

1 **ON THE DECAY OF CROSSING NUMBERS**
 2 **OF SPARSE GRAPHS**

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ABSTRACT. Richter and Thomassen proved that every graph has an edge e such that the crossing number $\text{cr}(G - e)$ of $G - e$ is at least $(2/5)\text{cr}(G) - O(1)$. Fox and Cs. Tóth proved that dense graphs have large sets of edges (proportional in the total number of edges) whose removal leaves a graph with crossing number proportional to the crossing number of the original graph; this result was later strengthened by Černý, Kynčl and G. Tóth. These results make our understanding of the decay of crossing numbers in dense graphs essentially complete. In this paper we prove a similar result for large sparse graphs in which the number of edges is not artificially inflated by operations such as edge subdivisions. We also discuss the connection between the decay of crossing numbers and expected crossing numbers, a concept recently introduced by Mohar and Tamon.

4 1. INTRODUCTION

5 The *crossing number* $\text{cr}(G)$ of a graph G is the minimum number of pair-
 6 wise crossings of edges in a drawing of G in the plane. A graph G is *k-*
 7 *crossing-critical* if $\text{cr}(G) \geq k$, but $\text{cr}(G - e) < k$ for every edge e of G . Since
 8 loops are totally irrelevant for crossing number purposes, all graphs under
 9 consideration are loopless.

10 1.1. **The decay of crossing numbers.** In this paper we are concerned
 11 with the effect of edge removal in the crossing number of a graph (following
 12 Fox and Tóth [10], this is referred to as the *decay of crossing numbers*).
 13 Richter and Thomassen [22] proved that every graph G has some edge e
 14 such that $\text{cr}(G - e) \geq (2/5)\text{cr}(G) - 37/5$. They conjectured that there
 15 always exist an edge e such that $\text{cr}(G - e) \geq \text{cr}(G) - c\sqrt{\text{cr}(G)}$, for some
 16 universal constant c . This conjecture was proved by Fox and Tóth [10] for
 17 dense graphs.

18 Fox and Tóth actually proved a much stronger result: the existence of a
 19 large subset of edges whose removal leaves a graph whose crossing number

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20 is at least a proportion of the crossing number of the original graph. More
 21 precisely, they proved that for every fixed $\epsilon > 0$, there is a constant $n_0 =$
 22 $n_0(\epsilon)$ such that if G is a graph with $n > n_0$ vertices and $m > n^{1+\epsilon}$ edges,
 23 then G has a subgraph G' with at most $(1 - \frac{\epsilon}{24})m$ edges such that $cr(G') \geq$
 24 $(\frac{1}{28} - o(1))cr(G)$.

25 This result was further strengthened by Černý, Kynčl and G. Tóth [5],
 26 who proved that for every $\epsilon, \gamma > 0$ there is an $n_0 = n_0(\epsilon, \gamma)$ such that if G is
 27 a graph with $n > n_0$ vertices and $m > n^{1+\epsilon}$ edges, then G has a subgraph
 28 G' with at most $(1 - \frac{\epsilon\gamma}{1224})m$ edges such that $cr(G') \geq (1 - \gamma)cr(G)$.

29 **1.2. The decay of crossing numbers of sparse graphs.** Due to the Fox-
 30 Tóth and the Černý-Kynčl-Tóth results, our understanding of the decay of
 31 crossing numbers of dense graphs is essentially complete. The situation
 32 for sparse graphs is quite different. Although the Richter and Thomassen
 33 result is fully general, it only guarantees the existence of a single edge whose
 34 deletion leaves a graph with crossing number substantially large. As pointed
 35 out in [10], by combining the following two facts one obtains an improvement
 36 to the Richter-Thomassen result for graphs with n vertices and $m > 8.1n$
 37 edges: (i) every graph with $m \geq \frac{103}{16}n$ satisfies $cr(G) \geq 0.032\frac{m^3}{n^2}$ [20]; and
 38 (ii) for any graph G and any edge e of G , $cr(G - e) \geq cr(G) - m + 1$ [21].

39 In this paper we investigate the decay of crossing numbers of sparse
 40 graphs. We are particularly interested in establishing results as similar as
 41 possible as those in [10] and [5]: the existence of large sets of edges whose
 42 removal leaves a graph whose crossing number is at least some (constant)
 43 fraction of the crossing number of the original graph.

44 In contrast with dense graphs, in a sparse graph it is possible to artificially
 45 increase the number of edges of a graph, while maintaining its crossing
 46 number, without adding any substantial topological feature. Consider, for
 47 instance, a graph consisting of a large planar grid plus an additional edge
 48 e joining two vertices far apart; subdivide this additional edge r times (for
 49 some integer $r > 0$) to get a path P , and let G denote the resulting graph.
 50 For any given $\alpha > 0$, we can make r sufficiently large so that any set of at
 51 least $\alpha|E(G)|$ edges of G contains at least an edge of P . That is, for any set
 52 E_0 of at least $\alpha|E(G)|$ edges of G , the crossing number of $G - E_0$ is 0.

53 This example shows that no general result can possibly be established if
 54 we allow the number of edges to be artificially inflated. In particular, degree
 55 2 vertices need to be precluded from the graphs under consideration. This
 56 is a particular instance of a more general way to spuriously increase the
 57 number of edges, by substituting a set of (possibly just one) edges joining
 58 the same two vertices by a plane connected graph, as we now describe.

59 We first recall the definition of a bridge. Let G be a graph, and let u, v
 60 be distinct vertices of G . Following Tutte, a uv -bridge is either a single edge
 61 joining u and v , together with u and v (in which case it is *trivial*), or a
 62 subgraph of G obtained by adding to a connected component K of $G \setminus \{u, v\}$

63 all the edges attaching K to u or v , together with their ends. A uv -bridge is
 64 uv -planar if it can be embedded in the plane with u and v in the same face.

65 Suppose that u, v are distinct vertices incident with the same face in a
 66 connected plane graph H with $|V(H)| > 2$, and let k be the maximum
 67 number of pairwise edge-disjoint uv -paths in H . We say that (H, u, v) is
 68 a uv -blob of width k . Now consider a graph G , and let u, v be vertices of
 69 G , joined by $k \geq 1$ edges. It is easy to see that we may substitute the
 70 edges joining u and v by an arbitrarily large uv -blob of width k , leaving
 71 the crossing number (and the criticality of G , if G is critical) unchanged.
 72 Conversely, if G is a graph with a vertex cut $\{u, v\}$, and for some $\{u, v\}$ -
 73 bridge H we have that (H, u, v) is a uv -blob of width k , then G may be
 74 simplified, leaving its crossing number (and its criticality, if G is critical)
 75 unchanged, by substituting H by k parallel uv -edges.

76 Note that the concept of uv -blob captures, in particular, the operation of
 77 edge subdivision. Indeed, a subdivided edge is simply a uv -blob of width 1,
 78 all of whose vertices, other than u and v , have degree 2.

79 **1.3. The main result.** Since we are interested in proving the existence of
 80 large sets of edges (linear in the crossing number) with a special property
 81 (their removal does not decrease the crossing number arbitrarily), we need
 82 to preclude the existence of $\{u, v\}$ -bridges (for any pair u, v of vertices) that
 83 are uv -blobs, since they inflate the number of edges of a graph, while adding
 84 no topologically interesting structure whatsoever to the graph itself.

85 As it happens, such objects are the *only* structure that needs to be
 86 avoided. A graph is *irreducible* if there do not exist vertices u, v and a
 87 $\{u, v\}$ -bridge H such that (H, u, v) is a uv -blob. We prove that if G is irre-
 88 reducible, then a large set of its edges (linear in the crossing number) may be
 89 removed, and still leave a graph whose crossing number is at least a fraction
 90 of the crossing number of the original graph. More precisely:

91 **Theorem 1.** *For each $\epsilon > 0$ and each positive integer k there exist $m_0 :=$
 92 $m_0(\epsilon, k)$ and $\gamma := \gamma(\epsilon)$ with the following property. Every 2-connected ir-
 93 reducible graph G with $\text{cr}(G) = k$ and at least m_0 edges has a set E_0 of at
 94 least γk edges such that $\text{cr}(G - E_0) > (1/2 - \epsilon)\text{cr}(G)$.*

95 Trivially, 3-connected graphs are irreducible, so in particular Theorem 1
 96 applies to all 3-connected graphs.

97 We also apply our techniques to improve (for sufficiently large graphs)
 98 the Richter and Thomassen result on crossing-critical graphs. Richter and
 99 Thomassen proved in [22] that every graph G has an edge e such that $\text{cr}(G -$
 100 $e) \geq (2/5)\text{cr}(G) - 37/5$.

101 In order to improve on this result, again we need to be careful not to allow
 102 the artificial inflation in the number of edges. However, we do not need
 103 the full condition of irreducibility: it suffices to require that each vertex is
 104 adjacent to at least 3 other vertices. A slight variant of this requirement
 105 (namely X -minimality) was introduced by Ding, Oporowski, Thomas, and

106 Vertigan in [6], with the same motivation of not allowing a graph with given
 107 crossing number (in their case, a 2-crossing-critical graph) to spuriously grow
 108 its number of edges.

109 **Theorem 2.** *For each positive integer k , there is an integer $m_1 := m_1(k)$
 110 with the following property. Let G be a 2-connected graph in which each
 111 vertex is adjacent to at least 3 vertices. If $\text{cr}(G) = k$ and G has at least m_1
 112 edges, then G has an edge e such that $\text{cr}(G - e) > (2/3)\text{cr}(G) - 10^8$.*

113 We conclude this section with a brief overview of the proofs of Theorems 1
 114 and 2, and of the rest of this paper.

115 As in [5], [10], and [22], we make essential use of the embedding method.
 116 This technique consists of finding a set E_0 of edges in a graph G , and for
 117 each $e = uv \in E_0$ a set of pairwise edge-disjoint uv -paths $\mathcal{P}(e)$, with the
 118 aim of drawing $G - E_0$ (with $\text{cr}(G - E_0)$ crossings) and then embedding each
 119 $e \in E_0$ very closely to some path in $\mathcal{P}(e)$. The idea is to choose the set E_0
 120 so that the embedding can be done without adding too many crossings.

121 Richter and Thomassen proved the existence of an edge $e = uv$ (so that
 122 $E_0 = \{e\}$) with the property that there is a uv -path (that avoids e) of length
 123 at most 4, all of whose internal vertices have degree less than 12. Fox and
 124 Tóth, and Černý-Kynčl-Tóth used the density of G to show the existence
 125 of a large set E_0 of edges, such that each edge $e = uv$ of E_0 has a large
 126 collection $\mathcal{P}(e)$ of short edge-disjoint paths, and such that the collections
 127 $\mathcal{P}(e)$ are pairwise edge-disjoint.

128 In our current setup (sparse graphs) for all we know the graphs under
 129 consideration may have maximum degree 3, and so in general we cannot
 130 expect to find collections $\mathcal{P}(e)$ of more than two edge-disjoint paths, for
 131 each $e \in E_0$. We prove that, indeed, each graph under consideration has
 132 large set E_0 of edges such that each $e = uv \in E_0$ has two short uv -paths
 133 $P(e), Q(e)$ whose internal vertices have bounded degree, and if $e \neq f$ then
 134 $P(e) \cup Q(e)$ and $P(f) \cup Q(f)$ are edge-disjoint. As it happens, $P(e)$ and $Q(e)$
 135 are not necessarily edge-disjoint, but this turns out to be unimportant. To
 136 be slightly more precise, let us mention that each graph $\Xi = e \cup P(e) \cup Q(e)$
 137 has the property that $P(e)$ and $Q(e)$ have length at most ℓ , and the degree
 138 of their internal vertices is less than Δ . Following the lively notation in [5],
 139 we call each Ξ an (ℓ, Δ) -*earring*.

140 Most of the rest of this paper is devoted to proving the result described
 141 in the previous paragraph. We start by establishing, in Section 2, several
 142 assorted statements on planar graphs; these are, in one way or another,
 143 elementary consequences of Euler's formula. The existence of a large set
 144 of edge-disjoint (ℓ, Δ) -earrings (for certain values of ℓ and Δ) is proved
 145 in Section 3 for planar graphs, and in Section 4 for irreducible nonplanar
 146 graphs.

147 In Section 5 we establish the version of the embedding method that we
 148 need. The proofs of Theorems 1 and 2 are in Section 6.

