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Hypergraphs with Zero Chromatic Threshold

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Abstract

Let F be an r-uniform hypergraph. The *chromatic threshold* of the family of F-free, r-uniform hypergraphs is the infimum of all non-negative reals c such that the subfamily of F-free, r-uniform hypergraphs H with minimum degree at least $c\binom{|V(H)|}{r-1}$ has bounded chromatic number. The study of chromatic thresholds of various graphs has a long history, beginning with the early work of Erdős-Simonovits. One interesting question, first proposed by Łuczak-Thomassé and then solved by Allen-Böttcher-Griffiths-Kohayakawa-Morris, is the characterization of graphs having zero chromatic threshold, in particular the fact that there are graphs with non-zero Turán density that have zero chromatic threshold. In this paper, we make progress on this problem for r-uniform hypergraphs, showing that a large class of hypergraphs have zero chromatic threshold in addition to exhibiting a family of constructions showing another large class of hypergraphs have non-zero chromatic threshold. Our construction is based on a special product of the Bollobás-Erdős graph defined earlier by the authors.

1 Introduction

In 1973, Erdős and Simonovits [8] asked the following question: "If G is non-bipartite, what bound on $\delta(G)$ forces G to contain a triangle?" This question was answered by Andrásfai, Erdős, and Sós [2], where they showed that if G is a triangle-free, n-vertex graph with $\delta(G) > \frac{2n}{5}$, then G is bipartite. The blowup of C_5 shows that this is sharp. Erdős and Simonovits [8] generalized this problem to the following conjecture: if $\delta(G) > (1/3+\epsilon)\left|V(G)\right|$ and G is triangle-free, then $\chi(G) < k_\epsilon$, where k_ϵ is a constant depending only on ϵ. The conjecture was proven by Thomassen [12], but interest remained in generalizing the problem to other graphs and to hypergraphs.

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Definition. An *r-uniform hypergraph* H is a pair $(V(H), E(H))$ where $V(H)$ is any finite set and $E(H)$ is a family of r-subsets of $V(H)$. A set of vertices $X \subseteq V(H)$ is an *independent set* if every hyperedge (element of $E(H)$) contains at least one vertex outside X and X is a *strong independent set* if every hyperedge intersects X in at most one vertex. The *degree* of a vertex $x \in V(H)$, denoted $d(x)$, is the number of hyperedges containing x. The *minimum degree of* H, denoted $\delta(H)$, is the minimum degree of a vertex of H. A hypergraph H is *k-colorable* if there exists a partition of the vertex set of H into k sets $V_1, \dots, V_k$ such that each V_i is an independent set. The *chromatic number of* H, denoted $\chi(H)$, is the minimum k such that H is k-colorable. If F and H are r-uniform hypergraphs, then H is *F-free* if F does not appear as a subhypergraph of H, i.e. there does not exist an injection $\alpha : V(F) \to V(H)$ such that if $\{f_1, \dots, f_r\}$ is a hyperedge of F then $\{\alpha(f_1), \dots, \alpha(f_r)\}$ is a hyperedge of H. If $X \subseteq V(H)$, the *induced hypergraph on* X, denoted $H[X]$, is the hypergraph with vertex set X and its hyperedges are all hyperedges of H which are completely contained inside X.

Definition. Let $\mathcal{F}$ be a family of r-uniform hypergraphs. The *chromatic threshold* of $\mathcal{F}$ is the infimum of $c \geq 0$ such that the subfamily of $\mathcal{F}$ consisting of hypergraphs H with minimum degree at least $c\binom{|V(H)|}{r-1}$ has bounded chromatic number.

Note that Erdős and Simonovits' [8] conjecture is that the chromatic threshold of the family of triangle-free graphs is 1/3. The chromatic threshold of the family of F-free graphs for various F have been studied by several researchers [1, 7, 9, 10, 11, 12, 13], eventually culminating in a theorem of Łuczak and Thomassé [11] and a theorem of Allen, Böttcher, Griffiths, Kohayakawa, and Morris [1] where they determined the chromatic threshold of the family of F-free graphs for all F. An interesting consequence of [1] is the solution to a conjecture of Łuczak and Thomassé [11]: the family of F-free graphs has chromatic threshold zero if and only if F is near acyclic. (A graph G is *near acyclic* if there exists an independent set S in G such that $G - S$ is a forest and every odd cycle has at least two vertices in S.) This is surprising because the Turán density is a trivial upper bound on the chromatic threshold, but the family of graphs with zero Turán density (bipartite graphs) differs from the family of near-acyclic graphs.

Balogh, Butterfield, Hu, Lenz, and Mubayi [3] initiated the study of chromatic thresholds of r-uniform hypergraphs and among other things proposed the following problem, again interested in comparing zero Turán density with zero chromatic threshold.

Problem 1. *Characterize the r-uniform hypergraphs F for which the chromatic threshold of the family of F-free hypergraphs has chromatic threshold zero.*

Balogh, Butterfield, Hu, Lenz, and Mubayi [3] made partial progress on this problem by proving that for a large class of hypergraphs F, the family of F-free hypergraphs has chromatic threshold zero. In the other direction, [3] also contains constructions of families of F-free hypergraphs with non-zero chromatic threshold for various hypergraphs F. Our main result in this paper is to extend both of these results, enlarging the class of Fs for which we can prove the family of F-free hypergraphs has chromatic threshold zero in addition to giving a more general construction of families with non-zero chromatic threshold. One of our key ideas is to use a hypergraph extension of the Bollobás-Erdős graph [6] defined by the authors in [4, 5] instead of the Borsuk-Ulam graph in a Hajnal type construction used by Łuczak and Thomassé [11]. To state our results, we need some definitions.

