r-1}^{r-1}$. □

The following consequence of Theorem 3 gives a condition on the degree sequence of a graph G that forces G to contain a copy of K_{r+1}.

Corollary 17 *Suppose that $n, r \in \mathbb{N}$ where $n \geq r \geq 2$. Let $n = mr + s$ where $m, s \in \mathbb{N}$ such that $0 \leq s \leq r-1$. Let G be a graph on n vertices whose degree sequence $d_1 \leq \dots \leq d_n$ satisfies the following conditions:*

(a) $d_{m+s} \geq n - m$;

(b) $d_n \geq n - m + 1$.

Then G contains a copy of K_{r+1}.

Proof. In the case when $s = 0$, (a) gives that $d_{n/r} \geq (r-1)n/r$ and (b) gives that $d_n \geq (r-1)n/r+1$. This latter condition implies that $G \not\subseteq T(n,r)$. Hence, Theorem 3 implies that G contains a copy of K_{r+1}.

Now consider the case when $s > 0$. Consider s distinct vertices $x_1, \ldots, x_s$ where $d_G(x_i) = d_i$ for each $1 \leq i \leq s$. Set $G' := G \backslash \{x_1, \ldots, x_s\}$. So $n' := n - s = mr = |V(G')|$. Let $d'_1 \leq \cdots \leq d'_{n'}$ denote the degree sequence of G'. By choice of the x_i,

$$d'_j \geq d_{j+s} - s \quad \text{for all} \quad 1 \leq j \leq n'.$$

In particular,

$$d'_{n'/r} = d'_m \geq d_{m+s} - s \geq n - m - s = (r-1)n'/r$$

and

$$d'_{n'} = d'_{n-s} \geq d_n - s \geq n - m - s + 1 = (r-1)n'/r + 1.$$

Thus, Theorem 3 implies that G' and therefore G contains a copy of K_{r+1}. □

In the case when r divides n, Corollary 17 is best possible in the sense that neither condition can be lowered here. Indeed, the Turán graph $T(n,r)$ shows that we cannot omit condition (b). Further, $T^*(n,r)$ does not contain a copy of K_{r+1} but satisfies (b) and only just fails to satisfy (a). (Recall that the graph $T^*(n,r)$ was defined in Section 5.)

8 Possible extensions of Conjecture 2

If one can prove Conjecture 2, it seems likely it can be used to prove the following conjecture.

Conjecture 18 *Suppose $\gamma > 0$ and H is a graph with $\chi(H) = r$. Then there exists an integer $n_0 = n_0(\gamma, H)$ such that the following holds. If G is a graph whose order $n \geq n_0$ is divisible by $|H|$, and whose degree sequence $d_1 \leq \cdots \leq d_n$ satisfies*

- $d_i \geq (r-2)n/r + i + \gamma n$ *for all* $i < n/r$,

then G contains a perfect H-packing.

We also suspect that the following 'Chvátal-type' degree sequence condition forces a graph to contain a perfect K_r-packing.

Question 19 *Let $n, r \in \mathbb{N}$ such that $r \geq 2$ divides n. Suppose that G is a graph on n vertices with degree sequence $d_1 \leq \cdots \leq d_n$ such that for all $i \leq n/r$:*

- $d_i \geq (r-2)n/r + i$ *or* $d_{n-i(r-1)+1} \geq n - i$.

Does this condition imply that G contains a perfect K_r-packing?

Note that Theorem 15 answers this question in the affirmative when $r = 2$. The following example shows that we cannot have a lower value in the second part of the condition in Question 19.

Proposition 20 *Suppose that $n, r, k \in \mathbb{N}$ such that $r \geq 2$ divides n and $1 \leq k \leq n/r$. Then there exists a graph G on n vertices whose degree sequence $d_1 \leq \cdots \leq d_n$ satisfies*

- $d_{n-i(r-1)+1} \geq n - i$ *for all* $i \in [n/r] \backslash \{k\}$;

- $d_{n-k(r-1)+1} = n-k-1$,

but such that G does not contain a perfect K_r-packing.