149 In Section 7 we discuss the connection between the decay of crossing
 150 numbers and the concept, recently introduced by Mohar and Tamon [18], of
 151 expected crossing numbers. Finally, in Section 8 we present some concluding
 152 remarks and open questions.

153 2. ASSORTED LEMMAS ON PLANAR GRAPHS

154 A *branch* in a graph is a path whose endpoints have degree at least 3, and
 155 all whose internal vertices have degree 2.

156 **Lemma 3.** *Let $G = (V, E)$ be a planar graph with minimum degree at least*
 157 *2, and let $B \subseteq V$ be a set of vertices of degree at least 3. Suppose that the*
 158 *number of branches with both endpoints in B is at most s . Then there are at*
 159 *least $|V|/2 - s/2 - (3/2)|B|$ edges with both endpoints in $V \setminus B$.*

160 *Proof.* Let $W := V \setminus B$. To help comprehension, we color white (respectively,
 161 black) the vertices in W (respectively, B). A branch is *black* if its endpoints
 162 are both black. A white vertex is *black-covered* if all its adjacent vertices are
 163 black. A black-covered vertex is *of Type I* if it has degree 2; otherwise (that
 164 is, if it has degree ≥ 3) it is *of Type II*.

165 Since there are no black vertices of degree 2, then no black branch can
 166 contain more than one Type I vertex. Thus there are at most s Type I
 167 vertices.

168 Let W' denote the set of black-covered vertices of Type II, and let G'
 169 denote the subgraph of G induced by the edges incident with a vertex in
 170 W' . This is a bipartite graph with bipartition (W', B') , for some $B' \subseteq B$.
 171 A standard Euler formula argument yields that $|E(G')| \leq 2|V(G')| - 4 =$
 172 $2|W'| + 2|B'| - 4$. Since each vertex in W' has degree at least 3 (in G' , as
 173 well as in G) it follows that $|E(G')| = \sum_{v \in W'} d(v) \geq 3|W'|$. Thus $3|W'| \leq$
 174 $2|W'| + 2|B'| - 4 \leq 2|W'| + 2|B| - 4$, and so $|W'| \leq 2|B| - 4$. Thus, there
 175 are at most $2|B| - 4$ Type II vertices.

176 Therefore, the total number of black-covered vertices is at most $s + 2|B| - 4$.
 177 It follows that there are at least $|W| - s - 2|B| + 4 > |W| - s - 2|B|$ white
 178 vertices that are adjacent to at least one white vertex, and so there are at
 179 least $|W|/2 - s/2 - |B| = |V|/2 - s/2 - (3/2)|B|$ edges with both endpoints
 180 in W . \square

181 The *length* of a face in a plane graph is the length of its boundary walk.

182 A *digon* in an embedded graph consists of two parallel edges, together
 183 with their common endpoints. If the endpoints are u and v , then it is a *uv-*
 184 *digon*. A plane embedding of a graph G is *clean* if for each pair of vertices
 185 u, v joined by parallel edges, there exist edges e, e' with endpoints u and v ,
 186 such that the disc bounded by the digon formed by e and e' contains all
 187 edges parallel to e and e' , and no other edges.

188 **Lemma 4.** *Let G be a connected plane graph in which each vertex is adjacent*
 189 *to at least 3 vertices. Suppose that the embedding of G is clean. Let $r \geq 0$*

190 be an integer. Let F be the set of faces of G , and let F' be the set of those
 191 faces whose length is at most $r + 5$. Then $|F'| \geq \frac{r|F|+12}{r+3}$.

192 *Proof.* Let H be a graph obtained from G as follows: for each pair (u, v) of
 193 vertices joined by parallel edges, contract to a single all the parallel edges
 194 between u and v . Let F_H denote the set of faces of H , and let F'_H denote
 195 the set of faces of H with length at most $r + 5$. Our first task is to show
 196 that $|F'_H| \geq \frac{r|F_H|+12}{r+3}$.

197 For each $f \in F_H$ the sum $w(f) := \sum_{v \sim f} 1/d(v)$ is the *weight* of f , where
 198 $d(v)$ denotes the degree of the vertex v and $v \sim f$ means that v is incident
 199 with f . (A vertex v contributes to $w(f)$ as many times as the boundary
 200 walk of f passes through v .) Since H is simple and has minimum degree
 201 at least 3, then, letting $l(f)$ denote the length of f , we have $l(f) \geq 3$ and
 202 $w(f) \leq l(f)/3$. It is easy to see that $|V(H)| = \sum_{f \in F_H} w(f)$ and $2|E(H)| =$
 203 $\sum_{f \in F_H} l(f)$. From the last two equations and Euler's formula it follows that
 204 $2 = \frac{1}{2} \sum_{f \in F_H} \{2w(f) - l(f) + 2\}$.

205 Since $w(f) \leq l(f)/3$, we have

$$12 \leq \sum_{f \in F_H} \{-l(f) + 6\} = \sum_{f \in F'_H} \{-l(f) + 6\} + \sum_{f \in F_H - F'_H} \{-l(f) + 6\}.$$

206 Since $l(f) \geq 3$ for each $f \in F_H$, then $-l(f) + 6 \leq 3$ and thus $\sum_{f \in F'_H} \{-l(f) +$
 207 $6\} \leq 3|F'_H|$. If $f \in F_H - F'_H$ then $l(f) - 6 \geq r$, that is, $-l(f) + 6 \leq -r$, and so
 208 $\sum_{f \in F_H - F'_H} \{-l(f) + 6\} \leq -r(|F_H| - |F'_H|)$. Thus, $12 \leq 3|F'_H| - r(|F_H| - |F'_H|)$,
 209 and so $|F'_H| \geq \frac{r|F_H|+12}{r+3}$, as required.

210 Now as we inflate back H to G , each face in F'_H becomes a face in F' .
 211 The other faces in F' are precisely the $t := |E(G) \setminus E(H)|$ faces created
 212 in the inflation process, that is, those bounded by parallel edges. Thus
 213 $|F| = |F_H| + t$ and $|F'| = |F'_H| + t$. Thus $|F'| - t \geq \frac{r(|F|-t)+12}{r+3}$, and so
 214 $|F'| \geq \frac{r|F|+12}{r+3} + t(1 - \frac{r}{r+3}) \geq \frac{r|F|+12}{r+3}$. \square

215 If D is a digon in a plane graph, then the open (respectively, closed) disc
 216 bounded by D will be denoted $\Delta(D)$ (respectively, $\overline{\Delta}(D)$). If D, D' are
 217 digons, then we write $D' \leq D$ if $\Delta(D') \subseteq \Delta(D)$. We recall that a vertex of
 218 degree 0 is an *isolated vertex*.

219 **Proposition 5.** *Let $G = (V, E)$ be a plane graph, and let Z be a set of*
 220 *isolated vertices of G . Suppose that for each digon D in G , the disc bounded*
 221 *by D contains at least one vertex in Z . Then G has at most $3|V \setminus Z| + |Z|$*
 222 *edges.*

223 *Proof.* Let $Y := V \setminus Z$. To help comprehension, we colour the vertices in Y
 224 and Z black and green, respectively.

225 We prove the stronger statement that G has at most $3|Y| + |Z| - 6$ edges.
 226 We proceed by induction on the number of digons in G . In the base case G

227 has no digons, and so by Euler's Formula it has at most $3|Y| - 6$ edges, as
 228 required. For the inductive step, we assume that G has at least one digon,
 229 and let D be a \leq -minimal digon in G .

230 Suppose first that D is also \leq -maximal. Then let G' be the graph obtained
 231 from G by removing one edge of D and one green vertex contained in $\Delta(D)$.
 232 Now G' contains one fewer edge and one fewer green vertex than G . It is
 233 easy to see that the induction hypothesis can be applied to G' , and so the
 234 inductive step follows.

235 Therefore we may assume that D is not \leq -maximal. Among all digons
 236 that contain D , let D' be a \leq -minimal one.

237 Suppose that D and D' have an edge e in common, and let \bar{e} be the other
 238 edge of D . It is easy to see that the induction hypothesis can be applied to
 239 the graph obtained from G by removing \bar{e} and a green vertex contained in
 240 $\Delta(D)$, and once again the inductive step follows. Thus we may assume that
 241 D and D' do not have an edge in common.

242 If $\Delta(D')$ contains a green vertex not contained in $\Delta(D)$, the situation is
 243 again straightforward: the induction hypothesis can be applied to the graph
 244 G' obtained by removing one edge of D and one green vertex contained in
 245 $\Delta(D)$, and the inductive step follows. Thus we may assume that every green
 246 vertex contained in $\Delta(D')$ is contained in $\Delta(D)$.

247 In this case, there are no digons other than D' and D contained in $\bar{\Delta}(D')$.
 248 Now let G' be the graph obtained by removing from G the black vertices
 249 and all the edges contained in $\Delta(D')$. Let Y' and Z' denote the sets of black
 250 and green vertices of G' , respectively, and let E' denote the set of edges
 251 of G' (note that $Z' = Z$). We may clearly apply the induction hypothesis
 252 to G' , obtaining that $|E'| \leq 3|Y'| + |Z| - 6$. Let $Y'' := Y \setminus Y'$, and $E'' :=$
 253 $E \setminus E'$. Let x, y be the vertices of D' . Consider the graph G'' that consists
 254 of the vertices in $Y'' \cup \{x, y\}$ and the edges in E'' . Since G'' has exactly
 255 one digon (namely D), the usual Euler formula argument yields $|E(G'')| \leq$
 256 $3|V(G'')| - 5$. However, this inequality is tight only if G'' is maximally
 257 planar, that is, if no edge can be added between two nonadjacent vertices
 258 while maintaining planarity; thus, since x and y are not adjacent in G'' , it
 259 follows that $|E(G'')| \leq 3|V(G'')| - 6$. Thus $|E''| \leq 3(|Y''| + 2) - 6$. That is,
 260 $|E| - |E'| \leq 3(|Y| - |Y'| + 2) - 6$, and so $|E| \leq 3|Y| + |Z| - 6$, as required. \square

261 A set Z of vertices in a 2-connected planar graph G is an *anchor* if the
 262 following hold:

- 263 (1) no vertex in Z is part of a 2-vertex-cut in G ; and
- 264 (2) if $\{u, v\}$ is a 2-vertex-cut in G , then every nontrivial uv -bridge con-
 265 tains a vertex in Z .

266 **Lemma 6.** *Let G be a 2-connected plane graph in which each vertex is*
 267 *adjacent to at least 3 distinct vertices, and let Z be an anchor of G . Let*
 268 *$Y \subseteq V(G) \setminus Z$, and let E_Y denote the set of edges of G with both endpoints*
 269 *in Y . Then the number of faces of G that are incident with exactly 2 vertices*
 270 *of Y is at most $3|Y| + |Z| + |E_Y|$.*

271 *Proof.* We may assume that $|Y| \geq 2$, as otherwise there is nothing to prove.
 272 Let F_2 denote the set of faces of G that are incident with exactly two vertices
 273 of Y .

274 We start by coloring red each edge in E_Y , and green each vertex in Z .
 275 Now for each $f \in F_2$, join the two vertices in Y incident with f by a simple
 276 blue arc contained (except, obviously, for its endpoints) in f . Let H denote
 277 the plane graph that consists of the vertices in Y plus all the red edges and
 278 the blue arcs (now seen as edges), as well as the set Z of green vertices.
 279 Note that the green vertices are isolated in H . We remark that $|F_2|$ is the
 280 number of blue edges in H .

281 Note that if D is a blue digon in H (that is, both edges of D are blue),
 282 with vertices u and v , then $\overline{\Delta}(D)$ contains a uv -bridge in G . This bridge
 283 may be trivial (in which case it is a red edge) or nontrivial (in which case,
 284 by hypothesis, $\Delta^o(D)$ contains a green vertex).

285 Finally, let K denote the graph that results from H by substituting each
 286 red edge by an isolated red vertex (placed in the interior of the red edge).
 287 Note that $|E(K)| = |F_2|$, that the vertex set of K is the union of Y with
 288 the set of all green or red vertices, and that there are $|Z|$ green and $|E_Y|$
 289 red vertices.

