

# The convex hull of every optimal pseudolinear drawing of $K_n$ is a triangle

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## Abstract

A pseudolinear (respectively, rectilinear) drawing of a graph  $G$  is *optimal* if it has the smallest number of crossings among all pseudolinear (respectively, rectilinear) drawings of  $G$ . We show that the convex hull of every optimal pseudolinear drawing of the complete graph  $K_n$  is a triangle. This is closely related to the recently announced result that the convex hull of every optimal rectilinear drawing of  $K_n$  is a triangle.

## 1 Introduction

### 1.1 Our main result

A rectilinear (respectively, pseudolinear) drawing of a graph  $G$  is *optimal* if it has the smallest number of crossings among all rectilinear (respectively, pseudolinear) drawings of  $G$ . The following statement remained an important, open conjecture for a long time. Recently, a proof was announced by Aichholzer, Orden, and Ramos [3].

**Theorem 1 ([3])** *The convex hull of every optimal rectilinear drawing of  $K_n$  is a triangle.*

Extending this conjecture to non-rectilinear drawings of  $K_n$  does not make much sense: there is no distinguished unbounded face if the rectilinear condition is altogether dropped, so a meaningful convex hull cannot even be defined. On the other hand, since the convex hull is well-defined for pseudolinear (which lie in between rectilinear and arbitrary) drawings, it makes sense to ask if a similar property holds for pseudolinear drawings. Our main result is that an analogous statement holds for pseudolinear drawings.

**Theorem 2 (Main result)** *The convex hull of every optimal pseudolinear drawing of  $K_n$  is a triangle.*

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## 1.2 Pseudolinear drawings

Recall that a *pseudoline* in the projective plane  $\mathbb{P}^2$  is a simple closed curve whose removal does not disconnect  $\mathbb{P}^2$ . A collection of pseudolines is a *pseudoline arrangement* if each two pseudolines intersect (necessarily cross) in exactly one point. A *generalized configuration*  $\Omega_P$  with point set  $P$  consists of a finite set  $P$  of points, together with a pseudoline joining each pair, and it is *simple* if there is a single pseudoline for each pair.

Consider a good drawing  $\mathcal{D}$  of  $K_n$  in the plane  $\mathbb{R}^2$  (thus, every edge is represented by a simple curve), contained in a disk with radius  $< R$  centered at the origin. Let  $D$  denote the disk with radius  $R$ , centered at the origin. By identifying antipodal points in the boundary of  $D$  (and discarding  $\mathbb{R}^2 \setminus D$ ), we may regard  $\mathcal{D}$  as (a new drawing  $\mathcal{D}'$ , as the host surface has changed) lying in the projective plane. Now if each edge  $e$  in  $\mathcal{D}'$  can be extended to a pseudoline (an *extension of  $e$* ) so that the resulting structure is a simple generalized configuration  $\Omega_P$  in which  $P$  is the set of  $n$  vertices, then the original drawing  $\mathcal{D}$  is a *pseudolinear drawing of  $K_n$* . The *pseudosegments* are the edges of a pseudolinear drawing; in pseudolinear drawings we use the term “edge” and “pseudosegment” interchangeably. If we start with a pseudolinear drawing of  $K_n$  (which, we emphasize, lies in  $\mathbb{R}^2$ ), it is easy to see that we may equivalently stay (all along) in  $\mathbb{R}^2$ , and for each edge  $e$  construct an  $\mathbb{R}^2$ -extension  $\ell_e$ , a set of points homeomorphic to a straight line, which contains  $e$ , whose removal disconnects  $\mathbb{R}^2$  into two unbounded sets, and so that every pair of  $\mathbb{R}^2$ -extensions cross at exactly one point.