Proof. Let G be the graph on n vertices with vertex classes V_1, V_2 an V_3 of sizes k, $(r-1)k-1$ and $n-rk+1$ respectively and with the following edges: There are all possible edges between V_1 and V_2 and between V_2 and V_3. Further add all possible edges in V_2 and all edges in V_3. Thus, V_1 is an independent set and there are no edges between V_1 and V_3.

The degree sequence of G is

$$\underbrace{(r-1)k-1,\dots,(r-1)k-1}_{k\text{ times}},\underbrace{n-k-1,\dots,n-k-1}_{n-rk+1\text{ times}},\underbrace{n-1,\dots,n-1}_{(r-1)k-1\text{ times}}.$$

Hence G satisfies our desired degree sequence condition. Every copy K'_r or K_r in G that contains a vertex from V_1 must contain $r-1$ vertices from V_2. But since $|V_1|(r-1) > |V_2|$ this implies that G does not contain a perfect K_r-packing. □

The rth power of a Hamilton cycle C is obtained from C by adding an edge between every pair of vertices of distance at most r on C. Seymour [13] conjectured the following strengthening of Dirac's theorem.

Conjecture 21 (Pósa-Seymour, see [13]) *Let G be a graph on n vertices. If $\delta(G) \geq \frac{r}{r+1}n$ then G contains the rth power of a Hamilton cycle.*

Pósa (see [4]) had earlier proposed the conjecture in the case of the square of a Hamilton cycle (that is, when $r=2$). Komlós, Sárközy and Szemerédi [9] proved Conjecture 21 for sufficiently large graphs. More recently, Chau, DeBiasio and Kierstead [2] proved Pósa's conjecture for graphs of order at least 2×10^8.

In the case when $r+1$ divides $|G|$, a necessary condition for a graph G to contain the rth power of a Hamilton cycle is that G contains a perfect K_{r+1}-packing. Further, notice that the minimum degree condition in Conjecture 21 is the same as the condition in the Hajnal-Szemerédi theorem with respect to perfect K_{r+1}-packings. Thus an obvious question is whether the condition in Conjecture 2 forces a graph to contain the $(r-1)$th power of a Hamilton cycle. Interestingly though, when $r=3$, this is not the case.

Proposition 22 *Suppose that $C, n \in \mathbb{N}$ such that $C \ll n$ and 3 divides n. Then there exists a graph G whose degree sequence $d_1 \leq \dots \leq d_n$ satisfies*

$$d_i \geq \frac{n}{3} + C + i \quad \textit{for all } 1 \leq i \leq \frac{n}{3}$$

but such that G does not contain the square of a Hamilton cycle.

Proof. Choose $C, K, n \in \mathbb{N}$ so that $C \ll K \ll n$. Let G denote the graph on n vertices consisting of three vertex classes $V_1 = \{v\}$, V_2 and V_3 where $|V_2| = n/3 + C + 1$ and $|V_3| = 2n/3 - C - 2$ which contains the following edges:

- All edges from v to V_2;
- All edges between V_2 and V_3 and all possible edges in V_3;

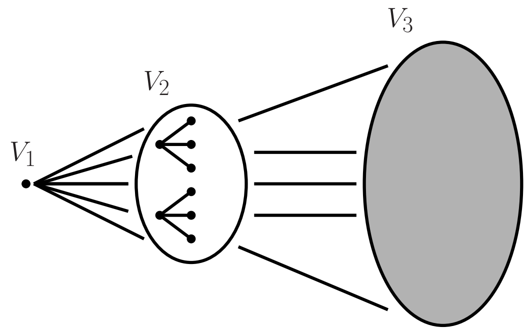

Figure 1: The example from Proposition 22 in the case when $K = 2$ and $|V_2| = 8$.