290 The graph K has the property that for each (necessarily blue) digon D in
 291 K , $\Delta^o(D)$ contains either a green or a red vertex. Applying Proposition 5
 292 we obtain that $|E(K)| \leq 3|Y| + |Z| + |E_Y|$. Thus $|F_2| \leq 3|Y| + |Z| + |E_Y|$,
 293 as required. \square

294 If G is a plane graph, then we let G^o denote its dual.

295 **Lemma 7.** *Let G be a 2-connected plane graph, and let Z be an anchor of*
 296 *G . Suppose that the embedding of G is clean. Let F' be a set of faces of G*
 297 *of length at least 3. Then the number of branches in G^o with both endpoints*
 298 *in F' is at most $3|F'| + |Z|$.*

299 *Proof.* Since the embedding is clean, we may as well assume (in the context
 300 of this lemma) that G has no parallel edges. It follows that all branches with
 301 both endpoints in F' are actual edges in G^o . Thus our goal is to show that
 302 there are at most $3|F'| + |Z|$ edges in G^o with both endpoints in F' .

303 Regarding G and G^o as simultaneously embedded, remove everything
 304 except for F' (seen as a set of vertices in G^o), the edges (in G^o) joining
 305 two vertices in F' , and the vertices in Z . The result is a graph G' in which
 306 each vertex in Z is isolated, and such that the disc bounded by every digon
 307 contains a vertex in Z . To see this last property, note that if e and f are
 308 the edges of a digon in G' , then the edges in G corresponding to e and f
 309 are a 2-edge-cut in G ; since Z is an anchor set of G , it then follows that the
 310 disc bounded by the digon must contain a vertex of Z in its interior.

311 Applying Proposition 5, we obtain that G' has at most $3|F'| + |Z|$ edges.
 312 This finishes the proof, since there is a bijection between the edges in G'
 313 and the edges in G^o with both endpoints in F' . \square

314

3. EARRINGS IN PLANAR GRAPHS

315 Černý, Kynčl and Tóth introduced the lively terminology of *earring of*
 316 *size* p to describe a graph consisting of an edge $e = uv$ plus a collection of
 317 p pairwise edge-disjoint, bounded-length uv -paths. In order to use the re-
 318 embedding method, the goal is to find many pairwise edge-disjoint earrings.

319 As we mentioned in Section 1, in our current context of sparse graphs,
 320 where (for all we know) the graphs under consideration may have maximum
 321 degree 3, the best we could hope for is to prove the existence of a large
 322 collection of earrings, each of size 2. As we also mentioned, in this discussion
 323 we do not need the two uv -paths of each earring to be edge-disjoint, but only
 324 a weaker condition (see (iii) in the following definition).

325 Let ℓ, Δ be positive integers. An (ℓ, Δ) -*earring* of a graph G is a subgraph
 326 of G that consists of a *base* edge $e = uv$ plus two distinct uv -paths P, Q
 327 (disjoint from e) with the following properties: (i) each of P and Q has at
 328 most ℓ edges; (ii) each internal vertex of P or Q has degree less than Δ ; and
 329 (iii) if f is an edge in both P and Q , then $\{e, f\}$ is a 2-edge-cut of G .

330 An edge $e = uv$ in a 2-connected plane graph is an (ℓ, Δ) -*edge* if each of
 331 its two incident faces has length at most $\ell + 1$, and no vertex incident with
 332 these two faces, other than possibly u or v , has degree Δ or greater. If e is an
 333 (ℓ, Δ) -edge, then the subgraph that consists of e plus the cycles that bound
 334 its two incident faces, is an (ℓ, Δ) -earring, the (ℓ, Δ) -earring $\Xi(e)$ *associated*
 335 *to* e .

336 The following lemma is the main workhorse in this paper.

337 **Lemma 8.** *Let $G = (V, E)$ be a 2-connected planar graph in which each*
 338 *vertex is adjacent to at least 3 other vertices. Let Z be an anchor of G ,*
 339 *where each vertex in Z has degree 4. Then G has at least $10^{-10}|E| - 10^{-5}|Z|$*
 340 *pairwise edge-disjoint $(5000, 500)$ -earrings.*

341 *Proof.* Throughout the proof, we make use of several constants that are
 342 either very small, very close to 1, or somewhat large. In order to simplify
 343 the whole discussion, we first proceed to introduce these constants. We let
 344 $\ell_0 = 5000$, $\Delta_0 = 500$, $\mathbf{c}_1 = 10^{-10}$, $\mathbf{c}_2 = 10^{-5}$, $\mathbf{c}_3 = 999/1000$, $\mathbf{c}_4 = 1/1000$,
 345 $\mathbf{c}_5 = 999$, $\mathbf{c}_6 = 36/5000$, and $\mathbf{z}_1 = 3(10^{-10})$.

346 It is a trivial observation that every planar graph has a clean plane em-
 347 bedding (clean embeddings are defined before Lemma 4). Throughout the
 348 proof we consider a fixed clean embedding of G in the plane. Let F denote
 349 the set of all faces of G , and let $t := |Z|$.

350 **Claim 9.** *It suffices to show that there are at least $(2\ell_0(2\ell_0 + 1) + 1) \cdot (\mathbf{z}_1|F| -$*
 351 *$\mathbf{c}_2t)$ (ℓ_0, Δ_0) -edges.*

352 *Proof.* Consider the graph H whose vertices are the (ℓ_0, Δ_0) -edges of G ,
 353 with two distinct (ℓ_0, Δ_0) -edges e, f adjacent if $\Xi(e)$ and $\Xi(f)$ have some
 354 edge in common.

355 We note that H has maximum degree at most $2\ell_0(2\ell_0 + 1)$. This follows
 356 at once from the following two easy observations: (i) for each (ℓ_0, Δ_0) -edge

357 e , $\Xi(e)$ has at most $2\ell_0$ edges other than e ; and (ii) each edge of G belongs
 358 to at most $2\ell_0 + 1$ (ℓ_0, Δ_0) -earrings of the form $\Xi(f)$ for some edge f .

359 Thus, $V(H)$ has a stable set of size at least $|V(H)|/(2\ell_0(2\ell_0 + 1) + 1)$.
 360 Suppose that G has at least $(2\ell_0(2\ell_0 + 1) + 1) \cdot (\mathbf{z}_1|F| - \mathbf{c}_2t)$ (ℓ_0, Δ_0) -edges;
 361 that is, $|V(H)| \geq (2\ell_0(2\ell_0 + 1) + 1) \cdot (\mathbf{z}_1|F| - \mathbf{c}_2t)$. Then H has a stable set S
 362 of size at least $\mathbf{z}_1|F| - \mathbf{c}_2t$; that is, there is a collection of at least $\mathbf{z}_1|F| - \mathbf{c}_2t$
 363 pairwise edge-disjoint (ℓ_0, Δ_0) -earrings.

364 Since G has minimum degree at least 3, a routine Euler formula argument
 365 yields that $|F| \geq |E|/3 + 2$. Thus there are at least $\mathbf{z}_1(|E|/3 + 2) - \mathbf{c}_2t >$
 366 $\mathbf{c}_1|E| - \mathbf{c}_2t$ pairwise edge-disjoint (ℓ_0, Δ_0) -earrings, as required in Lemma 8.
 367 \square

368 Let W be the set of those vertices of G with degree at least Δ_0 , and let
 369 F_W denote the set of faces of G that are incident with some vertex in W .
 370 For each integer $j \geq 1$, let F_j denote the set of those faces of G incident
 371 with exactly j vertices in W (and perhaps other vertices in $V \setminus W$), and let
 372 $f_j = |F_j|$. Note that F_W is the disjoint union $\bigcup_{i \geq 1} F_i$.

373 Let F_{long} (respectively, F_{short}) denote the collection of faces of G with
 374 length greater than (respectively, at most) $\ell_0 + 1$, and let $f_{\text{long}} := |F_{\text{long}}|$ and
 375 $f_{\text{short}} := |F_{\text{short}}|$. It follows immediately from Lemma 4 that

$$(1) \quad f_{\text{short}} \geq \mathbf{c}_3|F|.$$

376 Since F is the disjoint union of F_{long} and F_{short} , then $|F| = f_{\text{long}} + f_{\text{short}}$,
 377 and so $f_{\text{short}} \geq \mathbf{c}_3(f_{\text{long}} + f_{\text{short}})$ implies $f_{\text{short}} \geq (\mathbf{c}_3/(1 - \mathbf{c}_3))f_{\text{long}}$. Note that
 378 $\mathbf{c}_5 = \mathbf{c}_3/(1 - \mathbf{c}_3)$. Therefore,

$$(2) \quad f_{\text{short}} \geq \mathbf{c}_5 f_{\text{long}}.$$

379 We note that $\sum_{u \in W} d(u) = \sum_{i \geq 1} i f_i$. A routine application of Euler's
 380 formula yields that $\sum_{i \geq 3} i f_i \leq 2(3|W| - 6) = 6|W| - 12$. Since all vertices of
 381 Z have degree 4 it follows that $W \subseteq V \setminus Z$, and so we can apply Lemma 6,
 382 to obtain $f_2 \leq 3|W| + t + |E_W|$. Combining these observations we obtain

$$(3) \quad f_1 \geq \sum_{u \in W} d(u) - 12|W| - 2|E_W| - 2t + 12.$$

383 **Claim 10.** *If $|F_W| > 24t + 24\mathbf{c}_4 f_{\text{short}}$, then Lemma 8 follows.*

384 *Proof.* We establish four subclaims, and finally show that the proof follows
 385 easily from them.

386 **SUBCLAIM A** *If $|E_W| > 6|W| - 12 + \mathbf{c}_4 f_{\text{short}}$, then Lemma 8 follows.*

387 *Proof.* If e_1, e_2, e_3 are parallel edges with common endpoints u, v , and e_2 is
 388 in the disc bounded by the digon formed by e_1 and e_3 , then e_2 is a *sheltered*
 389 edge. By Euler's formula, a simple graph on $|W|$ vertices has at most $3|W| - 6$
 390 edges. Since the embedding of G is clean, it follows that the subgraph of G
 391 induced by W has at least $|E_W| - 2(3|W| - 6) = |E_W| - 6|W| + 12$ sheltered

392 edges. The fact that G is clean also implies that each sheltered edge is a
 393 (ℓ_0, Δ_0) -edge, and so G has at least $|E_W| - 6|W| + 12$ (ℓ_0, Δ_0) -edges.

394 Suppose that $|E_W| > 6|W| - 12 + \mathbf{c}_4 f_{\text{short}}$. Then G has at least $\mathbf{c}_4 f_{\text{short}}$
 395 (ℓ_0, Δ_0) -edges. Using (1), it follows that G has at least $\mathbf{c}_3 \mathbf{c}_4 |F|$ (ℓ_0, Δ_0) -
 396 edges. The result now follows from Claim 9, since $\mathbf{c}_3 \mathbf{c}_4 > (2\ell_0(2\ell_0 + 1) +$
 397 $1)\mathbf{z}_1$. \square

398 **SUBCLAIM B** *If $(1/6)(\sum_{u \in W} d(u)) \leq 12|W| + 2t + 2|E_W| - 12$, then $|F_W| <$*
 399 *$24t + 24\mathbf{c}_4 f_{\text{short}}$ or else Lemma 8 follows.*

400 *Proof.* By Subclaim A, under the given hypothesis we may assume that
 401 $\sum_{u \in W} d(u) \leq 72|W| + 12t + (72|W| - 144 + 12\mathbf{c}_4 f_{\text{short}}) - 72 = 144|W| + 12t +$
 402 $12\mathbf{c}_4 f_{\text{short}} - 216 < 144|W| + 12t + 12\mathbf{c}_4 f_{\text{short}}$.