As we observed above, the convex hull in a pseudolinear drawing of  $K_n$  is a well-defined object that naturally generalizes the definition of the convex hull of a rectilinear drawing (the definition actually applies to quite more general objects, namely the *CC*-systems introduced by Knuth; see [5] and [7]). Consider a pseudolinear drawing  $\mathcal{D}$  of  $K_n$ , and for each edge (pseudosegment)  $e$  construct an  $\mathbb{R}^2$ -extension  $\ell_e$  as described above. An edge in  $\mathcal{D}$  is a *convex hull edge* of  $\mathcal{D}$  if the  $n - 2$  points (vertices of  $K_n$ ) not incident with  $e$  lie on the same half-plane of  $\ell_e$ , and the *convex hull* of  $\mathcal{D}$  is the collection of all the convex hull edges and their incident vertices. It can be checked that convex hull edges are well-defined, that is, independent of the chosen  $\mathbb{R}^2$ -extensions.

It is readily verified that no convex hull edge can cross another edge. Therefore Theorem 2 states that the obvious extension of Theorem 1 to pseudolinear drawings is true: the unbounded face in any drawing that attains the pseudolinear crossing number of  $K_n$  is incident with (exactly) 3 vertices and 3 edges.

## 1.3 Pseudolinear and rectilinear crossing numbers

If  $\mathcal{D}$  is a drawing of  $K_n$ , then we let  $\text{cr}(\mathcal{D})$  denote the number of pairwise crossings of edges in  $\mathcal{D}$ . The *pseudolinear crossing number*  $\tilde{\text{cr}}(K_n)$  is the minimum of  $\text{cr}(\mathcal{D})$  over all pseudolinear drawings  $\mathcal{D}$  of  $K_n$ . The *rectilinear crossing number*  $\overline{\text{cr}}(K_n)$  of  $K_n$  is the minimum of  $\text{cr}(\mathcal{D})$  over all rectilinear drawings  $\mathcal{D}$  of  $K_n$ . Since every rectilinear drawing of  $K_n$  is also a pseudolinear drawing,  $\overline{\text{cr}}(K_n) \geq \tilde{\text{cr}}(K_n)$ .

If a pseudolinear drawing is combinatorially equivalent to a rectilinear drawing, then it is *stretchable*. Since almost all pseudolinear drawings are non-stretchable (see for instance [9]), it is conceivable that  $\tilde{\text{cr}}(K_n) < \overline{\text{cr}}(K_n)$  for some  $n$ . By computing good lower bounds for  $\tilde{\text{cr}}(K_n)$ , it is verified in [2] that  $\tilde{\text{cr}}(K_n) = \overline{\text{cr}}(K_n)$  for  $n \leq 12$  and  $n = 15$ . On this basis, we put forward the following.

**Conjecture 3** *For every  $n$ ,  $\tilde{\text{cr}}(K_n) = \overline{\text{cr}}(K_n)$ .*

Settling this conjecture in either direction would be quite interesting by itself: we would know whether or not there is anything to gain, with respect to crossing numbers, by considering non-stretchable pseudolinear drawings of  $K_n$  (over rectilinear ones).

## 2 Background: generalized configurations and allowable sequences

We recall that a *simple allowable sequence on  $n$  elements*  $\Pi$  is a doubly infinite sequence  $(\dots, \pi_{-1}, \pi_0, \pi_1, \dots)$  of permutations of an  $n$ -element *ground set* (say  $\{p_1, p_2, \dots, p_n\}$ ), such that (i) any two consecutive permutations differ by exactly one transposition of two elements in adjacent positions; and (ii) after a move in which  $i$  and  $j$  switch, they do not switch again until every other pair has switched. If a transposition  $\tau$  swaps elements  $p_i$  and  $p_j$ , so that  $p_i$  moves from position  $t$  to position  $t+1$ , and  $p_j$  moves from position  $t+1$  to position  $t$ , then we write  $\tau = [p_i|p_j]_t$ . An allowable sequence  $\Pi = (\dots, \pi_{-1}, \pi_0, \pi_1, \dots)$  on  $n$  elements is equivalently defined by its *transpositions sequence*  $T(\Pi) = (\dots, \tau_{-1}, \tau_0, \tau_1, \dots)$ , where  $\tau_i$  is the transposition that transforms  $\pi_{i-1}$  into  $\pi_i$ .

It is straightforward to see that a simple allowable sequence on  $n$  elements has period  $n(n-1)$ . We shall be particularly interested in halfperiods of  $\Pi$ , that is, finite subsequences  $(\pi_i, \pi_{i+1}, \dots, \pi_{i+\binom{n}{2}})$ . Note that the ending permutation of a halfperiod is the reverse permutation of the starting one.