Definition. A *cycle of length* $t \geq 2$ in a hypergraph is a collection of t distinct vertices $X = \{x_1, \ldots, x_t\}$ of H and t distinct edges $E_1, \ldots, E_t$ such that $\{x_i, x_{i+1}\} \in E_i$ for each $i = 1, \ldots, t$ (indices taken mod t.) A hypergraph is a *hyperforest* if it contains no cycles. A hypergraph is a *hypertree* if it is a connected hyperforest, where *connected* means for every two vertices x, y, there is a sequence of edges $E_1, \ldots, E_\ell$ for some ℓ such that $x \in E_1$, $y \in E_\ell$ and $E_i \cap E_{i+1} \neq \emptyset$ for all $i = 1, 2, \ldots, \ell - 1$. A *component* of H is a maximal connected subhypergraph of H. A hypergraph is *linear* if every pair of hyperedges intersect in at most one vertex.

The key definitions are the following.

Definition. An r-uniform hypergraph H is *unifoliate r-partite* if there exists a partition of the vertices into r classes $V_1, V_2, \ldots, V_r$ such that $H[V_1]$ is a linear hyperforest, every edge not in $H[V_1]$ has exactly one vertex in each V_i, and every cycle in H uses either vertices from at least two components of $H[V_1]$ or it contains zero or at least two edges of $H[V_1]$.

An r-uniform hypergraph H is *strong unifoliate r-partite* if it is *unifoliate r-partite* and in addition, in a witnessing partition there does not exist two vertices $x, y \in V_1$ such that x and y are in the same component of $H[V_1]$ and H contains a sequence of hyperedges $E_1, \ldots, E_\ell$ for some ℓ with $x \in E_1$, $y \in E_\ell$, and $E_i \cap E_{i+1} \cap (V_2 \cup \cdots \cup V_r) \neq \emptyset$ for all $1 \leq i \leq \ell - 1$. Note that this sequence of hyperedges is like a path between x and y which is required to "connect" using vertices outside V_1.

Strong and normal unifoliate r-partite are similar but not quite identical requirements; they differ only in the type of cycles that are allowed. To be strong unifoliate r-partite, a hypergraph is forbidden to have cycles which use arbitrary number of edges inside some hypertree of $H[V_1]$ combined with cross-edges which connect using vertices outside V_1. To be unifoliate r-partite, a hypergraph is forbidden to have cycles using exactly one edge E from $H[V_1]$ together with cross-edges which connect using vertices either outside V_1 or vertices that are in the same hypertree as the edge E. Note that the forbidden cycle condition for strong unifoliate r-partite is a stronger forbidden cycle condition, since if a cycle uses one edge E from $H[V_1]$ and some cross-edges which connected using vertices inside the same hypertree as E, the cycle could be rerouted through the hypertree containing E decreasing the number of cross-edges which connect using a vertex of V_1. Our main theorem is the following.

Theorem 2. *Fix $r \geq 3$. If F is an r-uniform, strong unifoliate r-partite hypergraph then the family of F-free hypergraphs has chromatic threshold zero. If F is not unifoliate r-partite then the family of F-free hypergraphs has chromatic threshold at least* $(r-1)!r^{-2r+2}$.

The constant $(r-1)!r^{-2r+2}$ could be slightly improved (see the comments in the proof of Lemma 8). Currently, proving sharpness seems out of reach so we make no effort to optimize the constant. Theorem 2 generalizes results of Balogh, Butterfield, Hu, Lenz, and Mubayi [3], since the classes of hypergraphs studied in [3] are subclasses of either non-unifoliate or strong unifoliate r-partite hypergraphs. For $r \geq 3$, there exist hypergraphs which are unifoliate r-partite but not strong unifoliate r-partite so Theorem 2 does not completely solve Problem 1. An example for $r = 3$ is the hypergraph with vertex set $\{a_1, a_2, a_3, a_4, b_1, b_2, c_1, c_2\}$ and hyperedges $a_1a_2a_3$, $a_1b_1c_1$, $a_2b_2c_2$, $a_4b_1c_2$, $a_4b_2c_1$. An illustrative vertex partition has a_1, a_2, a_3, a_4 in one part, b_1, b_2 in the second part, and c_1, c_2 in the third part.

Conjecture 3. *For $r \geq 3$ and an r-uniform hypergraph F, the family of F-free hypergraphs has chromatic threshold zero if and only if F is unifoliate r-partite.*

The definition of unifoliate 2-partite is not quite equivalent to the definition of a near-acyclic graph which is why Theorem 2 above is stated only for $r \geq 3$, but with a little extra work the proofs below extend to $r = 2$ with unifoliate 2-partite replaced by near-acylic. But since different behavior occurs for $r \geq 3$ compared to $r = 2$ (as evinced by the difference between unifoliate 2-partite and near-acyclic), we simplify the presentation by focusing only on $r \geq 3$. The remainder of this paper is organized as follows. In Section 2, using a construction built from the high-dimensional unit sphere, we prove that for every non unifoliate r-partite hypergraph F, the family of F-free hypergraphs has non-zero chromatic threshold. In Section 3, we show how the tools from [3] can be applied to prove that if F is a strong unifoliate r-partite hypergraph, then the family of F-free hypergraphs has chromatic theshold zero.

2 Construction for positive chromatic threshold

To prove a lower bound on the chromatic threshold of the family of F-free hypergraphs, we need to construct an infinite sequence of F-free hypergraphs with large chromatic number and large minimum degree. First, we need a construction of Balogh and Lenz [4, 5] of a hypergraph with large chromatic number built from the high dimensional unit sphere. The construction is based on the celebrated Bollobás-Erdős Graph [6]. We only sketch its definition here, for details see [4, 5]. Throughout this section, an integer r is fixed and all hypergraphs considered are r-uniform.