403 Since each vertex in W has degree at least Δ_0 , it follows that $\Delta_0|W| \leq$
 404 $\sum_{u \in W} d(u)$. Hence, $|W| < (12t + 12\mathbf{c}_4 f_{\text{short}})/(\Delta_0 - 144)$. On the other hand,
 405 obviously $|F_W| \leq \sum_{u \in W} d(u)$, and so $|F_W| < 144(12t + 12\mathbf{c}_4 f_{\text{short}})/(\Delta_0 -$
 406 $144) + 12t + 12\mathbf{c}_4 f_{\text{short}}$. Since $144/(\Delta_0 - 144) \leq 1$, this implies $|F_W| <$
 407 $12t + 12\mathbf{c}_4 f_{\text{short}} + 12t + 12\mathbf{c}_4 f_{\text{short}} = 24t + 24\mathbf{c}_4 f_{\text{short}}$. \square

408 **SUBCLAIM C** *If $(1/6)(\sum_{u \in W} d(u)) \leq f_{\text{long}}$, then $|F_W| \leq 6f_{\text{short}}/\mathbf{c}_5$.*

409 *Proof.* Suppose that $(1/6)(\sum_{u \in W} d(u)) \leq f_{\text{long}}$. The obvious inequality
 410 $|F_W| \leq \sum_{u \in W} d(u)$ then implies that $|F_W| \leq 6 \cdot f_{\text{long}}$. The required in-
 411 equality follows from (2). \square

412 **SUBCLAIM D** *If $(1/6)(\sum_{u \in W} d(u)) > 12|W| + 2t + 2|E_W| - 12$ and*
 413 *$(1/6)(\sum_{u \in W} d(u)) > f_{\text{long}}$, then $|F_W| \leq \mathbf{c}_6 f_{\text{short}}$ or else Lemma 8 follows.*

414 *Proof.* We show that, under the given hypotheses, if $|F_W| > \mathbf{c}_6 f_{\text{short}}$, then
 415 there are at least $(\mathbf{c}_3 \mathbf{c}_6/3)|F|$ (ℓ_0, Δ_0) -edges; the subclaim then follows from
 416 Claim 9, since $(\mathbf{c}_3 \mathbf{c}_6)/3 \geq (2\ell_0(2\ell_0 + 1) + 1) \cdot \mathbf{z}_1$.

417 It follows that, under the current hypotheses,

$$(4) \quad f_{\text{long}} < (1/3) \sum_{u \in W} d(u) - 12|W| - 2t - 2|E_W| + 12.$$

Since $|F_1 \setminus F_{\text{long}}| \geq f_1 - f_{\text{long}}$, using (3) and (4) we obtain

$$|F_1 \setminus F_{\text{long}}| \geq \sum_{u \in W} d(u) - 12|W| - 2|E_W| - 2t + 12 - f_{\text{long}} > (2/3) \sum_{u \in W} d(u).$$

418 Since each face in F_1 is (by definition) incident with exactly one ver-
 419 tex in W , the inequality $|F_1 \setminus F_{\text{long}}| > (2/3) \sum_{u \in W} d(u)$ implies that at least
 420 $1/3$ of the edges incident with W have their two incident faces in $F_1 \setminus F_{\text{long}}$.
 421 Note that all such edges are (ℓ_0, Δ_0) -edges. We conclude that there are
 422 at least $(1/3) \sum_{u \in W} d(u)$ (ℓ_0, Δ_0) -edges incident with W . Since obviously
 423 $\sum_{u \in W} d(u) \geq |F_W|$, this implies that there are at least $|F_W|/3$ (ℓ_0, Δ_0) -edges.

424 Using the assumption $|F_W| > \mathbf{c}_6 f_{\text{short}}$ and (1), it follows that there are at
 425 least $(\mathbf{c}_3 \mathbf{c}_6 / 3) |F|$ (ℓ_0, Δ_0) -edges, as required. \square

426 We now complete the proof of Claim 10.

427 Since the hypotheses of Subclaims B, C, and D are exhaustive, it fol-
 428 lows from these subclaims that either we may assume that $|F_W| < 24t +$
 429 $24\mathbf{c}_4 f_{\text{short}}$, or $|F_W| \leq 6f_{\text{short}}/\mathbf{c}_5$, or we may assume that $|F_W| \leq \mathbf{c}_6 f_{\text{short}}$.
 430 Since $\max\{24\mathbf{c}_4, 6/\mathbf{c}_5, \mathbf{c}_6\} = 24\mathbf{c}_4$, it follows that we may assume that $|F_W| <$
 431 $24t + 24\mathbf{c}_4 f_{\text{short}}$. \square

432 We now complete the proof of Lemma 8.

433 A face is *white* if it is either in $F_{\text{short}} \setminus F_W$ or has length exactly 2, and
 434 is *black* otherwise. We let F_\circ (respectively, F_\bullet) denote the set of all white
 435 (respectively, black) faces. Let $f_\circ := |F_\circ|$, and $f_\bullet := |F_\bullet|$.

436 Now consider the dual G° of G . The 2-connectivity of G implies that G°
 437 is also 2-connected. Let us say that an edge in G° is *white* if its endpoints
 438 are both white (faces in G).

439 The key (and completely straightforward) observation is that the edge of
 440 G associated to each white edge is an (ℓ_0, Δ_0) -edge. Our final goal is to
 441 prove that there are many white edges.

442 Every face in F_\bullet is either in F_{long} or in F_W , and so $f_\bullet \leq f_{\text{long}} + |F_W|$.
 443 Using (2), Claim 10, and the obvious inequality $f_{\text{short}} \leq |F|$, we obtain

$$(5) \quad f_\bullet \leq 24t + (24\mathbf{c}_4 + 1/\mathbf{c}_5)|F|.$$

444 By Lemma 7, G° has at most $3f_\bullet + t$ branches with both endpoints black.
 445 Lemma 3 (applied to G°) then implies that there are at least $|F|/2 - (3f_\bullet +$
 446 $t)/2 - (3/2)f_\bullet = |F|/2 - 3f_\bullet - t/2 \geq (1/2 - 3(24\mathbf{c}_4 + 1/\mathbf{c}_5))|F| - (145/2)t$
 447 white edges.

448 As we have observed, the edge of G associated to each white edge is an
 449 (ℓ_0, Δ_0) -edge. Thus there are at least $(1/2 - 3(24\mathbf{c}_4 + 1/\mathbf{c}_5))|F| - (145/2)t$
 450 (ℓ_0, Δ_0) -edges. Since $1/2 - 3(24\mathbf{c}_4 + 1/\mathbf{c}_5) \geq (2\ell_0(2\ell_0 + 1) + 1) \cdot \mathbf{z}_1$ and
 451 $145/2 \leq (2\ell_0(2\ell_0 + 1) + 1) \cdot \mathbf{c}_2$, then we are done by Claim 9. \square

452 4. EARRINGS IN NONPLANAR GRAPHS

453 **Lemma 11.** *Let $G = (V, E)$ be a 2-connected irreducible graph. Then G*
 454 *has at least $10^{-10}|E| - (10^{-5} + 2)\text{cr}(G)$ pairwise edge-disjoint $(5000, 500)$ -*
 455 *earrings.*

456 *Proof.* Let $\ell_0 := 5000$, $\Delta_0 := 500$, $\mathbf{c}_1 := 10^{-10}$, $\mathbf{c}_2 := 10^{-5}$, and $\mathbf{c}_7 :=$
 457 $(10^{-5} + 2)$. Let $t := \text{cr}(G)$, and let \mathcal{D} be a drawing of G with exactly t
 458 crossings. Let H denote the plane graph that results by regarding the t
 459 crossings as degree 4 vertices (this is the *crossings-to-vertices conversion*),
 460 which we colour green to help comprehension (the other vertices of H , each
 461 of which corresponds to a vertex in G , are coloured black). We claim that
 462 (i) each vertex in H is adjacent to at least 3 other vertices; (ii) no green

463 vertex is part of a 2-vertex-cut; (iii) H is 2-connected; and (iv) the set of
 464 green vertices is an anchor set for H .

465 We start by noting that (i) follows easily from the irreducibility of G , plus
 466 the observation that in any crossing-minimal drawing of any graph, the two
 467 edges involved in any crossing cannot have a common endpoint.

468 By way of contradiction, suppose that u, v are green vertices such that
 469 $\{u, v\}$ is a 2-vertex-cut in H . It is easy to see that then there are exactly two
 470 uv -bridges. Let B be any of these uv -bridges, and let H' denote the plane
 471 graph obtained from H by performing a Whitney switching on B around
 472 u and v . Now by reversing the crossings-to-vertices conversion, we obtain
 473 from H' a drawing of G in which the edge intersections corresponding to
 474 u and v are tangential, not crossings. Each of these two tangential edge
 475 intersections may be removed with a small perturbation, yielding a drawing
 476 of G with two fewer crossings than \mathcal{D} , contradicting the crossing-minimality
 477 of \mathcal{D} . This contradiction shows that $\{u, v\}$ cannot be a 2-vertex-cut in H .
 478 A similar contradiction is obtained from the assumption that H has a 2-
 479 vertex-cut with exactly one green vertex (in this case one obtains a drawing
 480 of G with one fewer crossing than \mathcal{D}). This proves (ii).

481 The 2-connectedness of G readily implies that no black vertex can be a
 482 cut vertex of H . On the other hand, a similar switching argument as in
 483 the proof of (ii) shows that no green vertex can be a cut vertex of H . This
 484 proves (iii).

485 Now let u, v be black vertices such that $\{u, v\}$ is a 2-vertex-cut in H , and
 486 let B be a nontrivial uv -bridge. If B does not contain any green vertex, then
 487 (B, u, v) is clearly a uv -blob of G . Since this contradicts the irreducibility
 488 of G , (iv) follows.

489 We can thus apply Lemma 8 to H , and obtain that H has a collection \mathcal{E} of
 490 at least $\mathbf{c}_1|E(H)| - \mathbf{c}_2t$ pairwise edge-disjoint (ℓ_0, Δ_0) -earrings. If any such
 491 earring contains a green vertex, then it obviously contains at least two edges
 492 incident with a green vertex. Since these earrings are pairwise edge-disjoint,
 493 it immediately follows that \mathcal{E} has a subcollection \mathcal{E}' , with $|\mathcal{E}'| \geq |\mathcal{E}| - 2t$
 494 pairwise edge-disjoint (ℓ_0, Δ_0) -earrings that do not contain any green vertex.
 495 That is, each earring in \mathcal{E}' is an (ℓ_0, Δ_0) -earring of G .

496 Therefore, \mathcal{E}' is a collection at least $|\mathcal{E}| - 2t \geq \mathbf{c}_1|E(H)| - (\mathbf{c}_2 + 2)t$
 497 pairwise edge-disjoint (ℓ_0, Δ_0) -earrings in G . Since $|E(H)| \geq |E|$, it follows
 498 that $|\mathcal{E}'| \geq \mathbf{c}_1|E| - (\mathbf{c}_2 + 2)t = \mathbf{c}_1|E| - \mathbf{c}_7t$. \square

499 5. THE EMBEDDING METHOD: ADDING EDGES WITH FEW CROSSINGS

500 Our main goal is to show that every (sufficiently large) irreducible graph
 501 has a large collection of edges whose removal leaves a graph with large cross-
 502 ing number. The first main ingredient is the existence of a large collection
 503 of pairwise edge-disjoint (ℓ, Δ) -earrings (for some fixed ℓ and Δ); this is
 504 Lemma 8. The second main ingredient is the *embedding method*, which was
 505 used under similar circumstances by Richter and Thomassen [22], Fox and

506 Tóth [10], and Černý, Kynčl and Tóth [5] (see also [13, 24, 26]). We use the
507 embedding method to prove the following.

508 **Lemma 12.** *Let G be a graph, and let ℓ, Δ , and r be positive integers.*
509 *Suppose that G has a collection of r pairwise edge-disjoint (ℓ, Δ) -earrings.*
510 *Then G has a set E_0 of r edges such that $\text{cr}(G - E_0) > (1/2)\text{cr}(G) -$
511 $(1/2)(\Delta\ell + \ell^2)r$.*

512 *Proof.* Let $\Xi_1, \Xi_2, \dots, \Xi_r$ be a collection of pairwise edge-disjoint (ℓ, Δ) -
513 earrings in G . For $i = 1, 2, \dots, r$, let $e_i = u_i v_i$ be the base edge of Ξ_i ,
514 and let P_i, Q_i be the $u_i v_i$ -paths such that $\Xi_i = P_i \cup Q_i \cup \{e_i\}$. We shall show
515 that $E_0 := \{e_1, e_2, \dots, e_r\}$ satisfies the required property.

516 Let $t := \text{cr}(G - E_0)$, and let \mathcal{D} be a drawing of $G - E_0$ with t crossings.
517 The strategy is to extend \mathcal{D} to a drawing of G by drawing e_i very close to
518 either P_i or Q_i , for $i = 1, 2, \dots, r$. Our aim is to show that this can be done
519 while adding relatively few crossings.