- There are K vertex-disjoint stars in V_2, each of size $\lfloor |V_2|/K \rfloor$, $\lceil |V_2|/K \rceil$, which cover all of V_2 (see Figure 1).

Let $d_1 \leq \cdots \leq d_n$ denote the degree sequence of G. There are $n/3 + C - K + 1 \leq n/3 - 2C - 1$ vertices in V_2 of degree $2n/3 - C$. Since $C \ll K \ll n$, the remaining K vertices in V_2 have degree at least $2n/3 - C - 2 + \lfloor |V_2|/K \rfloor \geq 2n/3 + C + 1$. Since $d_G(v) = n/3 + C + 1$ and $d_G(x) = n - 2$ for all $x \in V_3$, we have that $d_i \geq \frac{n}{3} + C + i$ for all $1 \leq i \leq \frac{n}{3}$.

A necessary condition for a graph G to contain the square of a Hamilton cycle is that, for every $x \in V(G)$, $G[N(x)]$ contains a path of length 3. Note that $N(v) = V_2$ and $G[V_2]$ does not contain a path of length 3. So G does not contain the square of a Hamilton cycle. $\square$

Notice that we can set $C = o(\sqrt{n})$ in Proposition 22. We finish by raising the following question.

Question 23 *What can be said about degree sequence conditions which force a graph to contain the rth power of a Hamilton cycle? In particular, can one establish a degree sequence condition that ensures a graph G on n vertices contains the rth power of a Hamilton cycle and which allows for "many" vertices of G to have degree "much less" than $rn/(r+1)$?*

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Appendix

Here we give proofs of Proposition 9 and Lemma 11. The following fact will be used in both of these proofs.

Fact 24 *Fix $n, r \in \mathbb{N}$ such that $r \geq 3$ and r divides $n \geq 2r$. Define*

$$h(x) := x + \frac{(n-x-1)^2}{2(r-2)} - \frac{1}{2}(n-x-1).$$

Then $h(x)$ is a decreasing function for $x \in [0, n/(r-1)]$. Moreover, if $n \geq 3r$ then $h(x)$ is a decreasing function for $x \in [0, (n+r)/(r-1)]$.

Proof. Notice that

$$h'(x) = \frac{3}{2} - \frac{(n-x-1)}{r-2} = \frac{x}{r-2} + \frac{1-n}{r-2} + \frac{3}{2}.$$

So for $x \le n/(r-1)$,

$$h'(x) \le \frac{n}{(r-1)(r-2)} + \frac{1-n}{r-2} + \frac{3}{2} = -\frac{n}{r-1} + \frac{1}{r-2} + \frac{3}{2}.$$

Note that $3(r-1)/2 + (r-1)/(r-2) < n$ since $n \ge 2r$ and $r \ge 3$. Thus,

$$h'(x) \le -\frac{n}{r-1} + \frac{1}{r-2} + \frac{3}{2} < 0.$$

If $x \le (n+r)/(r-1)$ then

$$h'(x) \le \frac{n+r}{(r-1)(r-2)} + \frac{1-n}{r-2} + \frac{3}{2} = -\frac{n}{r-1} + \frac{1}{r-2} + \frac{r}{(r-1)(r-2)} + \frac{3}{2}.$$

If $n \ge 3r$ then $n > 3r/2 + 4$. So $n > 3(r-1)/2 + (2r-1)/(r-2)$. Thus,

$$h'(x) \le -\frac{n}{r-1} + \frac{1}{r-2} + \frac{r}{(r-1)(r-2)} + \frac{3}{2} < 0,$$

as desired. $\square$

Proof of Proposition 9. We need to show that, for all $D \in \mathbb{N}$ such that $n/r \le D \le n/(r-1)$,

$$\frac{n^2}{2r^2} + \frac{n}{2r} = \binom{n/r+1}{2} \le D + e(\overline{T}(n-D-1, r-2)).$$