Simple allowable sequences, introduced by Goodman and Pollack in an influential paper [6], are a fruitful tool to encode any generalized configuration of points: to each generalized configuration of points  $\Omega_P$  with point set  $P$ , one can naturally associate a simple allowable sequence  $\Pi_{\Omega_P}$  with ground set  $P$ , and, reciprocally, given a simple allowable sequence  $\Pi$  with ground set  $P$  one can obtain a generalized configuration of points  $\Omega_P$  whose associated sequence is  $\Pi_{\Omega_P} = \Pi$ . The details of this relationship have been lucidly explained in [6] and in subsequent surveys (more recently in [1] or [8], precisely in the context of crossing numbers), so we shall omit them, and refer the interested reader to these sources.

Suppose that  $\mathcal{D}$  is a pseudolinear drawing of  $K_n$ , with underlying  $n$ -point set  $P$ . Thus (since  $\mathcal{D}$  is pseudolinear)  $P$  is the point set of a simple generalized configuration  $\Omega_P$ . We say that  $\Omega_P$  is a generalized configuration *associated to*  $\mathcal{D}$ . Although  $\Omega_P$  is not unique (as there are infinitely many ways to extend the pseudoedges to form pseudolines), the induced simple allowable sequence  $\Pi_{\Omega_P}$  is unique, and thus it is consistent to call  $\Pi_{\mathcal{D}} := \Pi_{\Omega_P}$  *the simple allowable sequence associated to*  $\mathcal{D}$ .

## 3 Allowable sequences and convex hulls: proof of Theorem 2

The encoding scheme from generalized configurations of points to simple allowable sequences [6] makes it particularly easy to identify the convex hull of a pseudolinear drawing of  $K_n$ , as follows.

**Proposition 4** *Let  $\mathcal{D}$  be a pseudolinear drawing of  $K_n$ , and let  $P$  denote the underlying  $n$ -point set. Let  $\Pi_0$  be any halfperiod of the associated simple allowable sequence. Then a point  $p$  in  $P$  is in the convex hull of  $\mathcal{D}$  iff it occupies either position 1 or position  $n$  in a permutation of  $\Pi_0$ .*

In view of this, in order to establish Theorem 2 it suffices to show that if  $\mathcal{D}$  is *optimal* among pseudolinear drawings (that is,  $\tilde{cr}(\mathcal{D}) = \tilde{cr}(K_n)$ ), then at most 3 elements in  $P$  ever occupy position 1 or position  $n$  in some permutation in  $\Pi_0$  (any halfperiod of  $\Pi_{\mathcal{D}}$ ). In order to prove such a result, we need a useful characterization of which simple allowable sequences are induced from optimal pseudolinear drawings of  $K_n$ .

Such a characterization can be obtained from results in [1] and [8] that give the crossing number in a pseudolinear drawing of  $K_n$  in terms of properties of its associated simple allowable sequence. In order to present this result, we need to define one local and one global function. Let  $\tau = [p_i|p_j]_t$  be a transposition in the transpositions sequence of a simple allowable sequence  $\Pi$ . The *impact*  $f(\tau)$  of  $\tau$  is defined as follows:

$$f(\tau) = f([a|b]_t) = \left( \frac{n-2}{2} - (t-1) \right)^2. \quad (1)$$

Now if  $\Pi_0$  is a halfperiod of a simple allowable sequence, then its *weight*  $F(\Pi_0)$  is

$$F(\Pi_0) = \sum_{\tau} f(\tau), \quad (2)$$

where the summation is over all the  $\tau_i$ 's in the transpositions sequence of  $\Pi_0$ . That is, the weight of  $\Pi_0$  is simply the sum of the impacts of all the transpositions in its transpositions sequence.

The relevance of the weight of a halfperiod of a simple allowable sequence induced by a pseudolinear drawing of  $K_n$  comes from the following result.