Definition. [4, Section 2 with $s = 2$] Given integers n, L, and d and a real number $\theta > 0$, we construct the r-uniform hypergraph $H(n, L, d, \theta)$ as follows. Let P be n "evenly distributed" points on the d-dimensional unit sphere $\mathbb{S}^d$, let $\ell = \lceil \log_2 r \rceil$, and let $B_1, \ldots, B_\ell$ be complete bipartite graphs on the vertex set $[r]$ defined as follows. Assign bit strings of length $\lceil \log_2 r \rceil$ to the elements of $[r]$ and define B_i as the complete bipartite graph consisting of the edges between vertices differing in coordinate i. Note that the union $\cup B_i$ covers the complete graph $K_{[r]}$. If $\bar{x} \in P^\ell$, for $1 \leq i \leq \ell$ denote by x_i the point of the sphere appearing in the i-th coordinate of tuple $\bar{x}$. Let $\|\cdot\|$ be the Euclidean norm in $\mathbb{R}^{d+1}$. Now we define an r-uniform hypergraph H' as follows.

- $V(H') = P^\ell$,

- $E \in \binom{V(H')}{r}$ is a hyperedge if there exists some ordering $\bar{x}^1, \ldots, \bar{x}^r$ of the elements of E such that for all $1 \le i \le \ell$ and all $ab \in E(B_i)$, then $\left\|x_i^a - x_i^b\right\| > 2 - \theta$.

Now form the hypergraph H by applying [5, Theorem 16] to H'; this operation consists of blowing up every vertex in H' into a strong independent set, randomly sparsening the hypergraph, and finally deleting one edge from each cycle on at most L vertices.

Theorem 4. *([4, Lemma 13]) Given any real $\delta > 0$ and any integer $L \ge 3$, it is possible to select d, $\theta_0 > 0$, and n_0 so that $\alpha(H(n, L, d, \theta)) \le \delta |V(H(n, L, d, \theta))|$ for all $0 < \theta \le \theta_0$ and $n \ge n_0$. Also, every L vertices in $H(n, L, d, \theta)$ induce a linear hyperforest.*

We will use a subhypergraph of H for our application. Define

$$V_0' = \left\{ (p_1, \ldots, p_\ell) \in P^\ell : d(p_i, p_j) \le \sqrt{2} \text{ for all } i \ne j \right\}.$$

Note that V_0' is a subset of the vertices of H'. Define V_0 as the set of vertices of H which came from a blowup of a vertex of V_0'. Let $H_0 = H[V_0]$. An easy consequence of Theorem 4 and some properties of the unit sphere is the following.

Corollary 5. *Given any integers $L \ge 3$ and $k \ge 2$, it is possible to select d, $\theta_0 > 0$, and n_0 so that $\chi(H_0(n, L, d, \theta)) \ge k$ for all $0 < \theta \le \theta_0$ and $n \ge n_0$. Also, every L vertices in $H_0(n, L, d, \theta)$ induce a linear hyperforest.*

Proof. First, $|V_0'| \ge 2^{-\ell^2} n^\ell$ for large n. Indeed, for each point $p \in P$, there are roughly $n/2$ points within distance $\sqrt{2}$ of p. In the worst case, to form a vertex $(p_1, \ldots, p_\ell)$ of H_0 there are n ways to choose p_1, $n/2$ ways to choose p_2, $n/4$ ways to choose p_3, and so on. Thus there are certainly at least $2^{-\ell^2} n^\ell$ vertices of H_0', even taking into account that points are evenly distributed on the sphere and there might not be exactly $n/2$ points within distance $\sqrt{2}$ of p for every p (n is large).

Since $n^\ell = |V(H')|$ and each vertex in H' is blown up into the same number of vertices, $|V(H_0)| \ge 2^{-\ell^2} |V(H)|$. Now select $\delta = k^{-1} 2^{-\ell^2}$ and apply Theorem 4 to obtain d, θ_0, and n_0. Now for $0 < \theta \le \theta_0$ and $n \ge n_0$ (and n large enough for the previous paragraph to hold), Theorem 4 implies that $\alpha(H_0) \le \alpha(H) \le k^{-1} 2^{-\ell^2} |V(H)| \le \frac{1}{k} |V(H_0)|$. Thus $\chi(H_0) \ge k$ and the proof is complete. □

We also need a simple property of the unit sphere (see [5, Property (P1)].)

Lemma 6. *Let μ be the Lebesgue measure on the unit sphere normalized so that $\mu(\mathbb{S}^d) = 1$. For any $\epsilon > 0$ there is a $\beta > 0$ depending only on ϵ so that for any $d \ge 3$ and any fixed $p \in \mathbb{S}^d$,*

$$\mu\left(\left\{ q \in \mathbb{S}^d : \|p - q\| \le \sqrt{2} - \beta \right\}\right) \ge \frac{1}{2} - \epsilon.$$

Definition. Given an r-uniform hypergraph F, integers $k, n \geq 2$, and an $\epsilon > 0$, we define an r-uniform hypergraph $G = G(F, k, \epsilon, n)$ as follows. First, pick β according to Lemma 6, let $L = |V(F)|$, and pick d, θ_0, and n_0 according to Corollary 5. For $n \leq n_0$, let G be the edgeless hypergraph. Now assume $n > n_0$. Let $f = |E(F)|$. Next, define $\theta > 0$ so that $\theta < \theta_0$, $4^f\theta^{2^{-f}} < \frac{1}{10}$, and for any fixed $p \in \mathbb{S}^d$

$$\operatorname{diam}\left(\left\{q \in \mathbb{S}^d : \|p - q\| \leq \sqrt{2} - \beta\right\}\right) < 2 - 4^f\theta^{2^{-f}}. \tag{1}$$

This is possible since once $\beta > 0$ is chosen, the spherical cap centered at p will not be the entire hemisphere so θ can be selected smaller than θ_0, smaller than the solution to $4^f\theta^{2^{-f}} = \frac{1}{10}$, and small enough to have $2 - 4^f\theta^{2^{-f}}$ between the diameter of the spherical cap centered at p and 2. All the parameters for our hypergraph are now chosen.