520 We analyze several types of crossings of P_i and Q_i , for $i = 1, 2, \dots, r$. A
521 crossing in \mathcal{D} is (i) *of Type 1* if one edge is in P_i and the other edge is in
522 Q_i , for some $i \in \{1, \dots, r\}$; (ii) *of Type 2A* if one edge is in $P_i \cup Q_i$ and
523 the other edge is in $P_j \cup Q_j$, for some $i \neq j$, $i, j \in \{1, \dots, r\}$; and (iii) *of*
524 *Type 2B* if one edge is in $P_i \cup Q_i$ for some $i \in \{1, \dots, r\}$ and the other
525 in $E(G) \setminus \bigcup_{j=1}^r (P_j \cup Q_j)$. Note that if a crossing \times involving an edge of
526 $\bigcup_{i=1}^r P_i \cup Q_i$ is neither of Type 1, nor 2A, nor 2B, then the edges involved
527 in \times must be both in P_i or both in Q_i , for some $i \in \{1, 2, \dots, r\}$. As we shall
528 see, this last type of crossing is irrelevant to our discussion.

529 For $i = 1, 2, \dots, r$ and $k \in \{1, 2\}$, let $\chi_k(P_i)$ (respectively, $\chi_k(Q_i)$) denote
530 the number of crossings of Type k that involve an edge in P_i (respectively,
531 Q_i).

532 In every crossing-minimal drawing of any graph, no pair of edges cross
533 each other more than once. Since each of P_i and Q_i has at most ℓ edges, it
534 follows that

$$(6) \quad \chi_1(P_i) \leq \ell^2, \text{ for } i = 1, \dots, r.$$

535 Now let \mathcal{R} be the set of all sequences (R_1, R_2, \dots, R_r) , with $R_i \in \{P_i, Q_i\}$
536 for $i = 1, 2, \dots, r$, and consider the sum $\Sigma := \sum_{R \in \mathcal{R}} (\sum_{i=1}^r \chi_2(R_i))$.

537 We claim that a crossing of Type 2A contributes in exactly 2^r to Σ . To see
538 this, first note that such a crossing involves an edge of an $R_i \in \{P_i, Q_i\}$ and
539 an edge of an $R_j \in \{P_j, Q_j\}$ for some $i \neq j$. Let T_i (respectively, T_j) be the
540 element in $\{P_i, Q_i\} \setminus R_i$ (respectively, $\{P_j, Q_j\} \setminus R_j$). There are 2^{r-2} sequences
541 in \mathcal{R} that include both R_i and R_j , and so for each such sequence, the crossing
542 contributes in 2 to Σ . There are 2^{r-2} sequences in \mathcal{R} that include R_i and do
543 not include R_j , and so for each such sequence, the crossing contributes in 1
544 to Σ . Analogously, there are 2^{r-2} sequences in \mathcal{R} that include R_j and do not
545 include R_i , and so for each such sequence, the crossing contributes in 1 to Σ .
546 Therefore each crossing of Type 2A contributes in $2 \cdot 2^{r-2} + 2^{r-2} + 2^{r-2} = 2^r$
547 to Σ , as claimed. Note that this reasoning assumes that no crossing of Type

548 $2A$ is in both P_i and Q_i for the same i . This is immediate if P_i and Q_i are
 549 edge-disjoint, but we recall from our definition of earring that P_i and Q_i may
 550 share edges. However, the validity of our reasoning follows since (again, by
 551 the definition of earring) any edge $f \in E(P_i) \cap E(Q_i)$ is a cut edge of $G - e_i$,
 552 from which it follows that f cannot be crossed in any optimal drawing of
 553 $G - E_0$.

554 We also note that a crossing of Type 2B contributes to Σ in exactly 2^{r-1} .
 555 Indeed, such a crossing involves (for some fixed i) an edge of R_i and an
 556 edge that belongs to no R_j ; it contributes in 1 to $\chi(R_i)$, and there are 2^{r-1}
 557 sequences in \mathcal{R} that include R_i . (As in the previous paragraph, we remark
 558 that we are making use of the valid assumption that no crossing is in both
 559 P_i and Q_i for the same i).

560 In conclusion, each crossing of Type 2A or 2B contributes to Σ in at
 561 most 2^r . Since only crossings of Types 2A and 2B contribute to Σ , and
 562 \mathcal{D} has t crossings in total, we conclude that $\sum_{R \in \mathcal{R}} (\sum_{i=1}^r \chi_2(R_i)) \leq 2^r t$.
 563 Since $|\mathcal{R}| = 2^r$, it follows that for some sequence $(R_1, R_2, \dots, R_r) \in \mathcal{R}$,
 564 $\sum_{i=1}^r \chi_2(R_i) \leq t$. By relabeling (exchanging) P_i and Q_i if necessary, we may
 565 assume without any loss of generality that $R_i = P_i$ for each $i = 1, 2, \dots, r$,
 566 and so

$$(7) \quad \sum_{i=1}^r \chi_2(P_i) \leq t.$$

567 Now note that some P_i may have self-crossings. However, for each i there
 568 is a simple curve α_i , contained in P_i , joining u_i and v_i . The definition
 569 of crossings of types 1, 2A, and 2B obviously extend to the crossings on
 570 each α_i , and so (6) and (7) imply that $\chi_1(\alpha_i) \leq \ell^2$ for $i = 1, 2, \dots, r$, and
 571 $\sum_{i=1}^r \chi_2(\alpha_i) \leq t$. Moreover (this is the effect of having obtained α_i by
 572 avoiding the self-crossings of its corresponding P_i), for $i = 1, 2, \dots, r$, each
 573 crossing of α_i is of one of these types.

574 The idea is to draw each e_i very close to its corresponding α_i . There are
 575 two kinds of crossings on the resulting drawings of e_i , $i = 1, \dots, r$. Some
 576 crossings occur as we traverse e_i and pass very close to a crossing of α_i .
 577 The inequalities in the previous paragraph imply that there are, in total, at
 578 most $\ell^2 r + t$ crossings of this first kind. The second kind of crossing occurs
 579 as we pass very close to a vertex in α_i , and cross some edges incident with
 580 this vertex. Since each such vertex is an internal vertex of some P_i (that is,
 581 has degree $< \Delta$) and there are at most $\ell - 1$ internal vertices in each P_i , we
 582 conclude that each e_i has fewer than $\Delta \ell$ crossings of this second kind. Thus
 583 in total there are fewer than $\Delta \ell r$ crossings of the second kind.

584 We conclude that all the edges e_1, e_2, \dots, e_r may be added to the drawing
 585 \mathcal{D} of $G - E_0$ by introducing fewer than $(\Delta \ell + \ell^2)r + t$ crossings. Since $t =$
 586 $\text{cr}(G - E_0)$, it follows that $\text{cr}(G) < 2\text{cr}(G - E_0) + (\Delta \ell + \ell^2)r$ or, equivalently,
 587 $\text{cr}(G - E_0) > (1/2)\text{cr}(G) - (1/2)(\Delta \ell + \ell^2)r$. \square

588 If we are interested in removing only one edge (as we are in Theorem 2),
 589 we can improve the $1/2$ coefficient in Lemma 12 to $2/3$, as the following
 590 statement shows.

591 **Lemma 13.** *Let G be a graph, and let ℓ and Δ be positive integers. Suppose*
 592 *that G has an (ℓ, Δ) -earring. Then G has an edge e such that $\text{cr}(G - e) >$*
 593 *$(2/3)\text{cr}(G) - (2/3)(\Delta\ell + \ell^2)$.*

594 *Proof.* The proof is essentially the same as the proof of Lemma 13, with
 595 the following favourable exception. If we consider only one earring, then
 596 $r = 1$, and so there are no crossings of Type 2A. Each crossing of Type 2B
 597 contributes to Σ in at most 1, and so $\chi_2(P_1) + \chi_2(Q_1) \leq t$. By exchanging
 598 P_1 and Q_1 if necessary, we may assume that $\chi_2(P_1) \leq t/2$.

599 In parallel to the last paragraph of the proof of Lemma 13, in the present
 600 case we conclude that the edge e_1 may be added to the drawing \mathcal{D} of $G - E_0 =$
 601 $G - e_1$ by introducing fewer than $(\Delta\ell + \ell^2) + t/2$ crossings. Since $t =$
 602 $\text{cr}(G - e_1)$, it follows that $\text{cr}(G) < (3/2)\text{cr}(G - e_1) + \Delta\ell + \ell^2$ or, equivalently,
 603 $\text{cr}(G - e_1) > (2/3)\text{cr}(G) - (2/3)(\Delta\ell + \ell^2)$. \square

604

6. PROOF OF THEOREMS 1 AND 2

605 *Proof of Theorem 1.* Let $\ell_0 := 5000$ and $\Delta_0 := 500$, $\mathbf{c}_1 := 10^{-10}$, and
 606 $\mathbf{c}_7 := (10^{-5} + 2)$. Let k be a positive integer and let $\epsilon > 0$. Define
 607 $\gamma := \epsilon / ((1/2)(\Delta_0\ell_0 + \ell_0^2))$ and $m_0 := ((\mathbf{c}_7 + \gamma)k) / \mathbf{c}_1$. Let $G = (V, E)$ be
 608 a 2-connected irreducible graph with $\text{cr}(G) = k$ and at least m_0 edges.

609 Lemma 11 implies that G has a collection of at least $\mathbf{c}_1|E| - \mathbf{c}_7k$ pairwise
 610 edge-disjoint (ℓ_0, Δ_0) -earrings. Since $|E| \geq ((\mathbf{c}_7 + \gamma)k) / \mathbf{c}_1$, it follows that
 611 G has a collection of at least γk pairwise edge-disjoint (ℓ_0, Δ_0) -earrings.
 612 Thus, by Lemma 12, G has a collection E_0 of at least γk edges such that
 613 $\text{cr}(G - E_0) > (1/2)\text{cr}(G) - (1/2)(\Delta_0\ell_0 + \ell_0^2)\gamma k = (1/2)\text{cr}(G) - \epsilon k = ((1/2) -$
 614 $\epsilon)\text{cr}(G)$. \square

615 If u, v are vertices of a graph G , a *double uv -path* is a subgraph of G that
 616 consists of a uv -path with all its edges doubled.

617 *Proof of Theorem 2.* Let $\ell_0 := 5000$, $\Delta_0 := 500$, $\mathbf{c}_1 := 10^{-10}$, and $\mathbf{c}_7 :=$
 618 $(10^{-5} + 2)$. Let k be a positive integer, and let $m_1 := (\mathbf{c}_7k) / \mathbf{c}_1 + 1$. We
 619 prove that if $G = (V, E)$ is a 2-connected graph in which each vertex is
 620 adjacent to at least 3 vertices, $\text{cr}(G) = k$, and G has at least m_1 edges, then
 621 G has an edge e such that $\text{cr}(G - e) > (2/3)\text{cr}(G) - 10^8$.

622 Suppose first that G is not irreducible, and let (B, u, v) be a minimal
 623 blob in G , (that is, G has no blob (B', u', v') such that B' is a subgraph of
 624 B). The minimality of B implies that B has no cut edges, and so its width
 625 $w(B)$ is at least 2. It is easy to see that if every edge of B is in a 2-edge-cut
 626 separating u and v , then B is a double uv -path. This clearly contradicts the
 627 X -minimality of G , and so we conclude that there is an edge e in B such
 628 that the uv -blob (in $G - e$) $B - e$ has width at least 2.

629 By way of contradiction, suppose that $\text{cr}(G - e) < (2/3)\text{cr}(G)$. It is
 630 straightforward to see that there is a crossing-minimal drawing \mathcal{D} of $G - e$
 631 in which the set E' of edges crossed in $B - e$ form a smallest *uv-edge cut* (that
 632 is, a minimum size edge cut in $B - e$ separating u and v), with each edge
 633 in E' crossed the same number (say s) of times. In particular, $\text{cr}(G - e) \geq$
 634 $|E'|s \geq 2s$. The planarity of $B - e$ (with u, v in the same face) implies that:
 635 (i) if e is in distinct components of $(B - e) - E'$, then e can be added to \mathcal{D}
 636 by introducing exactly s crossings; and (ii) otherwise, e can be added to \mathcal{D}
 637 without introducing any crossings. In either case, the result is a drawing of
 638 G with at most $\text{cr}(G - e) + s$ crossings, and so $\text{cr}(G) \leq \text{cr}(G - e) + s$. The
 639 assumption $\text{cr}(G) > (3/2)\text{cr}(G - e)$ then implies $\text{cr}(G - e) < 2s$, contradicting
 640 that $\text{cr}(G - e) \geq 2s$. Thus $\text{cr}(G - e) \geq (2/3)\text{cr}(G) > (2/3)\text{cr}(G) - 10^8$.

