On 2-detour subgraphs of the hypercube

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Abstract. A spanning subgraph H of a graph G is a 2-detour subgraph of G if for each $x, y \in V(G)$, $d_H(x,y) \leq d_G(x,y)+2$. We prove a conjecture of Erdős, Hamburger, Pippert, and Weakley by showing that for some positive constant c and every n, each 2-detour subgraph of the n-dimensional hypercube Q_n has at least $c \log_2 n \cdot 2^n$ edges.

Key words. graph, extremal subgraph, hypercube, detour subgraph, additive spanner.

1. Introduction

Let $d_G(x,y)$ denote the distance between vertices x and y in the graph G. A spanning subgraph H of a graph G is a k-*additive spanner* if for each pair (u,v) of vertices of G, $d_H(u,v) \leq d_G(u,v)+k$. Studying k-additive spanners was motivated by a number of problems in communication networks, broadcasting, routing, etc., see [7,9,10]. Additive spanners were studied in [1–6,8]. Sometimes, k-additive spanners of the n-dimensional hypercube Q_n are also called k-*detour subgraphs*.

Let $f_k(n)$ denote the minimum number of the edges of a k-detour subgraph of Q_n. Since Q_n is a bipartite graph, it is enough to consider $f_k(n)$ only for even k. Erdős, Hamburger, Pippert, and Weakley [3] studied 2-detour subgraphs of Q_n. They constructed a 2-detour subgraph of Q_n with at most $3/(2\sqrt{2})\sqrt{n}2^n$ edges. Since each k-detour subgraph of Q_n is connected, it has at least 2^n-1 edges. The best known lower bound on $f_2(n)$ is due to Hamburger, Kostochka and Sidorenko [4], who proved that

$$f_2(n) \geq (3.000013 - o(1)) \cdot 2^n. \tag{1}$$

Arizumi, Hamburger and Kostochka [1] proved that $f_4(n) < (3+o(1)) \cdot 2^n$ for large n, that is, that the average degree of some 4-detour subgraphs of Q_n is less than 7 for large n. In contrast, it was conjectured in [3] that the function $f_2(n) \cdot 2^{-n}$ is unbounded. The main result of our note is a proof of the conjecture.

Theorem 1. *If $H = H_n$ is a 2-detour subgraph of the n-dimensional hypercube Q_n, then*

$$2\,e(H) > 10^{-6}\,2^n\,\log_2 n. \tag{2}$$

* *Present address:* Research supported in part by NSF grants DMS-0302804, DMS-0603769 and DMS-0600303, UIUC Campus Reseach Board #06139 and #07048, and OTKA 049398.
† *Present address:* Research supported in part by NSF grants DMS-0400498 and DMS-0650784, and grant 06-01-00694 of the Russian Foundation for Basic Research.

The idea of the proof is the following: If H is a 2-detour subgraph of Q_n with $10^{-6}\ 2^n\ \log_2 n$ edges, then we claim several properties of H. First, that 'most' vertices of H have a 'low' degree, second that for some m between 'most' pairs of low vertices at distance m in Q_n the shortest path in H has length $m+2$, additionally the first and the last edges in these paths are 'parallel'. After claiming these properties, a simple counting argument provides the result. The difficulty arises from that we do not know the precise value of m, we could only prove its existence.

We introduce the necessary notation in the next section and prove a series of preparatory statements in Section 3. Theorem 1 will be proved in Section 4.

2. Notation

Let $\ell = \lfloor \log_2 n \rfloor$. Let $H = H_n$ be a 2-detour subgraph of Q_n, with

$$e(H) = t \cdot 2^n \tag{3}$$

for some t. By the definition of a 2-detour subgraph, H has no isolated vertices. By $d(v)$ we denote the degree of v in H, but by $d(u,v)$ we denote the distance between u and v in Q_n. For $i = 0, \dots, \ell$, let

$$A_i := \{x \in V(H) : 2^i \le d(x) < 2^{i+1}\}, \quad \widetilde{A}_i = \cup_{j=i}^{\ell} A_j, \tag{4}$$

$$a_i := |A_i| \quad \text{and} \quad \widetilde{a}_i = |\widetilde{A}_i|. \tag{5}$$