Define a hypergraph $G = G(F, k, \epsilon, n)$ as follows. Let A be a set of size $|V(H_0(n_0, L, d, \theta))|$, let $C_1, \dots, C_{r-1}$ be sets of size $\frac{n}{r}$, and let D be a set of size $\frac{n}{r}$. The vertex set of G is the disjoint union $A \dot\cup C_1 \dot\cup \cdots \dot\cup C_{r-1} \dot\cup D$. To form the hyperedges of G, put a copy of $H_0(n_0, L, d, \theta)$ on A, add all hyperedges with one vertex in each C_i and one vertex in D, and add the following hyperedges between A and C_i. Think of C_i as a set of "evenly spaced" points on the unit sphere $\mathbb{S}^d$. Make a hyperedge on $w = (p_1, \dots, p_\ell) \in A$, $c_1 \in C_1, \dots, c_{r-1} \in C_{r-1}$ if $\|p_i - c_j\| < \sqrt{2} - \beta$ for all $1 \leq i \leq \ell$ and all $1 \leq j \leq r-1$.

To complete the first half of the proof of Theorem 2, we need to prove that G has large chromatic number, large minimum degree, and if F is non-unifoliate r-partite then $F \nsubseteq G$.

Lemma 7. *Given F, k, and ϵ, it is possible to select n large enough so that $\chi(G(F, k, \epsilon, n)) \geq k$.*

Proof. If $n \geq n_0$, then Corollary 5 implies that $\chi(H_0) \geq k$. Since $G[A] = H_0$, $\chi(G) \geq k$. ☐

Lemma 8. *Given F, k, and ϵ, it is possible to select n large enough so that the minimum degree of $G = G(F, k, \epsilon, n)$ is at least $\left(\frac{(r-1)!}{r^{2r-2}} - \xi_r\epsilon\right)\binom{|V(G)|}{r-1}$, where ξ_r is a constant depending only on r.*

Proof. Vertices in C_i have degree at least $|C_1| \cdot \ldots \cdot |C_{i-1}||C_{i+1}| \cdot \ldots \cdot |D| = \left(\frac{n}{r}\right)^{r-1}$ and vertices in D have degree $\prod |C_i| = \left(\frac{n}{r}\right)^{r-1}$. Consider some vertex $(p_1, \dots, p_\ell)$ in A. By Lemma 6 there are roughly $(1/2 - \epsilon)|C_i|$ elements of C_i within distance $\sqrt{2} - \beta$ of each of p_i. In the worst case, there are only $(2^{-\ell} - \xi_1\epsilon)|C_i|$ elements of C_i within $\sqrt{2} - \beta$ of all of $p_1, \dots, p_\ell$, where ξ_1 is a constant depending only on ℓ with $\xi_1\epsilon$ much less than $2^{-\ell}$. Thus the vertex $(p_1, \dots, p_\ell)$ will have degree at least $(2^{-\ell(r-1)} - \xi_2\epsilon)\prod |C_i|$ where again ξ_2 is some constant depending only on ℓ. Therefore, if n is sufficiently large, the minimum degree of G is at least

$$\min\left\{\left(\frac{n}{r}\right)^{r-1}, (2^{-\ell(r-1)} - \xi_2\epsilon)\left(\frac{n}{r}\right)^{r-1}\right\} \geq \left(\frac{n}{r2^\ell}\right)^{r-1} - \epsilon\xi_3 n^{r-1}, \tag{2}$$

where ξ_3 is a constant depending only on r. Since $\ell = \lceil \log_2 r \rceil$, $|V(G)| = n + |V(H_0)|$, and $|V(H_0)|$ is fixed, we can choose n large enough so that

$$\delta(G) \geq \frac{(r-1)!}{r^{2r-2}} \binom{|V(G)|}{r-1} - \epsilon \xi_4 n^{r-1},$$

for some constant ξ_4 depending only on r. The proof is now complete, but note that the relative sizes of C_i and D could be optimized to obtain equality in the minimum in (2) and thus a better bound on the minimum degree of G. □

We now turn our attention to proving that if F is non-unifoliate r-partite, then G does not contain a copy of F. For this, we need a couple of helpful lemmas. Throughout the rest of this section, for $x, y \in \mathbb{S}^d$, $\rho(x, y)$ denotes the Euclidean distance between x and y.

Lemma 9. *Let x, y, and z be points on $\mathbb{S}^d$ and let $0 < a < \frac{1}{10}$. If*

- *either $\rho(x, y) < a$ or $\rho(x, y) > 2 - a$,*
- *and either $\rho(y, z) < a$ or $\rho(y, z) > 2 - a$,*

then either $\rho(x, z) < 4\sqrt{a}$ or $\rho(x, z) > 2 - 4\sqrt{a}$.