641 Suppose finally that G is irreducible. Lemma 11 then implies that G has
 642 at least $\mathbf{c}_1|E| - \mathbf{c}_7k$ pairwise edge-disjoint (ℓ_0, Δ_0) -earrings. Since $|E| \geq$
 643 $(\mathbf{c}_7k)/\mathbf{c}_1 + 1$, it follows that G has at least one (ℓ_0, Δ_0) -earring. Thus, by
 644 Lemma 13, G has an edge e such that $\text{cr}(G - e) > (2/3)\text{cr}(G) - (2/3)(\Delta\ell + \ell^2) >$
 645 $(2/3)\text{cr}(G) - 10^8$. \square

646 7. BOUNDED DECAY AND EXPECTED CROSSING NUMBERS

647 The pioneering work of Richter and Thomassen, as well as our work in
 648 this paper, are naturally described as “bounded decay” results: the existence
 649 of sets of edges whose removal does not decrease arbitrarily the crossing
 650 number. The papers by Fox and Tóth [10] and by Černý, Kynčl and Tóth [5]
 651 concern themselves with “almost no decay” results: the existence of sets of
 652 edges whose removal results in a very small decrease of the crossing number.

653 As an additional motivation to bounded decay results, we discuss in this
 654 section a connection with expected crossing numbers, a concept recently
 655 introduced by Mohar and Tamon [18, 19].

656 7.1. Expected crossing numbers and decay of crossing numbers.

657 Given a drawing \mathcal{D} of a graph $G = (V, E)$, and a *weight function* $w :$
 658 $E \rightarrow \mathbb{R}_+$, define the *crossing weight* $\text{cr}(\mathcal{D}, w)$ as $\sum_{\{e, f\} \in \mathbb{X}(\mathcal{D})} w(e)w(f)$, where
 659 $\mathbb{X}(\mathcal{D})$ is the set of all pairs of edges that cross each other in \mathcal{D} . The pair
 660 (G, w) is a *weighted graph*, and the *weighted crossing number* of (G, w) is
 661 $\text{cr}(G, w) := \min_{\mathcal{D}} \text{cr}(\mathcal{D}, w)$, where the minimum is taken over all drawings \mathcal{D}
 662 of G . Now take the weights on the edges to be independently identically dis-
 663 tributed random variables, with uniform distributions on the interval $[0, 1]$.
 664 The expected value of $\text{cr}(G, w)$ under this distribution is the *expected cross-*
 665 *ing number* of G , and is denoted $\mathbb{E}(\text{cr}(G))$.

666 Let us say that a family \mathcal{G} of graphs is *robust* (or, more precisely, *ϵ -robust*)
 667 if there exist a constant $\epsilon := \epsilon(\mathcal{G})$ and an $n(\mathcal{G})$ such that $\mathbb{E}(\text{cr}(G)) \geq \epsilon \cdot \text{cr}(G)$
 668 for every graph G in \mathcal{G} with at least $n(\mathcal{G})$ vertices.

669 Mohar and Tamon proved in [18] that $\mathbb{E}(\text{cr}(K_n))$ is $\Theta(n^4)$. From this it
 670 follows immediately that the family of all complete graphs is robust. More-
 671 over, it follows from their Crossing Lemma for Expectations (Theorem 5.2

672 in [18]) that for each fixed $\gamma > 0$, the family of graphs with at least $\gamma \cdot n^2$
 673 edges is also robust (more precisely, ϵ -robust, where ϵ might depend on γ). It
 674 is thus natural to inquire about the robustness of families of sparser graphs.

675 Our aim in this subsection is to unveil and exploit the close connection
 676 between robustness and several results and conjectures, presented in [5], on
 677 the decay of crossing numbers.

678 In [5], Černý, Kynčl and Tóth proved the following: for each $\epsilon > 0$, there
 679 exist $\delta, \gamma > 0$ such that every sufficiently large graph G with n vertices and
 680 $m \geq n^{1+\epsilon}$ edges has a subgraph G' with at most $(1 - \delta)m$ edges such that
 681 $\text{cr}(G') \geq \gamma \cdot \text{cr}(G)$. This impressive “almost no decay” statement is best
 682 possible, in the sense that (as shown in [5]) one cannot require that *every*
 683 subgraph with $(1 - \delta)m$ edges has crossing number at least $\gamma \cdot \text{cr}(G)$. In this
 684 vein, Černý, Kynčl and Tóth also investigated the following closely related
 685 problem.

686 Let us say that a family \mathcal{G} of graphs is *stable* (or, more precisely, (δ, γ) -
 687 *stable*) if there exist positive constants $\delta := \delta(\mathcal{G})$, $\gamma := \gamma(\mathcal{G})$, and $n(\mathcal{G})$
 688 such that for every graph $G \in \mathcal{G}$ with at least $n(\mathcal{G})$ vertices (and m edges),
 689 a positive fraction of all subgraphs of G with $(1 - \delta)m$ edges has crossing
 690 number at least $\gamma \cdot \text{cr}(G)$. The requirement may be equivalently formulated as
 691 follows: if G' is a random subgraph of G obtained by deleting independently
 692 each edge with probability δ , then w.h.p. $\text{cr}(G') \geq \gamma \cdot \text{cr}(G)$.

693 In the earlier version [4] of [5], it was conjectured that for each $\epsilon > 0$,
 694 the family of graphs with $\Theta(n^{1+\epsilon})$ edges is stable. In [5], it was shown that
 695 this is false for $\epsilon < 1/3$ (we have slightly refined the construction in [5],
 696 and shown that it does not hold either for $\epsilon = 1/3$; see Theorem 17). The
 697 conjecture remains open for denser graphs:

698 **Conjecture 14.** *There exists an $\bar{\epsilon} \in (1/3, 1)$ such that, for each $\epsilon \in (\bar{\epsilon}, 1]$,*
 699 *the family of graphs with $\Theta(n^{1+\epsilon})$ edges is stable.*

700 (See also a weaker version put forward in [5]).

701 Before moving on to explore the close relationship between Conjecture 14
 702 and the robustness of dense graphs, we note the stability of random graphs:

703 **Remark 15.** *The family of all random graphs $G(n, p)$ with $p > 2/n$, is*
 704 *stable.*

705 *Proof.* We start by noting that $\mathbb{E}(\text{cr}(G(n, p))) \leq p^2 \text{cr}(K_n) \leq (1/10)p^2 n^4$.
 706 From the other side, Spencer and G. Tóth ([25], Section 4) proved that there
 707 is a $c > 0$ such that for n sufficiently large the lower bound $\mathbb{E}(\text{cr}(G(n, 2/n))) >$
 708 cn^2 holds. Standard sparsening of $G(n, p)$ (keeping each edge with proba-
 709 bility $2/(pn)$) gives that for $p > 2/n$, $\mathbb{E}(\text{cr}(G(n, p))) > (c/4)p^2 n^4$. Using
 710 these bounds, together with the observation that if each edge of a $G(n, p)$
 711 is removed with probability ϵ then we obtain a $G(n, (1 - \epsilon)p)$, the remark
 712 follows. \square

713 The key connection between expected crossing number (robustness) and
 714 the decay of crossing numbers (stability) is the following observation:

715 **Proposition 16.** *If a family \mathcal{G} of graphs is stable, then it is robust. More*
 716 *precisely: if \mathcal{G} is (δ, γ) -stable, then it is $\delta^2\gamma$ -robust.*

717 *Proof.* Suppose that \mathcal{G} is a (δ, γ) -stable family of graphs. Let G be a (suffi-
 718 ciently large) graph in \mathcal{G} , and let w be a random weight assignment (sampled
 719 from the uniform distribution) on the edges of G . Our aim is to show that
 720 the expected value of $\text{cr}(G, w)$ is at least $\delta^2\gamma \cdot \text{cr}(G)$.

721 Let G' be the subgraph of G that results by deleting the edges that re-
 722 ceive a weight smaller than δ under w . Let \mathcal{D} be a drawing of G that
 723 minimizes $\text{cr}(G, w)$, and let \mathcal{D}' be the restriction of G to G' . Clearly \mathcal{D}' has
 724 at most $\text{cr}(G, w)/\delta^2$ crossings, and so $\text{cr}(G') \leq \text{cr}(\mathcal{D}') \leq \text{cr}(G, w)/\delta^2$. Thus
 725 $\text{cr}(G, w) \geq \delta^2 \text{cr}(G')$.

726 Note that G' may be equivalently regarded as a graph obtained from G by
 727 deleting each edge independently with probability δ . Since \mathcal{G} is (δ, γ) -stable,
 728 it follows that w.h.p. $\text{cr}(G') \geq \gamma \cdot \text{cr}(G)$. Therefore the expected value of
 729 $\text{cr}(G, w)$ is at least $\delta^2\gamma \cdot \text{cr}(G)$, as required. \square

730 We now proceed with a concrete illustration of how the results and tech-
 731 niques on the decay of crossing numbers (specifically, those developed in [5])
 732 find an immediate application in expected crossing numbers.

733 As we observed above, Černý, Kynčl and Tóth [5] proved that, for each
 734 $\epsilon \in (0, 1/3)$, the family of graphs with $\Theta(n^{1+\epsilon})$ edges is *not* stable. We have
 735 slightly refined the construction in [5], and extended it to cover the case
 736 $\epsilon = 1/3$.

Theorem 17 (Non-stability of graphs with $\Theta(n^{4/3})$ edges). *For every $\delta, \gamma >$
 0 there exist $c := c(\delta, \gamma)$ and $n_0 := n_0(\delta, \gamma)$ such that there exist infinitely
 many graphs G with $n > n_0$ vertices and $c \cdot n^{4/3} < m < n^{4/3}$ edges, that
 satisfy the following. If G' is a random subgraph of G obtained by deleting
 independently each edge with probability δ , then w.h.p.*

$$\text{cr}(G') < \gamma \cdot \text{cr}(G).$$

737 We omit the proof of this result, since it closely resembles the proof of
 738 our next statement. Theorem 18 shows the non-robustness of graphs with
 739 $\Theta(n^{4/3})$ edges, and illustrates how the non-stability results and techniques
 740 in [5] can be extended to prove the non-robustness of graphs with $\Theta(n^{1+\epsilon})$
 741 edges for each $\epsilon \in (0, 1/3)$.

Theorem 18 (Non-robustness of graphs with $\Theta(n^{4/3})$ edges). *For every
 $\gamma > 0$ there exist $c := c(\gamma)$ and $n_0 := n_0(\gamma)$ such that there are infinitely
 many graphs G with $n > n_0$ vertices and $c \cdot n^{4/3} < m < n^{4/3}$ edges, and*

$$\mathbb{E}(\text{cr}(G)) < \gamma \cdot \text{cr}(G).$$

742 *Proof.* For readability purposes, we shall omit explicitly taking the integer
 743 part of several quantities involved. The integrality requirement will be, in
 744 every case, obvious from the context.

745 We may assume without loss of generality that γ is small enough so that
 746 $e^{-1200/\gamma} < \gamma/720$. Let $\alpha := \gamma/600$, $c := \alpha^2/100$, $r := \alpha^2 n^{1/3}/5$, $s := 1/\alpha^2$,
 747 and $t := \sqrt{n/s}$. Note that obviously $r > 5cn^{1/3}$.

748 Inspired by the construction in [5], G will be the disjoint union of two
 749 graphs G_1 and G_2 plus some isolated vertices. Let G_1 be $n/2r$ copies of the
 750 complete graph K_r . Clearly $|V(G_1)| \leq n/2$. Now let G_2 be obtained from a
 751 complete graph K_t by subdividing each edge $s-1$ times, i.e. replacing each
 752 edge by a path with s edges (these length s paths are the *branches*). It is
 753 easy to check that $|V(G_2)| \leq n/2$. Furthermore,

$$(8) \quad \alpha^4 n^2 = t^4 > \text{cr}(G_2) > \frac{t^4}{100} = \frac{n^2}{100s^2} = \frac{\alpha^4 n^2}{100},$$

754 where the inequalities $t^4 > \text{cr}(G_2) > t^4/100$ are easily derived bounds for
 755 the crossing number of the complete graph on t vertices.