By the definition of a_i,

$$\sum_{i=0}^{\ell} a_i \cdot 2^i \le 2e(H) < \sum_{i=0}^{\ell} a_i \cdot 2^{i+1}. \tag{6}$$

We say that a vertex x is *low* if $d(x) \le 2000t$. The set of low vertices is

$$\mathrm{L} := \{x : \ d(x) \le 2000t\}. \tag{7}$$

It is convenient to consider the vertices in a vector form, where they are $\{0,1\}$-vectors of length n. The coordinates of a vertex sometimes are referred to as *directions*. For each two vertices $u, v \in V(H)$, we define

$$\mathrm{co}(u,v) := \{\text{the set of coordinates where } u \text{ and } v \text{ differ}\}. \tag{8}$$

Note that $d(u,v) = |\mathrm{co}(u,v)|$ and that each (u,v)-path uses the edges of each direction in $\mathrm{co}(u,v)$ an odd number of times. For a vertex u and a positive integer r, let

$$\mathrm{Dir}(u,r) := \{\mathrm{co}(u,v) : \ v \in N(u)\} \cup \{\mathrm{co}(v,w) : \ v \in N(u),\ d(v) < 2^r, w \in N(v)\}. \tag{9}$$

In other words, $\mathrm{Dir}(u,r)$ is the union of the directions (in H) from u toward its neighbors, and the directions from not-very-high degree neighbors of u towards their neighbors. For a vertex u and positive integers m and r, we define the set $S(u,m,r)$ of vertices at distance m from u such that the shortest paths from u to them do not use any direction from $\mathrm{Dir}(u,r)$: Let

$$B(u,m) := \{v : \ d(u,v) = m\} \tag{10}$$

be the sphere of radius m about vertex u and

$$S(u,m,r) := \{x \in B(u,m) : \ \mathrm{co}(u,x) \cap \mathrm{Dir}(u,r) = \emptyset\}. \tag{11}$$

Finally, let $\widetilde{S}(u,m,r) := \{v \in S(u,m,r) \ : \ u \in S(v,m,r)\}$.

3. Preliminaries

Proposition 1

$$|L| > \frac{999}{1000} \cdot 2^n.$$

Proof. The chain of inequalities

$$2t \cdot 2^n = \sum_{x \in V(Q_n)} d(x) > \sum_{x \in V(Q_n) \setminus \mathrm{L}} d(x) > 2000t(2^n - |\mathrm{L}|)$$

implies our claim. □

The next observation follows from the definitions of L and Dir.

Proposition 2 *Let $u \in L$ and $r \le \ell$ be a positive integer. Then*

$$|\mathrm{Dir}(u, r)| < 2000 \cdot t \cdot 2^r.$$

□

For each $u \in \mathrm{L}$, let

$$T(u) := \{r \in \{0, 1, 2, \ldots, \ell\} \;:\; |\mathrm{Dir}(u, r)| \le 400000 \cdot t \cdot 2^r/\ell\}. \tag{12}$$

Proposition 3 *For each $u \in L$, $|T(u)| \ge 0.99(\ell + 1)$.*

Proof. Consider $\phi(u) := \sum_{r=0}^{\ell} |\mathrm{Dir}(u, r)|2^{-r}$. A neighbor $v \in A_i$ of u contributes to the summand $|\mathrm{Dir}(u, r)|2^{-r}$ the amount of $d(v)2^{-r} < 2^{i+1-r}$ if $i \le r - 1$, and contributes 0 otherwise. Hence, in total v contributes less than 2 to $\phi(u)$. Therefore,

$$\phi(u) = \sum_{r=0}^{\ell} |\mathrm{Dir}(u, r)|2^{-r} < 2d(u) \le 4000t. \tag{13}$$

In order (13) to hold, fewer than 0.01ℓ summands can exceed $400000\,t/\ell$. □

For each $r \in \{0, 1, 2, \ldots, \ell\}$, let

$$m(r) := \lfloor \min\{0.1n, 10^{-8}\frac{\ell \cdot n}{t}2^{-r}\}\rfloor. \tag{14}$$

Proposition 3 will be used to prove the following two claims.