Proof. There are four cases. If $\rho(x, y) < a$ and $\rho(y, z) < a$, then the triangle inequality implies that $\rho(x, z) < 2a$. If $\rho(x, y) > 2 - a$ and $\rho(y, z) < a$, then let x' be the point antipodal to x on the sphere. Since x, x', y forms a right triangle (with right angle at y),

$$\rho^2(x', y) = \rho^2(x, x') - \rho^2(x, y) < 4 - (2-a)^2 \leq 4a,$$

so by the triangle inequality, $\rho(x', z) \leq \rho(x', y) + \rho(y, z) \leq 2\sqrt{a} + a \leq 3\sqrt{a}$. Now using that x, x', z forms a right triangle (with right angle at z),

$$\rho^2(x, z) = \rho^2(x, x') - \rho^2(x', z) \geq 4 - 9a > 4 - 16\sqrt{a} + 16a = (2 - 4\sqrt{a})^2.$$

(The last inequality used that $-9a > -16\sqrt{a} + 16a$ since $a < \frac{1}{10}$.) This finishes the case that $\rho(x, y) > 2 - a$ and $\rho(y, z) < a$ and also finishes the case $\rho(x, y) < a$ and $\rho(y, z) > 2 - a$ by symmetry. The last case is when $\rho(x, y) > 2 - a$ and $\rho(y, z) > 2 - a$. Let y' be the point antipodal to y on the sphere. Then

$$\begin{aligned}
\rho^2(x, y') &= \rho^2(y, y') - \rho^2(x, y) < 4 - (2-a)^2 \leq 4a, \\
\rho^2(z, y') &= \rho^2(y, y') - \rho^2(z, y) < 4 - (2-a)^2 \leq 4a, \\
\rho(x, z) &\leq \rho(x, y') + \rho(y', z) < 4\sqrt{a}.
\end{aligned}$$

□

Lemma 10. *Let $x = (x_1, \ldots, x_\ell)$ and $y = (y_1, \ldots, y_\ell)$ be two distinct vertices of $H_0 = H_0(n_0, L, d, \theta)$ contained together in a hyperedge of H_0. Then for every $1 \leq i \leq \ell$, either $\rho(x_i, y_i) > 2 - 4\sqrt{\theta}$ or $\rho(x_i, y_i) < 4\sqrt{\theta}$.*

Proof. Let E be the edge containing x and y and let $\bar{z}^1, \dots, \bar{z}^r$ be an ordering of the vertices of E such that if $ab \in E(B_i)$ then $\rho(z_i^a, z_i^b) > 2 - \theta$ (recall that the vertex $\bar{z}^j$ consists of an ℓ-tuple $(z_1^j, \dots, z_\ell^j)$ with each z_i^j a point on the unit sphere). Let $1 \le a \neq b \le r$ such that $x = \bar{z}^a$ and $y = \bar{z}^b$. If $ab \in E(B_i)$ then $\rho(x_i, y_i) > 2 - \theta$. If $ab \notin E(B_i)$, they appear in the same part of the bipartite graph B_i so let c be a common neighbor of a and b in B_i. Now since $ac, cb \in E(B_i)$, we have that $\rho(x_i, z_i^c) > 2 - \theta$ and $\rho(y_i, z_i^c) > 2 - \theta$ so that Lemma 9 implies that $\rho(x_i, y_i) < 4\sqrt{\theta}$ or $\rho(x_i, y_i) > 2 - 4\sqrt{\theta}$ (in fact the proof of Lemma 9 implies that $\rho(x_i, y_i) < 4\sqrt{\theta}$). □

Lemma 11. *Let $x = (x_1, \dots, x_\ell)$ and $y = (y_1, \dots, y_\ell)$ be two distinct vertices of $H_0 = H_0(n_0, L, d, \theta)$ which appear in the same component of H_0 and are at distance at most $f = |E(F)|$. That is, there exist hyperedges $E_1, \dots, E_q$ of H_0 where $q \le f$, $x \in E_1$, $y \in E_q$, and $E_i \cap E_{i+1} \neq \emptyset$. Then for every $1 \le i \le \ell$, either $\rho(x_i, y_i) > 2 - 4^f \theta^{2^{-f}}$ or $\rho(x_i, y_i) < 4^f \theta^{2^{-f}}$.*

Proof. Consider a path $E_1, \dots, E_q$ from x to y with $q \le f$ and fix $1 \le i \le \ell$. For $1 \le j \le q-1$, let $e^j \in E_j \cap E_{j+1}$. Recall that $x = (x_1, \dots, x_\ell)$, $y = (y_1, \dots, y_\ell)$, and $e^j = (e_1^j, \dots, e_\ell^j)$ where x_i, y_i, and e_i^j are points on the sphere. We will prove by induction on j that either $\rho(x_i, e_i^j) < 4^j \theta^{2^{-j}}$ or $\rho(x_i, e_i^j) > 2 - 4^j \theta^{2^{-j}}$. The base case of $j = 1$ is just Lemma 10, since x and e^1 are contained together in a hyperedge. For the inductive step, the fact that e^j and e^{j+1} are contained together in a hyperedge, Lemma 10, and induction imply that

$$\begin{aligned} &\text{either} \quad \rho(x_i, e_i^j) < 4^j \theta^{2^{-j}} \quad \text{or} \quad \rho(x_i, e_i^j) > 2 - 4^j \theta^{2^{-j}}, \\ &\text{either} \quad \rho(e_i^j, e_i^{j+1}) < 4\sqrt{\theta} \quad \text{or} \quad \rho(e_i^j, e_i^{j+1}) > 2 - 4\sqrt{\theta}. \end{aligned}$$

Since $4^j \theta^{2^{-j}} \ge 4\sqrt{\theta}$, Lemma 9 shows that (note we selected θ so that $4^f \theta^{2^{-f}} < \frac{1}{10}$)

$$\text{either} \quad \rho(x_i, e_i^{j+1}) < 4\sqrt{4^j \theta^{2^{-j}}} \quad \text{or} \quad \rho(x_i, e_i^{j+1}) > 2 - 4\sqrt{4^j \theta^{2^{-j}}}.$$

Since $4\sqrt{4^j \theta^{2^{-j}}} \le 4^{j+1} \theta^{2^{-j-1}}$, the proof of the inductive step is complete.