**Theorem 5 ([1],[8])** *Let  $\mathcal{D}$  be a pseudolinear drawing of  $K_n$ , and let  $\Pi$  be a halfperiod of its associated simple allowable sequence. Then*

$$\tilde{c}r(\mathcal{D}) = 3 \binom{n}{4} - F(\Pi_0).$$

Our last required result, which is proved in Section 4, gives us a crucial piece of information on halfperiods that maximize  $F$ .

**Proposition 6** *Let  $\Pi_0$  be a halfperiod of a simple allowable sequence on  $n$  elements. Suppose that  $\Pi_0$  maximizes  $F$  over all halfperiods of simple allowable sequences on  $n$  elements. Then there are (exactly) 3 elements that occupy either position 1 or position  $n$  in a permutation of  $\Pi_0$ .*

*Proof of Theorem 2.*

Since every simple allowable sequence can be induced from a pseudolinear drawing of  $K_n$ , it follows from Theorem 5 that a (pseudolinear) drawing attains the pseudolinear crossing number of  $K_n$  iff any halfperiod of its associated simple allowable sequence maximizes  $F$  over all possible halfperiods of simple allowable sequences. Propositions 6 and 4 complete the proof. ■

## 4 Proof of Proposition 6

Throughout this proof,  $\Pi_0 = (\pi_0, \pi_1, \pi_2, \dots, \pi_{\binom{n}{2}})$  is a halfperiod of a simple allowable sequence that minimizes  $F$ . Unless otherwise stated, all transpositions and permutations hereby mentioned occur are associated to  $\Pi_0$ .

Let us label the points so that the initial permutation is  $a_1 a_2 \dots a_n$ .

**Claim A** *Let  $i$  satisfy  $\lceil n/2 \rceil \leq i < n$ . Let  $\tau_s$  be the transposition that moves  $a_n$  to position  $i$ . Suppose that  $a_\ell$  is to the right of  $a_n$  in  $\pi_s$ . Then, after  $\tau_s$  occurs, the first transposition that involves  $a_\ell$  moves  $a_\ell$  to the left, and the other element involved in the transposition is to the left of  $a_n$  in  $\pi_s$ .*

*Proof.* Seeking a contradiction, let  $i$  be smallest possible so that the statement is false. Label  $b_1, b_2, \dots, b_{n-i}$  the last  $n-i$  elements in  $\pi_s$ , in the order in which they appear in  $\pi_s$ . Note that  $\tau_s = [b_1 | a_n]_i$ .

We claim that the first transposition  $\tau_t$  after  $\tau_s$  that involves an element in  $\{b_1, b_2, \dots, b_{n-i}\}$  must be the transposition swapping elements  $b_1$  and  $b_2$ . Recall that Claim A holds if we substitute  $i$  by  $i-1$ . This implies, in particular, that the first element in  $\{b_2, \dots, b_{n-i}\}$  that gets involved in a transposition after  $\tau_s$  must be  $b_2$ , and that the other element involved in the transposition is to the left of  $b_2$  in  $\pi_s$ . Now the first transposition after  $\tau_s$  that involves  $b_1$  cannot involve an element to the left of  $b_1$  in  $\pi_s$ , as otherwise (it is easy to check) Claim A would then also hold for  $i$ . Thus  $\tau_t$  must involve  $b_1$  and  $b_2$ , that is,  $\tau_t = [b_1 | b_2]_{i+1}$ . Again using the assumption that Claim A holds for  $i-1$ , it follows that the last transposition  $\tau_r$  before  $\tau_s$  that involves an element in  $b_1, b_2, \dots, b_{n-i}$  is precisely the transposition that swaps  $b_2$  and  $a_n$ , that is,  $\tau_r = [b_2 | a_n]_{i+1}$ .

Thus, the following transpositions occur in the given order:  $\tau_r = [b_2 | a_n]_{i+1}$ ,  $\tau_s = [b_1 | a_n]_i$ , and  $\tau_t = [b_1 | b_2]_{i+1}$ . Moreover, the only transposition between  $\tau_r$  and  $\tau_t$  that involves an element in position  $i+1$  or further right is precisely  $\tau_s$ . This last observation implies that if we modify the transpositions sequence by delaying  $\tau_r$  (if necessary) and letting it occur immediately before  $\tau_s$ , and then accelerating  $\tau_t$  (if necessary) and letting it occur immediately after  $\tau_s$ , and leaving the transposition sequence otherwise unchanged, the resulting transpositions sequence will still correspond to a (valid) halfperiod  $\tilde{\Pi}_0$  of a simple allowable sequence.