Proposition 4 *Let $u \in L$. Then for each $r \in T(u)$,*

$$|S(u, m(r), r)| \ge 0.99\binom{n}{m(r)}. \tag{15}$$

Proof. Let $u \in \mathrm{L}$, $r \in T(u)$ and $m = m(r)$. By the definitions of $T(u)$ and $S(u, m(r), r)$,

$$\frac{|S(u, m, r)|}{\binom{n}{m}} = \frac{\binom{n-|\mathrm{Dir}(u,r)|}{m}}{\binom{n}{m}} \ge \frac{\binom{n-4\cdot 10^5 t\cdot 2^r/\ell}{m}}{\binom{n}{m}} \ge \left(\frac{n - m - 4 \cdot 10^5 t \cdot 2^r/\ell}{n - m}\right)^m \ge 1 - m\frac{4 \cdot 10^5 t \cdot 2^r}{\ell(n - m)}.$$

By the definition of $m = m(r)$, we have

$$m\frac{4 \cdot 10^5 t \cdot 2^r}{\ell(n-m)} \leq \frac{\ell \cdot n \cdot 4 \cdot 10^5 \cdot t \cdot 2^r}{10^8 t\ell(n-m)2^r} = \frac{0.004n}{n-m} \leq 0.01.$$

This proves the proposition. □

Proposition 5 *For at least $0.5 \cdot 2^n$ vertices $u \in L$, there are at least $0.8(\ell+1)$ values of $r \in \{0, 1, 2, \dots, \ell\}$ such that*

$$|\{v \in B(u, m(r)) \ : \ u \in S(v, m(r), r)\}| > 0.75\binom{n}{m(r)}. \tag{16}$$

Proof. Propositions 3 and 4 imply that

$$\sum_{v \in \mathrm{L}} \sum_{r=0}^{\ell} \frac{|S(v, m(r), r)|}{\binom{n}{m(r)}} \geq (0.99)^2(\ell+1)|\mathrm{L}| > 0.98(\ell+1)|\mathrm{L}|.$$

By Proposition 1, the last expression is greater than $0.979(\ell+1)2^n$. But this sum is less than

$$\sum_{u \in V(H)} \sum_{r=0}^{\ell} \frac{|\{v \in B(u, m(r)) \ : \ u \in S(v, m(r), r)\}|}{\binom{n}{m(r)}}.$$

It follows that for at least $0.51 \cdot 2^n$ vertices $u \in V(H)$,

$$\sum_{r=0}^{\ell} \frac{|\{v \in B(u, m(r)) \ : \ u \in S(v, m(r), r)\}|}{\binom{n}{m(r)}} > 0.95(\ell+1). \tag{17}$$

Hence, by Proposition 1, (17) holds for at least $0.5 \cdot 2^n$ vertices $u \in \mathrm{L}$. And (17) cannot hold for a vertex $u \in \mathrm{L}$ if for more than $0.2(\ell+1)$ values of $r \in \{0, 1, 2, \dots, \ell\}$, (16) fails. □

By (6),

$$\widetilde{a}_r = |\cup_{j=r}^{\ell} A_j| \leq 2e(H) \cdot 2^{-r} = t \cdot 2^{n-r+1}. \tag{18}$$

Let

$$R := \{r \in \{0, 1, 2, \dots, \ell\} \ : \ \widetilde{a}_r < 20t \cdot 2^{n-r}/\ell\}. \tag{19}$$

Proposition 6 $|R| \geq 4(\ell+1)/5$.