Therefore either $\rho(x_i, e_i^{q-1}) < 4^{q-1} \theta^{2^{-q+1}}$ or $\rho(x_i, e_i^{q-1}) > 2 - 4^{q-1} \theta^{2^{-q+1}}$. But since e^{q-1} and y are contained together in the hyperedge E_q, one last application of Lemmas 9 and 10 show that either $\rho(x_i, y_i) < 4^q \theta^{2^{-q}}$ or $\rho(x_i, y_i) > 2 - 4^q \theta^{2^{-q}}$. Since $q \le f$, the proof is complete. □

Lemma 12. *If F is not unifoliate r-partite, then G does not contain a copy of F.*

Proof. Assume G contains a copy of F; we will prove that F is unifoliate r-partite, a contradiction. Any copy of F must use vertices of A since otherwise it would be r-partite. It cannot be completely contained in A since by construction any L vertices in A induce a linear hyperforest and we set $L = |V(F)|$. Let F' be the subhypergraph of F formed by restricting F to $A \cup C_1 \cup \dots \cup C_{r-1}$. We first show that F' is unifoliate r-partite and then secondly show that this implies that F is unifoliate r-partite.

Claim 1: F' is unifoliate r-partite.

Proof. Let $X = V(F') \cap A$ so $G[X]$ is a linear hyperforest. We claim that $V_1 = X$, $V_2 = V(F') \cap C_1, \ldots, V_r = V(F') \cap C_{r-1}$ is a partition witnessing that F' is unifoliate r-partite. First, $F'[X]$ is a linear hypertree since $G[X]$ is a linear hypertree. Also, all other edges cross the partition since edges in G not completely contained in A and not using vertices of D use one vertex of A and one vertex from each C_i. Now consider a cycle C using exactly one edge E of $G[X]$ and cross-edges which intersect A only in vertices in the same component of the hyperforest as E. We will show that such a cycle does not exist by deriving a contradiction.

Let $E_1, \ldots, E_m$ be the edges of the cycle C labeled so that E_1 is the edge of C in $G[X]$ and $E_i \cap E_{i+1} \neq \emptyset$. Let $x \in E_1 \cap E_2$ and $y \in E_1 \cap E_m$ so that $x \neq y$. Since the bipartite graphs $B_1, \ldots, B_\ell$ cover $K_{[r]}$, there is some $1 \leq i \leq \ell$ such that $\rho(x_i, y_i) > 2 - \theta$. Fix such an i for the remainder of this proof. For $2 \leq s \leq m$, let $e^s \in E_s \cap A$ (there is a unique such e^s since these are all cross-edges). Note that $x = e^2$ and $y = e^m$. Finally, let $f = |E(F)|$.

We now claim that for all $2 \leq s \leq m$, $\rho(x_i, e^s_i) < s4^f\theta^{2^{-f}}$. Since $e^m = y$, this will contradict that $\rho(x_i, y_i) > 2 - \theta$. We prove that $\rho(x_i, e^s_i) < s4^f\theta^{2^{-f}}$ by induction. First, since C uses vertices in $H_0[X]$ within the same component as E_1, the vertices e^s are all within the same component of H_0 so Lemma 11 implies that either their ith coordinates (pairwise) are within distance $4^f\theta^{2^{-f}}$ of each other or have distance at least $2 - 4^f\theta^{2^{-f}}$. In particular, either $\rho(e^s_i, e^{s+1}_i) < 4^f\theta^{2^{-f}}$ or $\rho(e^s_i, e^{s+1}_i) > 2 - 4^f\theta^{2^{-f}}$ and we claim that the latter is impossible. First, if $e^s = e^{s+1}$ then obviously $\rho(e^s_i, e^{s+1}_i) = 0 < 4^f\theta^{2^{-f}}$. For $e^s \neq e^{s+1}$, since $E_s \cap E_{s+1} \neq \emptyset$, there exists a vertex $z \in E_s \cap E_{s+1} \cap (C_1 \cup \cdots \cup C_{r-1})$. Since z is contained together with e^s in the cross-hyperedge E_s, by the definition of cross-edges of G we have that $\rho(e^s_i, z_i) < \sqrt{2} - \beta$. Similarly, $\rho(e^{s+1}_i, z_i) < \sqrt{2} - \beta$. By (1) (with $p = z_i$), $\rho(e^{s+1}_i, e^s_i) < 2 - 4^f\theta^{2^{-f}}$ since both are within distance $\sqrt{2} - \beta$ of z_i. By Lemma 11, this implies that $\rho(e^s_i, e^{s+1}_i) < 4^f\theta^{2^{-f}}$. By induction and the triangle inequality, $\rho(e^2_i, e^s_i) < s4^f\theta^{2^{-f}}$. Since $x = e^2$ and $y = e^m$, $\rho(x_i, y_i) < m4^f\theta^{2^{-f}}$, a contradiction of the coordinate choice i. This contradiction proves that no such cycle can exist so that $V_1, \ldots, V_r$ witnesses that F' is unifoliate r-partite. □

Claim 2: F is unifoliate r-partite.

Proof. We will show that we can extend $V_1, \ldots, V_t$ to be a witness to the fact that F is unifoliate r-partite, contradicting the choice of F. Indeed, let $W_1 = V_1 \cup (V(F) \cap D), W_2 = V_2, \ldots, W_t = V_t$ be a partition of $V(F)$. That is, add all vertices which appear in $V(F) \cap D$ to V_1. Note that edges of F are either contained in W_1 or use one vertex from each W_i, since the edges of F touching D use one vertex from D and one vertex from each C_i and $W_{i+1} \subseteq C_i$. Also, $F[W_1]$ is still a linear hyperforest since no edges of $F - F'$ were added inside W_1. Finally, consider a cycle C. If C uses no edges intersecting D then it is a cycle in F' so it is good. If C uses an edge E of $F - F'$, then let $\{x\} = E \cap W_1$ and notice that x is an isolated vertex in $F[W_1]$. Since C must use at least one more edge and x is isolated, the cycle either uses another vertex of W_1 so uses vertices from at least two components of $F[W_1]$, or C uses only the vertex x from W_1 and so uses zero edges of $F[W_1]$. Thus $W_1, \ldots, W_t$ witnesses that F is unifoliate r-partite. □