More precisely, if we let  $\tau'_i = \tau_i$  for  $1 \leq i < r$ ,  $\tau'_i = \tau_{i+1}$  for  $r \leq i \leq s-2$ ,  $\tau'_{s-1} = [b_1|b_2]_i$ ,  $\tau'_s = [b_1|a_n]_{i+1}$ ,  $\tau'_{s+1} = [b_2|a_n]_i$ ,  $\tau'_i = \tau_{i-1}$  for  $s+2 \leq i \leq t$ , and  $\tau'_i = \tau_i$  for  $i > t$ , then  $\tau'_0, \tau'_1, \dots, \tau'_{\binom{n}{2}}$  is the transpositions sequence of a simple allowable sequence  $\bar{\Pi}_0$ . Clearly,  $\sum_{\tau_i \notin \{\tau_r, \tau_s, \tau_t\}} f(\tau_i) = \sum_{\tau'_i \notin \{\tau'_{s-1}, \tau'_s, \tau'_{s+1}\}} f(\tau'_i)$ . Moreover,  $f(\tau_r) = f(\tau'_s)$  and  $f(\tau_s) = f(\tau'_{s-1})$ , and so  $\sum_{\tau_i \neq \tau_t} f(\tau_i) = \sum_{\tau'_i \neq \tau'_{s+1}} f(\tau'_i)$ . However,  $f(\tau_t) = (\frac{n-2}{2} - ((i+1) - 1))^2 < (\frac{n-2}{2} - (i-1))^2 = f(\tau'_{s+1})$  (note that here we are using that  $i \geq \lceil n/2 \rceil$ ). Therefore  $F(\Pi_0) = \sum_{\tau_i} f(\tau_i) < \sum_{\tau'_i} f(\tau'_i) = F(\bar{\Pi}_0)$ , contradicting the assumption that  $\Pi_0$  maximizes  $F$  over all halfperiods of simple allowable sequences of size  $n$ . ■

**Claim B** *Either  $a_1$  moves  $a_n$  from position  $n$  or  $a_n$  moves  $a_1$  from position 1.*

*Proof of Claim B.* We suppose that  $a_1$  reaches position  $\lceil n/2 \rceil$  before  $a_n$  reaches position  $\lfloor n/2 \rfloor + 1$  (it is readily checked that these cannot occur simultaneously), and show that in this case  $a_1$  moves  $a_n$  out of position  $n$ . The other possibility, that  $a_n$  reaches position  $\lfloor n/2 \rfloor + 1$  before  $a_1$  reaches position  $\lceil n/2 \rceil$  (in which case the conclusion is that  $a_n$  moves  $a_1$  from position 1), is dealt with in a totally analogous manner.

Let  $m+1$  be the position of  $a_1$  immediately after it swaps with  $a_n$ . Thus, the transposition between  $a_1$  and  $a_n$  is  $[a_1|a_n]_m = \tau_q$  for some  $q$ . Since  $a_1$  only moves right, and  $a_n$  only moves left, it follows that  $a_1$  is in position  $m \geq \lfloor n/2 \rfloor$  just before this permutation, that is, in  $\pi_{q-1}$ .

To prove the statement, for the rest of the proof we assume that  $m < n-1$ , and derive a contradiction.

Let  $b$  denote the element in position  $m+2$  in  $\pi_{q-1}$  (and still there in  $\pi_q$ ). Now  $b$  is to the right of  $a_n$  already in  $\pi_{q-1}$ . An application of Claim A with  $i = m+1$  (that is, when  $a_n$  first moved into position  $m+1$ ) yields that  $b$  could not have arrived to position  $m+2$  (in  $\pi_{q-1}$ ) by transposing with an element other than  $a_n$ . Thus  $b$  and  $a_n$  swap when  $b$  is in position  $m+1$  (and  $a_n$  is in position  $m+2$ ). Thus this transposition is  $[b|a_n]_{m+1} = \tau_p$  for some  $p < q$ .