756 Now let w be a random weight assignment on the edges of G . Let $E_{<\alpha}$
 757 denote the set of edges of G that receive a weight smaller than α under
 758 w . Let us say that a branch is *weak* if at least one of its edges is in $E_{<\alpha}$;
 759 otherwise the branch is *strong*.

The probability that any fixed branch is strong is

$$(1 - \alpha)^s \approx e^{-\alpha s} = e^{-1/\alpha}.$$

760 Using Chernoff's bound, w.h.p. at most $t^2 e^{-1/\alpha}$ branches are strong. That
 761 is, w.h.p. at least $\binom{t}{2} - t^2 e^{-1/\alpha} \approx t^2(1/2 - e^{-1/\alpha})$ branches are weak.

762 Now consider the drawing of G_2 in which the t vertices of degree $t-1$ are
 763 in convex position, and the edges are the straight segments joining them.
 764 This drawing of G_2 has $\binom{t}{4} \approx t^4/24$ crossings (this is by no means a crossing-
 765 minimal drawing of G_2 , but it is enough for our purposes). Moreover, by
 766 adjusting the drawing of each branch if needed, we may ensure that each
 767 branch is crossed in exactly one edge, namely the edge with smallest weight.
 768 It follows that the number of crossings involving two strong branches (and
 769 thus, in particular, the number of crossings of weight $\geq \alpha$) is w.h.p. at most
 770 $(t^2 e^{-1/\alpha})^2$, and so w.h.p.

$$(9) \quad \begin{aligned} \text{cr}(G_2, w) &< t^4 e^{-2/\alpha} + \alpha \cdot t^4(1/24 - e^{-2/\alpha}) < t^4(\alpha/24 + e^{-2/\alpha}) \\ &< 100\text{cr}(G_2)(\alpha/24 + e^{-2/\alpha}) \leq 5\alpha \cdot \text{cr}(G_2), \end{aligned}$$

771 where for this last inequality we used that $e^{-1200/\gamma} = e^{-2/\alpha} < \gamma/720 =$
 772 $(5/6)\alpha$.

We finally move on to G . First we note that

$$|E(G)| = |E(G_1)| + |E(G_2)| \geq |E(G_2)| = (n/2r)r(r-1)/2 > nr/5 > cn^{4/3}.$$

773 Using (8), we obtain

$$(10) \quad \text{cr}(G) = \text{cr}(G_1) + \text{cr}(G_2) > \text{cr}(G_2) > \alpha^4 n^2 / 100.$$

774 From the other side, using (8) and (9) and the trivial bound $\text{cr}(K_r) \leq r^4$,
 775 we get

$$(11) \quad \text{cr}(G, w) \leq \text{cr}(G_1) + \text{cr}(G_2, w) \leq (n/2r)r^4 + 5\alpha^5 n^2 < 6\alpha^5 n^2,$$

776 where for the last inequality we used the (easily checked) inequality $(n/2r)r^4 <$
 777 $\alpha^5 n^2$.

Finally, using (10) and (11) and recalling that $\alpha = \gamma/600$, we obtain

$$\text{cr}(G, w) < 6\alpha^5 n^2 = (600\alpha)(\alpha^4 n^2/100) < \gamma \cdot \text{cr}(G),$$

778 as required. \square

779 We close this subsection with two constructions that further illustrate
 780 the discrepancy between the crossing number of a graph and its expected
 781 crossing number.

782 First we describe a construction that highlights the fact that the crossing
 783 number (of a family of graphs) may grow with the number of vertices, and yet
 784 the expected crossing number (of all graphs in the family) may be bounded
 785 by an absolute constant. For any graph G , let $n(G)$ and $m(G)$ denote the
 786 number of vertices and edges of G , respectively, and let $s \cdot G$ the graph that
 787 consists of s disjoint copies of G . Let $K_5(t)$ denote the graph obtained by
 788 replacing each edge of K_5 with a path of length t (a *branch*). Trivially, for
 789 any positive integer s , $n(s \cdot K_5(t)) = s(10(t-1) + 5) = 10st - 5s$, $m(s \cdot$
 790 $K_5(t)) = 10st$, and $\text{cr}(s \cdot K_5(t)) = s$. However, the weighted crossing number
 791 of $K_5(t)$ is $\min w(e)w(f)$, where the minimum is taken over all pairs of
 792 edges e, f that lie on branches that correspond to nonincident edges. A
 793 fairly standard calculation shows that $\mathbb{E}(\text{cr}(s \cdot K_5(t))) \leq (s/t^2) \log^2 s$. It is
 794 worthwhile to explore the consequences of plugging in various values of s .
 795 Probably the most interesting case occurs when $s = n^{2/3}/\log n$, for this
 796 shows the following:

797 **Proposition 19.** *There exists an infinite family of graphs G with crossing*
 798 *number $n^{2/3}/\log n$ and expected crossing number at most 1. \square*

799 Our final construction pertains a family of graphs that seem more natural
 800 than the graphs constructed above. We recall that $C_3 \square C_n$ denotes the
 801 Cartesian product of the cycles of sizes 3 and n (see Figure 1).

Proposition 20. *The Cartesian products $C_3 \square C_n$ satisfy*

$$\text{cr}(C_3 \square C_n) = n,$$

and yet

$$\mathbb{E}(\text{cr}(C_3 \square C_n)) \leq 2n^{2/3} \log^{1/3} n + 3.$$

802 *Proof.* The vertices of $C_3 \square C_n$ can be labeled $v_{i,j}$, $0 \leq i \leq 2$, $0 \leq j \leq n-1$,
 803 so that there is an edge joining $v_{i,j}$ and $v_{i',j'}$ if and only if either (i) $j = j'$
 804 and $|i - i'| = 1$ or (ii) $i = i'$ and $|j - j'| = 1$ (indices are modulo n). For
 805 $j = 0, 1, \dots, n-1$, let $V_j := \{v_{i,j} \mid i \in \{0, 1, 2\}\}$. That is, the V_j s are the
 806 vertex sets of the 3-cycles. For $j = 0, 1, \dots, n-1$, let $E(j)$ denote the set of
 807 (three) edges with an endpoint in V_j and another endpoint in V_{j+1} .

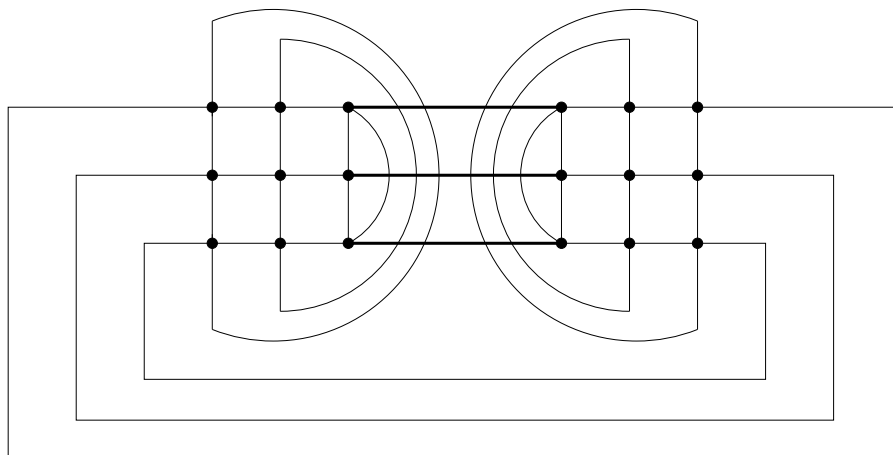


FIGURE 1. A drawing of $C_3 \square C_6$ with 14 crossings, where the thick edges are the edges of one particular $E(j)$. This is easily generalized to obtain, for every even integer $n \geq 2$, a (not crossing-minimal) drawing of $C_3 \square C_n$ with $3n - 4$ crossings with the following property: there exists a $j \in \{0, 1, 2, \dots, n - 1\}$ such that each crossing involves an edge in $E(j)$.

808

809 It is known that $\text{cr}(C_3 \square C_n) = n$ for every $n \geq 3$ [23]. In Figure 1 we
 810 depict how to produce a (not crossing-minimal) drawing of $C_3 \square C_n$ with
 811 $3n - 4$ crossings, for every even integer $n \geq 2$, with the following property:
 812 there is a $j \in \{0, 1, 2, \dots, n - 1\}$ such that every crossing involves an edge in
 813 $E(j)$ (the edges in $E(j)$ are the thick edges in Figure 1). Thus,

- 814 (A) if the edges in $C_3 \square C_n$ are weighted, and there exists a j such
 815 that the sum of the weights of the edges in $E(j)$ is r , then such a
 816 weighted $C_3 \square C_n$ has crossing number at most $r \cdot n$.

For $j = 0, 1, \dots, n - 1$, denote the weights of the edges in $E(j)$ by x_1^j, x_2^j, x_3^j . We have for $t \leq 1$ that $\Pr(x_1^j + x_2^j + x_3^j > t) = 1 - t^3/3!$. Using independence,

$$\Pr(\exists j : x_1^j + x_2^j + x_3^j \leq t) = 1 - (1 - t^3/6)^n \approx 1 - \exp[-nt^3/6].$$

817 Choosing $t = 6^{1/3}n^{-1/3} \log^{1/3} n$, this is at least $1 - 1/n$.

818 Now let $s := \min\{x_1^j + x_2^j + x_3^j \mid j \in \{0, 1, \dots, n - 1\}\}$. Thus $s \leq t$ with prob-
 819 ability at least $1 - 1/n$. In the complementary scenario (which occurs with
 820 probability $< 1/n$), s is obviously at most 3. Using this observation together
 821 with (A), it follows that $\mathbb{E}(\text{cr}(C_3 \square C_n)) < [(1 - 1/n)((6)^{1/3}n^{-1/3} \log^{1/3} n) +$
 822 $(1/n)3] \cdot n < 2n^{2/3} \log^{1/3} n + 3$. \square

823 **7.2. Concentration of the expected crossing number and the cross-**
 824 **ing number of randomly sparsened graphs.** Continuing in the theme
 825 of expected crossing numbers and its interplay with the decay of crossing
 826 numbers, we finally explore the concentration around the crossing number
 827 of a randomly sparsened graph, as well as the concentration around the
 828 expected crossing number of a graph.

829 Denote $R = R(G, p)$ the random graph obtained from G by randomly
 830 and independently removing edges, each with probability p . Using a stan-
 831 dard martingale concentration inequality we show that $\text{cr}(R)$ is concentrated
 832 around its mean. Let $E(G) = \{e_1, \dots, e_m\}$, and consider the random vari-
 833 able $\text{cr}(R)$ as a Doob's martingale, where the edges are exposed one by one.
 834 The length of the martingale is $|E(G)|$. Removing or adding an edge changes
 835 the crossing number by at most $|E(G)|$. Thus, by the Azuma-Hoeffding's
 836 inequality, for every $\lambda > 0$ we have

$$(12) \quad \Pr[|\mathbb{E}(\text{cr}(R)) - \text{cr}(R)| > \lambda] \leq \exp\left[\frac{-\lambda^2}{2|E(G)|^3}\right].$$

837 Let $\beta(n)$ be any function tending to infinity. Inequality (12) shows con-
 838 centration with radius $\lambda = \beta(n)|E(G)|^{3/2}$:

$$(13) \quad \Pr[|\mathbb{E}(\text{cr}(R)) - \text{cr}(R)| > \beta(n)|E(G)|^{3/2}] \leq \exp\left[\frac{-\beta(n)^2}{2}\right].$$

839 Similary, we can get concentration around the expected crossing number.
 840 Assign to each edge a random variable taking values from $[0, 1]$ (which could
 841 be different for each edge), which provides to each of them a random weight.
 842 Formally, it could be a function $w : E(G) \rightarrow \mathcal{F}$, where \mathcal{F} is a collection of
 843 random variables taking values from $[0, 1]$. Then $\mathbb{E}(\text{cr}(G, w))$ is the expected
 844 crossing number for a given w , and $\text{cr}(G, w)$ is a random variable, which is
 845 the crossing number of a weighted graph G . As with the random graph
 846 R above, resampling the weight of one edge changes the weighted crossing
 847 number by at most $|E(G)|$, and so we obtain:

$$(14) \quad \Pr[|\mathbb{E}(\text{cr}(G)) - \text{cr}(G, w)| > \beta(n)|E(G)|^{3/2}] \leq \exp\left[\frac{-\beta(n)^2}{2}\right].$$

848 These inequalities are meaningful only when G is dense enough, i.e. $|E(G)| \geq$
 849 $n^{5/4}$. Note that we could have obtained sharper concentration results for
 850 sparse graphs, under the assumption that removing any edge makes the
 851 crossing number drop by $o(|E(G)|)$.