Claim 2 contradicts the assumption that F is not unifoliate r-partite, completing the proof. □

We shall now prove half of Theorem 2, i.e. that if F is a non-unifoliate r-partite hypergraph, then the family of F-free hypergraphs has chromatic threshold at least $(r-1)!r^{-2r+2}$. We must prove that the infimum of the values c such that the family of F-free hypergraphs H with minimum degree at least $c\binom{|V(H)|}{r-1}$ has bounded chromatic number. So consider $c < \frac{(r-1)!}{r^{2r-2}}$ and assume the family

$$\mathcal{H} = \left\{ H : F \nsubseteq H, \delta(H) \geq c\binom{|V(H)|}{r-1} \right\}$$

has bounded chromatic number. That is, there exists some integer k so that $\chi(H) < k$ for every $H \in \mathcal{H}$. Now define $\epsilon > 0$ so that

$$c < \frac{(r-1)!}{r^{2r-2}} - \xi_r \epsilon$$

where ξ_r is the constant depending only on r from Lemma 8. Now that we have chosen F, k, and ϵ, by Lemmas 7, 8, and 12, we can select n large enough so that $G = G(F, k, \epsilon, n)$ is an element of $\mathcal{H}$ with chromatic number at least k, a contradiction.

3 Strong Unifoliate r-partite hypergraphs

In this section, we prove that if G is an r-uniform, strong unifoliate r-partite hypergraph, then the family of G-free hypergraphs has chromatic threshold zero.

Definition. Let d be an integer, H an r-uniform hypergraph, and $v \in V(H)$. The hypergraph H is (d, v)-*bounded* if there is a vertex set $A_v \subseteq V(H)$ with $|A_v| \leq d$ such that every edge of H containing v intersects A_v. A hypergraph H is d-*degenerate* if there exists an ordering $v_1, \ldots, v_n$ of the vertices such that for every $1 \leq i \leq n$, the hypergraph $H[v_1, \ldots, v_i]$ is (d, v_i)-bounded.

Lemma 13. *Let H be a d-degenerate hypergraph. Then $\chi(H) \leq d + 1$.*

Proof. Use a greedy coloring of the vertices in the order $v_1, \ldots, v_n$ given by the definition of d-degenerate. □

Lemma 14. *Let G be an r-uniform linear hyperforest on d vertices and let H be an r-uniform, non-d-degenerate hypergraph. Then H contains a copy of G.*

Proof. Observe that there is some induced subhypergraph H' of H such that for every $v \in V(H')$ and every subset $A \subseteq V(H')$ with $|A| \leq d$, there is a hyperedge of H' containing v and missing A. Indeed, delete vertices of H one by one until the condition is satisfied. If all vertices were deleted then H is d-degenerate, a contradiction.

Now embed the edges of G greedily into H'. Since G is a linear hyperforest, its edges can be ordered $E_1, \ldots, E_m$ so that E_i uses at most one vertex from $E_1 \cup \cdots \cup E_{i-1}$. To embed E_i, let A be the set of previously embedded vertices and x the previously embedded vertex which E_i must extend if $E_i \cap (E_1 \cup \cdots \cup E_{i-1}) \neq \emptyset$ and otherwise let x be any vertex of H' outside A. Since $|A| \leq d$, the definition of H' guarantees the existence of a hyperedge missing $A - x$ to which E_i can be embedded. □

Definition. The following definitions are from [3]. A *fiber bundle* is a tuple (B, γ, F) such that B is a hypergraph, F is a finite set, and $\gamma : V(B) \to 2^{2^F}$. That is, γ maps vertices of B to collections of subsets of F, which we can consider as hypergraphs on vertex set F. The hypergraph B is called the *base hypergraph* of the bundle and F is the *fiber* of the bundle. For a vertex $b \in V(B)$, the hypergraph $\gamma(b)$ is called the *fiber over* b. A fiber bundle (B, γ, F) is (r_B, r_γ)*-uniform* if B is an r_B-uniform hypergraph and $\gamma(b)$ is an r_γ-uniform hypergraph for each $b \in V(B)$. Given $X \subseteq V(B)$, the *section of* X is the hypergraph with vertex set F and edges $\cap_{x \in X} \gamma(x)$. In other words, the section of X is the collection of subsets of F that appear in the fiber over x for every $x \in X$. For a hypergraph H, define $\dim_H(B, \gamma, F)$ to be the maximum integer d such that there exist d pairwise disjoint edges $E_1, \ldots, E_d$ of B (i.e. a matching) such that for every $x_1 \in E_1, \ldots, x_d \in E_d$, the section of $\{x_1, \ldots, x_d\}$ contains a copy of H.

Balogh, Butterfield, Hu, Lenz, and Mubayi [3] proved the following theorem about fiber bundles.

Theorem 15. *Let $r_B \geq 2$, $r_\gamma \geq 1$, $d \in \mathbb{Z}^+$, $0 < \epsilon < 1$, and K be an r_γ-uniform hypergraph with zero Turán density. Then there exists constants $C_1 = C_1(r_B, r_\gamma, d, \epsilon, K)$ and $C_2 = C_2(r_B, r_\gamma, d, \epsilon, K)$ such that the following holds. Let (B, γ, F) be any (r_B, r_γ)-uniform fiber bundle where $\dim_K(B, \gamma, F) < d$ and for all $b \in V(B)$, $|\gamma(b)| \geq \epsilon\binom{|F|}{r_\gamma}$. If $|F| \geq C_1$, then $\chi(B) \leq C_2$.*

We will apply Theorem 15 to the following fiber bundle.