Proof. Let $S := \sum_{r=0}^{\ell} 2^r \widetilde{a}_r$. By definition of $\widetilde{a}_r$ and by (6),

$$S = \sum_{r=0}^{\ell} 2^r \sum_{j=r}^{\ell} a_j = \sum_{j=0}^{\ell} a_j \sum_{r=0}^{j} 2^r < \sum_{j=0}^{\ell} a_j 2^{j+1} \leq 2 \sum_{u \in V(H)} d(u) = 4t \cdot 2^n.$$

In order S to be less than $4t \cdot 2^n$, fewer than $\ell/5$ summands can be greater or equal to $20t \cdot 2^n/\ell$. □

Proposition 7 *For each $r \in R$ and for any given $1 < m < n$,*

$$\left|\left\{x:\ |B(x,m)\cap \widetilde{A}_r| \le 20000\frac{t}{\ell}2^{-r}\binom{n}{m}\right\}\right| > \frac{999}{1000}\cdot 2^n. \tag{20}$$

Proof. Observe that for each r and m,

$$\sum_{x\in V(H)} |B(x,m)\cap \widetilde{A}_r| = \widetilde{a}_r\binom{n}{m}. \tag{21}$$

Hence, if $r \in R$, then the number of summands on the left-hand side of (21) exceeding $20000\frac{t}{\ell}2^{-r}\binom{n}{m}$ is less than $10^{-3}2^n$. □

Proposition 7 immediately yields the following.

Proposition 8 *Let $m(r)$ be defined by (14). For each $r \in R$, the number of vertices $x \in V(H)$ such that*

$$|B(x,m(r)+1)\cap \widetilde{A}_r| \le 20000\frac{t}{\ell}2^{-r}\binom{n}{m(r)+1} \tag{22}$$

is at least $0.999\cdot 2^n$. □

We extend it as follows.

Proposition 9 *At least $0.99\cdot 2^n$ vertices $x \in V(H)$ possess the following property: For at least $3(\ell+1)/5$ values of $r \in \{0,1,2,\ldots,\ell\}$,*

$$|B(x,m(r)+1)\cap \widetilde{A}_r| \le 20000\frac{t}{\ell}2^{-r}\binom{n}{m(r)+1}.$$

Proof. For each $r \in R$, let $X(r)$ be the set of vertices of H for which (22) does not hold. By Proposition 8, $|X(r)| \le 0.001\cdot 2^n$ for each $r \in R$. Hence $\sum_{r\in R}|X(r)| \le 0.001\cdot 2^n|R|$ and the number of vertices $v \in V(H)$ that belong to at least $(\ell+1)/5$ sets $X(r)$ is at most

$$\frac{0.001\cdot 2^n|R|}{0.2(\ell+1)} \le 0.01\cdot 2^n.$$

Since $|R| \ge 4(\ell+1)/5$, this proves the proposition. □

The next easy observation is one of our key tools.

Proposition 10 *Let vertices $u,v \in L$ be such that $v \in \widetilde{S}(u,m,r)$ for some integers $m \ge 4$ and $r \ge 1$. Let P be a (u,v)–path in H of length at most $m+2$, with the first edge ux and the last edge yv. Then the following statements hold:*
(i) The length of P is exactly $m+2$;
(ii) $x,y \in \widetilde{A}_r$;
(iii) $\mathrm{co}(u,x) = \mathrm{co}(v,y)$;
(iv) $y \in B(u,m(r)+1)\cap \widetilde{A}_r$, and $\mathrm{co}(y,v) \in \mathrm{Dir}(u,r)$;
(v) $\mathrm{co}(y,u) = \mathrm{co}(v,u)\cup \mathrm{co}(y,v)$.

Proof. By the definition of $\widetilde{S}(u,m,r)$, $\mathrm{co}(u,x) \notin \mathrm{co}(u,v)$ and $\mathrm{co}(v,y) \notin \mathrm{co}(u,v)$. This implies that both edges, ux and vy, are additional to the shortest (u,v)-path in Q_n, implying (i). Furthermore, if $\mathrm{co}(u,x) \neq \mathrm{co}(v,y)$, then extra edges are needed in P for both directions, which makes the length of P at least $m+4$, a contradiction, yielding (iii).