Since $D \le n/(r-1)$, Facts 10 and 24 imply that

$$\begin{aligned} D + e(\overline{T}(n-D-1, r-2)) &\ge D + \frac{(n-D-1)^2}{2(r-2)} - \frac{(n-D-1)}{2} \\ &\ge \frac{n}{r-1} + \frac{1}{2(r-2)}\left[\frac{(r-2)}{r-1}n - 1\right]^2 - \frac{1}{2}\left[\frac{(r-2)}{r-1}n - 1\right] \\ &\ge \frac{(r-2)}{2(r-1)^2}n^2 - \frac{(r-2)}{2(r-1)}n. \end{aligned}$$

Thus, it suffices to show that

$$\frac{(r-2)}{2(r-1)^2}n - \frac{r-2}{2(r-1)} \ge \frac{n}{2r^2} + \frac{1}{2r}. \tag{23}$$

Notice that

$$\frac{r-2}{2(r-1)^2} - \frac{1}{2r^2} = \frac{(r-2)r^2 - (r-1)^2}{2r^2(r-1)^2} = \frac{r^3 - 3r^2 + 2r - 1}{2r^2(r-1)^2} \tag{24}$$

and

$$\frac{r-2}{2(r-1)} + \frac{1}{2r} = \frac{r^2 - r - 1}{2r(r-1)}.$$

Since $n \ge 2r$, (23) implies that it suffices to show that

$$\frac{r^3 - 3r^2 + 2r - 1}{r(r-1)^2} - \frac{r^2 - r - 1}{2r(r-1)} \ge 0. \tag{25}$$

Note that $r^3 \ge 4r^2 - 4r + 3$ as $r \ge 3$. Thus, $2(r^3 - 3r^2 + 2r - 1) \ge (r^2 - r - 1)(r - 1)$. So indeed (25) is satisfied, as desired. $\square$

Proof of Lemma 11. We need to show that, for all $D \in \mathbb{N}$ such that $n/r \leq D < (n+r)/(r-1)$,

$$\frac{n^2}{2r^2} + \frac{n}{2r} = \binom{n/r+1}{2} \leq D + e(\overline{T}(n-D-1, r-2)).$$

Since $D < (n+r)/(r-1)$ we have that $D \leq n/(r-1)+1$. So Facts 10 and 24 imply that

$$\begin{aligned} D + e(\overline{T}(n-D-1, r-2)) &\geq D + \frac{(n-D-1)^2}{2(r-2)} - \frac{(n-D-1)}{2} \\ &\geq \frac{n}{r-1} + 1 + \frac{1}{2(r-2)}\left[\frac{(r-2)}{r-1}n - 2\right]^2 - \frac{1}{2}\left[\frac{(r-2)}{r-1}n - 2\right] \\ &\geq \frac{(r-2)}{2(r-1)^2}n^2 - \frac{(r-2)}{2(r-1)}n - \frac{n}{r-1}. \end{aligned}$$

Thus, it suffices to show that

$$\frac{(r-2)}{2(r-1)^2}n - \frac{(r-2)}{2(r-1)} - \frac{1}{r-1} \geq \frac{n}{2r^2} + \frac{1}{2r}. \tag{26}$$

Notice that

$$\frac{r-2}{2(r-1)} + \frac{1}{r-1} + \frac{1}{2r} = \frac{r^2+r-1}{2r(r-1)}.$$

Since $n \geq 3r$, (24) and (26) imply that it suffices to show that

$$\frac{3(r^3-3r^2+2r-1)}{2r(r-1)^2} - \frac{r^2+r-1}{2r(r-1)} \geq 0. \tag{27}$$

Note that $2r^3 - 9r^2 + 8r - 4 \geq 0$ as $r \geq 4$. Thus, $3(r^3-3r^2+2r-1) \geq (r^2+r-1)(r-1)$. So indeed (27) is satisfied, as desired. $\square$