Definition. Given r-uniform hypergraphs T and H, define the *T-bundle of H* as the following fiber bundle. Let B be the hypergraph with vertex set $V(H)$, where a set $X \subseteq V(B)$ is a hyperedge of B if $|X| = |V(T)|$ and $H[X]$ contains a (not necessarily induced) copy of T. Let $F = V(H)$. Define $\gamma : V(B) \to 2^{2^F}$ as the map which sends $b \in V(B)$ to $\{A \subseteq F : A \cup \{b\} \in E(H)\}$. The *$T$-bundle of H* is the fiber bundle (B, γ, F).

A simple corollary of Lemmas 13 and 14 is the following.

Corollary 16. *Let T be an r-uniform linear hyperforest, let H be an r-uniform hypergraph, and let (B, γ, F) be the T-bundle of H. Then $\chi(H) \leq (|V(T)| + 1)\chi(B)$.*

Proof. Let $C \subseteq V(B)$ be a color class in a proper $\chi(B)$-coloring of B. If $H[C]$ is non-$|V(T)|$-degenerate, then by Lemma 14 $H[C]$ contains a copy of T. This copy of T becomes an edge of B contained in C contradicting that the coloring of B is proper. Thus $H[C]$ is $|V(T)|$-degenerate so Lemma 13 implies that $\chi(H[C]) \leq |V(T)| + 1$. Combining these two colorings produces a proper $(|V(T)| + 1)\chi(B)$-coloring of H. □

We are now ready to complete the proof of Theorem 2, i.e. prove that if G is an r-uniform, strong unifoliate r-partite hypergraph, then the family of r-uniform, G-free hypergraphs has chromatic threshold zero. Let G be an r-uniform, strong unifoliate r-partite hypergraph with m vertices. Let H be an r-uniform, G-free hypergraph with $\delta(H) \geq \epsilon\binom{|V(H)|}{r-1}$. We need to show that the chromatic number of H is bounded by a constant depending only on ϵ and G. Let $V_1, \ldots, V_r$ be a vertex partition of $V(G)$ guaranteed by the definition of strong unifoliate r-partite and let t be the number of components in $G[V_1]$. Let (B, γ, F) be the $G[V_1]$-bundle of H and let K be the complete $(r-1)$-uniform, $(r-1)$-partite hypergraph with $(rm)^m$ vertices in each part. Since $\delta(H) \geq \epsilon\binom{|V(H)|}{r-1}$, for every $x \in V(H) = V(B)$, we have $|\gamma(x)| = d(x) \geq \delta(H) \geq \epsilon\binom{|V(H)|}{r-1}$. Since $F = V(H)$, we have that for every $x \in V(B)$, $|\gamma(x)| \geq \epsilon\binom{|F|}{r-1}$.

First, assume that $\dim_K(B, \gamma, F) < t$. Notice that (B, γ, F) is $(|V_1|, r-1)$-uniform by definition and t depends on G, so Theorem 15 implies that if $|F| \geq C_1$ then $\chi(B) \leq C_2$, where C_1 and C_2 are constants depending only on ϵ and G. Since $F = V(H) = V(B)$, this implies that $\chi(B) \leq \max\{C_1, C_2\}$ so $\chi(B)$ is bounded by a constant depending only on ϵ and G. Now Corollary 16 shows that the chromatic number of H is bounded by a constant depending only on ϵ and G, exactly what we would like to prove.

Therefore, $\dim_K(B, \gamma, F) \geq t$. Let $T_1, \ldots, T_t$ be the components of $G[V_1]$. We now show how to find a copy of G in H. Since $\dim_K(B, \gamma, F) \geq t$, there exists edges $E_1, \ldots, E_t$ of B with large sections. Recall that by the definition of B, for every i $H[E_i]$ contains a copy of $G[V_1]$ so we can find a copy of T_i in $H[E_i]$ for each i. Pick any $x_1 \in V(T_1), \ldots, x_t \in V(T_t)$. The definition of $\dim_K(B, \gamma, F)$ implies that the section of $x_1, \ldots, x_t$ contains a copy of K, the complete $(r-1)$-uniform, $(r-1)$-partite hypergraph with $(rm)^m$ vertices in each class. These copies of K are used to embed the rest of G in H as follows.

Let G' be the $(r-1)$-uniform hypergraph with vertex set $V_2 \cup \cdots \cup V_r$ where $\{v_2, \ldots, v_r\}$ is a hyperedge of G' if there exists a cross-hyperedge of G containing $\{v_2, \ldots, v_r\}$. Let $C_1, \ldots, C_c$ be the components G' and define

$$N_{V_1}(C_i) = \{x \in V_1 : \exists \{v_2, \ldots, v_r\} \in E(C_i) \,\text{with}\, \{x, v_2, \ldots, v_r\} \in E(G)\}.$$

In other words, $N_{V_1}(C_i)$ is the "neighborhood" of C_i in G, the collection of vertices of V_1 which are contained in a hyperedge of G together with some $(r-1)$-edge of C_i. Note that since G is strong unifoliate r-partite, each C_i is an $(r-1)$-partite, $(r-1)$-uniform hypergraph. Also since G is strong unifoliate r-partite, for each C_i and each hypertree T_j, there is at most one vertex of T_j in $N_{V_1}(C_i)$. Let $x_{i,j}$ be such a vertex if it exists and otherwise define $x_{i,j}$ to be any vertex in T_j. We can now find a copy of G in H by embedding each C_i into the copy of K contained in the section of $x_{i,1}, \ldots, x_{i,t}$. This forms a copy of G in H since $V(C_1) \dot\cup \cdots \dot\cup V(C_c)$ is a partition of $V_2 \cup \cdots \cup V_r$ and each C_i is embedded into exactly one of the sections of vertices from the hypertrees in V_1.

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