If $x \notin \widetilde{A}_r$ and the second edge of P is xz, then $\mathrm{co}(x,z) \in \mathrm{Dir}(u,r)$, and hence by (11), the edge xz is extra to make P longer than $m+2$. Thus, $x \in \widetilde{A}_r$. Similar argument proves $y \in \widetilde{A}_r$, implying (ii). Part (ii) already gives $y \in \widetilde{A}_r$. As $\mathrm{co}(y,v) \notin \mathrm{co}(u,v)$ we have that $d(u,y) = m(r)+1$, yielding instantly (iv) and (v). □

4. Proof of the main result

Choose a vertex $u \in \mathrm{L}$ for which the statements of Propositions 9 and 5 hold. Then by Propositions 3, 4, 9, and 5, there are at least $(1-0.01-0.4-0.2)(\ell+1) = 0.39(\ell+1)$ values of $r \in \{0,1,2,\ldots,\ell\}$ such that

$$|B(u,m(r)+1) \cap \widetilde{A}_r| \leq 20000\frac{t}{\ell}2^{-r}\binom{n}{m(r)+1} \tag{23}$$

and (recalling that $\widetilde{S}(u,m,r) = \{v \in S(u,m,r) \ : \ u \in S(v,m,r)\}$ and combining (15) and (16))

$$|\widetilde{S}(u,m(r),r)| > (0.75-0.01)\binom{n}{m(r)}. \tag{24}$$

Choose some $r \in \{0,1,2,\ldots,\ell\}$ satisfying (23) and (24) with $r \geq 0.1\ell$. If n (and hence ℓ) is large enough, then $m(r) = \lfloor 10^{-8}\frac{\ell\cdot n}{t}2^{-r}\rfloor$.

Let $v \in \widetilde{S}(u,m(r),r)$. By the definition of H, there is a (u,v)-path P_{uv} of length at most $m(r)+2$. By Proposition 10 (iv), the neighbor $y = y(v)$ of v on the path P_{uv} is in $B(u,m(r)+1) \cap \widetilde{A}_r$, and $\mathrm{co}(y,v) \in \mathrm{Dir}(u,r)$. By the definition of $\widetilde{S}(u,m(r),r)$ we have that $\mathrm{co}(v,u) \cap \mathrm{Dir}(u,r) = \emptyset$. Proposition 10 (v) states that $\mathrm{co}(y,u) = \mathrm{co}(v,u) \cup \mathrm{co}(y,v)$, in particular that $\mathrm{co}(y,u) \cap \mathrm{Dir}(u,r) = \mathrm{co}(y,v)$. Therefore, given u and y, the direction $\mathrm{co}(y,v)$ is determined. So the only vertex in $\widetilde{S}(u,m(r),r)$ that can be reached from u by a path of length $m(r)+2$ passing vertex y is v. With other words, the number of choices for v is not larger than for y, i.e.,

$$|\widetilde{S}(u,m(r),r)| \leq |B(u,m(r)+1) \cap \widetilde{A}_r|.$$

Using (23) and (24), we get

$$0.74\binom{n}{m(r)} < 20000\frac{t}{\ell}2^{-r}\binom{n}{m(r)+1}.$$

Hence,

$$\frac{m(r)+1}{n-m(r)} < 30000\frac{t}{\ell}2^{-r}.$$

Plugging in the value $m(r) = \lfloor 10^{-8}\frac{\ell\cdot n}{t}2^{-r}\rfloor$ from the previous paragraph, we have

$$\frac{10^{-8}\frac{\ell n}{t}2^{-r}}{n} < \frac{m(r)+1}{n-m(r)} < 30000\frac{t}{\ell}2^{-r}.$$

This yields

$$t > \frac{1}{\sqrt{3}} 10^{-6} \ell = \frac{1}{\sqrt{3}} 10^{-6} \lfloor \log_2 n \rfloor > 0.5 \cdot 10^{-6} \log_2 n,$$

which proves our theorem, since $t = 2^{-n} e(H)$. □

Acknowledgements. We thank for the referee for careful reading of the first version of the manuscript.

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Received: February 21, 2008
Final version received: February 21, 2008