852 8. CONCLUDING REMARKS

853 Lemma 8 falls into the realm of light subgraphs. We recall that the
 854 *weight* of a subgraph H of a graph G is the sum of the degrees (in G) of its
 855 vertices. For a class \mathcal{G} of graphs, define $w(H, \mathcal{G})$ as the smallest integer w

856 such that each graph $G \in \mathcal{G}$ which contains a subgraph isomorphic to H has
 857 a subgraph isomorphic to H of weight at most w . If $w(H, \mathcal{G})$ is finite then
 858 H is *light* in \mathcal{G} .

859 Fabrici and Jendrol' [8] proved that paths (and no other connected graphs)
 860 are light in the class of 3-connected planar graphs. Fabrici et al. [9] proved
 861 that this remains true even if the minimum degree is at least 4, and Mo-
 862 har [16] extended this to 4-connected planar graphs.

863 Although some cycles are light in certain families of planar graphs (see for
 864 instance [11, 12, 15, 17]), it is easy to see that cycles are not light on the class
 865 of planar graphs (consider, for instance, a wheel W_n with n large: each cycle
 866 in W_n is either very long or incident with a large degree vertex). However,
 867 as Richter and Thomassen illustrated in [22], for some applications one does
 868 not need the full lightness condition. A cycle C in a graph is (ℓ, Δ) -*nearly*
 869 *light* if it has length less than ℓ and at most one of its vertices has degree
 870 Δ or greater. Richter and Thomassen proved that every planar graph has a
 871 $(6, 11)$ -nearly light cycle. This was later refined in [14], where it was shown
 872 that if the graphs under consideration are sufficiently large, then there is a
 873 $\Delta > 0$ such that a linear proportion of the face boundaries are $(6, \Delta)$ -nearly
 874 light.

875 The concept of (ℓ, Δ) -earrings extends the idea of nearly light cycles: we
 876 allow both vertices u, v incident with some edge e to have arbitrarily large
 877 degree, and ask for the existence of *two* cycles that contain e , have bounded
 878 length, and (other than u and v) bounded degree. The following imme-
 879 diate corollary (since every 3-connected graph is obviously irreducible) of
 880 Lemma 11 guarantees the existence of many pairwise edge-disjoint earrings
 881 in 3-connected planar graphs.

882 **Lemma 21.** *If $G = (V, E)$ is a 3-connected planar graph, then G has at*
 883 *least $10^{-10}|E|$ pairwise edge-disjoint $(5000, 500)$ -earrings.*

884 We remark that the linear dependence on $|E|$ in Lemma 21 is clearly best
 885 possible, since there cannot be more pairwise edge-disjoint earrings than
 886 edges in a graph.

887 Finally, it is natural to ask if the 3-connectedness requirement can be
 888 weakened. The construction illustrated in Figure 2 answers this in the neg-
 889 ative.

890

891 It might be argued that the graphs constructed in the proof of Theorem 18
 892 are somewhat artificial, since many edges are subdivided a large number of
 893 times. However, these graphs can be turned into 3-connected graphs, with
 894 equivalent properties, as follows. Consider the graph G_2 in the proof of
 895 Theorem 18, and some fixed drawing of G_2 (for instance, as in the proof
 896 of Theorem 18, draw the degree $t - 1$ vertices on a circumference, and the
 897 branches as the straight edges joining them). Let u_1, u_2, \dots, u_t be the *nodes*
 898 (degree $t - 1$ vertices) of G_2 . Thus each branch with endpoints u_i, u_j can be
 899 written as $u_i = u_{i,j}^0, u_{i,j}^1, \dots, u_{i,j}^{s-1}, u_{i,j}^s = u_j$ (the same branch, traversing the

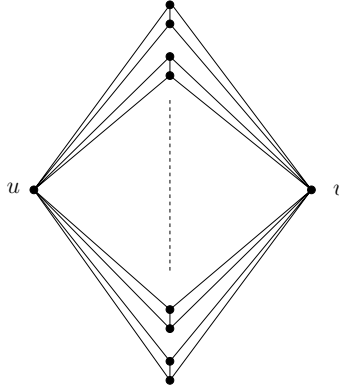


FIGURE 2. The graph H_n obtained by identifying n copies of $K_4 - e$ on their degree 2 vertices u, v . This family of 2-connected graphs shows that the 3-connectedness condition in Lemma 21 cannot be weakened: for each pair of integers ℓ, Δ there is an $n_0 := n_0(\ell, \Delta)$ such that for all $n \geq n_0$, H_n does not contain any (ℓ, Δ) -earring.

900 vertices in the reverse order, reads $u_j = u_{j,i}^0, u_{j,i}^1, \dots, u_{j,i}^{s-1}, u_{j,i}^s = u_i$, so that
 901 $u_{i,j}^k = u_{j,i}^{s-k}$ for $k = 0, 1, \dots, s$). Now for each branch $u_{i,j}^0, u_{i,j}^1, \dots, u_{i,j}^{s-1}, u_{i,j}^s$,
 902 add the edges $u_{i,j}^k$ and $u_{i,j}^{k+2}$, for $k = 0, 1, \dots, s-2$. The augmented graph is
 903 already 2-connected, but each pair of nodes (that is, degree $t-1$ vertices)
 904 is a 2-vertex-cut, so we need to strengthen the connectivity around each
 905 node. Consider the node u_1 , and suppose for simplicity that the edges
 906 $u_1 u_{1,2}^1, u_1 u_{1,3}^1, \dots, u_1 u_{1,t}^1$ leave u_1 in the given (say clockwise) cyclic order.
 907 Then, for each $j = 2, 3, \dots, s$, it is possible to draw an edge from one of
 908 $u_{1,j}^1$ and $u_{1,j}^2$ to one of $u_{1,j+1}^1$ and $u_{1,j+1}^2$ without introducing any crossings
 909 (indices are read modulo s). By performing this procedure around each
 910 node, we obtain a 3-connected graph that also witnesses Theorem 18. The
 911 proof is analogous to the proof of Theorem 18; the only difference is that
 912 instead of requiring a weak edge of a branch (say between u_i and u_j), we
 913 need weak triplets of edges of the form $(u_{i,j}^\ell, u_{i,j}^{\ell+1}), (u_{i,j}^{\ell-1}, u_{i,j}^{\ell+1}), (u_{i,j}^\ell, u_{i,j}^{\ell+2})$,
 914 where $3 \leq \ell \leq s-3$; we omit the details.

915

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917

REFERENCES

- 918 [1] M. Ajtai, V. Chvátal, M.M. Newborn, and E. Szemerédi, Crossing-free subgraphs,
 919 Theory and practice of combinatorics, North-Holland Math. Stud., vol. 60, North-
 920 Holland, Amsterdam, 1982, pp. 9-12.
 921 [2] D. Bokal, Infinite families of crossing-critical graphs with prescribed average degree
 922 and crossing number. *J. Graph Theory* **65** (2010), 139-162.

- 923 [3] D. Bokal, B. Oporowski, R.B. Richter, and G. Salazar, Classification of 2-crossing-
924 critical graphs. Manuscript (2011).
- 925 [4] J. Černý, J. Kynčl and G. Tóth, Improvement on the decay of crossing numbers.
926 *Lecture Notes in Comput. Sci.* **4875**, 25–30. Springer, Berlin, 2008.
- 927 [5] J. Černý, J. Kynčl and G. Tóth, Improvement on the decay of crossing numbers.
928 *Graphs and Combinatorics*, to appear.
- 929 [6] G. Ding, B. Oporowski, R. Thomas, and D. Vertigan, Large non-planar graphs and
930 an application to crossing-critical graphs. *J. Combin. Theory Ser. B* **101** (2011),
931 111-121,
- 932 [7] Z. Dvořák and B. Mohar, Crossing-critical graphs with large maximum degree. *J.*
933 *Combin. Theory Ser. B* **100** (2010), 413-417.
- 934 [8] I. Fabrici and S. Jendrol', Subgraphs with restricted degrees of their vertices in planar
935 3-connected graphs, *Graphs Comb.* **13** (1997) 245–250.
- 936 [9] I. Fabrici, E. Hexel, S. Jendrol', H. Walther, On vertex-degree restricted paths in
937 polyhedral graphs. Graph theory (Drnfeld, 1997). *Discrete Math.* 212 (2000), no. 1-2,
938 61-73.
- 939 [10] J. Fox and Cs. D. Tóth, On the decay of crossing numbers. *J. Combin. Theory Ser.*
940 *B* **98** (2008), no. 1, 33–42.
- 941 [11] S. Jendrol', T. Madaras, R. Soták, and Zs. Tuza, On light cycles in plane triangula-
942 tions. *Discrete Math.* **197/198** (1999), 453-467
- 943 [12] S. Jendrol', P.J. Owens, On light graphs in 3-connected plane graphs without trian-
944 gular or quadrangular faces. *Graphs Combin.* **17** (2001), 659-680.
- 945 [13] T. Leighton, *Complexity issues in VLSI*. MIT Press, Cambridge, MA, 1983.
- 946 [14] M. Lomeli and G. Salazar, Nearly light cycles in embedded graphs and crossing-
947 critical graphs. *J. Graph Theory* **53** (2006), 151-156.
- 948 [15] T. Madaras, R. Škrekovski, H.-J. Voss, The 7-cycle C_7 is light in the family of planar
949 graphs with minimum degree 5, *Discrete Math.* **307** (2007), no. 1430-1435.
- 950 [16] B. Mohar, Light paths in 4-connected graphs in the plane and other surfaces. *J. Graph*
951 *Theory* **34** (2000), 170-179.
- 952 [17] B. Mohar, R. Škrekovski, H.-J. Voss, Light subgraphs in planar graphs of minimum
953 degree 4 and edge-degree 9. *J. Graph Theory* **44** (2003), 261-295.
- 954 [18] B. Mohar and S. Tamon, Expected crossing numbers, *Electronic Notes in Discrete*
955 *Mathematics* **38** (2011), 651–656.
- 956 [19] B. Mohar and S. Tamon, Expected crossing numbers, [arXiv:1012.2579v1](https://arxiv.org/abs/1012.2579v1)
957 [\[math.CO\]](https://arxiv.org/abs/1012.2579v1).
- 958 [20] J. Pach, R. Radoičić, G. Tardos, and G. Tóth, Improving the crossing lemma by
959 finding more crossings in sparse graphs, *Discrete and Comp. Geom.* **36** (2006), 527–
960 552.
- 961 [21] J. Pach and G. Tóth, Which crossing number is it anyway? *Journal of Comb. Theory*
962 *Ser. B* **80** (2000), 225–246.
- 963 [22] R.B. Richter and C. Thomassen, Minimal graphs with crossing number at least k . *J.*
964 *Combin. Theory Ser. B* **58** (1993), 217-224.
- 965 [23] R.D. Ringeisen and L.W. Beineke, The crossing number of $C_3 \times C_n$, *Journal of*
966 *Combinatorial Theory, Series B* **24** (1978), 134–136.
- 967 [24] F. Shahrokhi, O. Sýkora, L. Székely, and I. Vrt'ó, Crossing numbers: bounds and
968 applications. In: *Intuitive geometry* (Budapest, 1995). Bolyai Soc. Math. Stud. **6**
969 Budapest, 1997. 179–206.
- 970 [25] J. Spencer and G. Tóth, Crossing numbers of random graphs, *Random Structures*
971 *and Algorithms* **21** (2002), 347–358.
- 972 [26] L. Székely, Short proof for a theorem of Pach, Spencer, and Tóth. In *Towards a theory*
973 *of geometric graphs*. Contemporary Mathematics **342**, AMS, Providence, RI, 2004,,
974 281–283.

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