We note again that  $a_1$  never moves left. Applying Claim A (again with  $i = m+1$ ), we obtain that the transposition  $\tau_r$  with  $r > q$  smallest possible that involves an element in position  $m+1$  or further right is the transposition that swaps  $a_1$  and  $b$ . That is,  $\tau_r = [a_1|b]_{m+1}$ .

Thus, the following transpositions occur in the given order:  $\tau_p = [b|a_n]_{m+1}$ ,  $\tau_q = [a_1|a_n]_m$ , and  $\tau_r = [a_1|b]_{m+1}$ . Moreover,  $\tau_q$  is the only transposition between  $\tau_p$  and  $\tau_r$  that involves an element in position  $m+1$  or further right (this follows again from Claim A). This observation implies that if we modify the transpositions sequence by delaying  $\tau_p$  (if necessary) and letting it occur immediately before  $\tau_q$ , and then accelerating  $\tau_r$  (if necessary) and letting it occur immediately after  $\tau_q$ , and leaving the transposition sequence otherwise unchanged, the resulting transpositions sequence will still induce a (valid) simple allowable sequence  $\tilde{\Pi}_0$ . More precisely, if we let  $\tau'_i = \tau_i$  for  $1 \leq i < p$ ,  $\tau'_i = \tau_{i+1}$  for  $p \leq i \leq q-2$ ,  $\tau'_{q-1} = [a_1|b]_m$ ,  $\tau'_q = [a_1|a_n]_{m+1}$ ,  $\tau'_{q+1} = [b|a_n]_m$ ,  $\tau'_i = \tau_{i-1}$  for  $q+2 \leq i \leq r$ , and  $\tau'_i = \tau_i$  for  $i > r$ , then  $\tau'_0, \tau'_1, \dots, \tau'_{\binom{n}{2}}$  is the transpositions sequence of a simple allowable sequence  $\bar{\Pi}_0$ . Clearly,  $\sum_{\tau_i \notin \{\tau_p, \tau_q, \tau_r\}} f(\tau_i) = \sum_{\tau'_i \notin \{\tau'_{q-1}, \tau'_q, \tau'_{q+1}\}} f(\tau'_i)$ . Moreover,  $f(\tau_p) = f(\tau'_q)$  and  $f(\tau_q) = f(\tau'_{q-1})$ , and so  $\sum_{\tau_i \neq \tau_r} f(\tau_i) = \sum_{\tau'_i \neq \tau'_{q+1}} f(\tau'_i)$ . However,  $f(\tau_r) = (\frac{n-2}{2} - ((m+1) - 1))^2 < (\frac{n-2}{2} - (m-1))^2 = f(\tau'_{q+1})$ . Therefore  $F(\Pi_0) = \sum_{\tau_i} f(\tau_i) < \sum_{\tau'_i} f(\tau'_i) = F(\bar{\Pi}_0)$  (here we are using that  $m \geq \lceil n/2 \rceil$ ), contradicting the assumption that  $\Pi_0$  maximizes  $F$  over all halfperiods of simple allowable sequences of size  $n$ . ■

*Conclusion of proof of Proposition 6.*

By Claim B, either  $a_1$  moves  $a_n$  from position  $n$  or  $a_n$  moves  $a_1$  from position 1. Suppose the former case holds. Let  $x$  be the element that moves  $a_1$  from position 1. Immediately after  $a_1$  and  $x$  transpose,  $x$  is in position 1, and  $a_n$  is in position  $n$ . Thus another application of Claim B (with the suitable relabeling) implies that either  $x$  moves  $a_n$  out of position  $n$  or  $a_n$  moves  $x$  out of position 1. The former case is impossible, since  $a_1 \neq x$  is the element that moves  $a_n$  out of position  $n$ . Thus  $a_n$  moves  $x$  out of position 1. Therefore, the only elements that ever occupy position 1 are  $a_1, x$ , and  $a_n$ , and the only elements that ever occupy position  $n$  are  $a_1$  and  $a_n$ . ■

A slightly different proof is given in [10].

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