The typical structure of graphs without given excluded subgraphs

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Abstract

Let $\mathcal{L}$ be a finite family of graphs. We describe the typical structure of $\mathcal{L}$-free graphs, improving our earlier results [2] on the Erdős-Frankl-Rödl theorem [6], by proving our earlier conjecture that, for $p = p(\mathcal{L}) = \min_{L \in \mathcal{L}} \chi(L) - 1$, the structure of almost all $\mathcal{L}$-free graphs is very similar to that of a random subgraph of the Turán graph $T_{n,p}$. The "similarity" is measured in terms of graph theoretical parameters of $\mathcal{L}$.

1. Introduction

Notation. We restrict our attention to simple graphs and the notation we use is standard. Thus $V(G)$ denotes the set of vertices of a graph G, and for a vertex set

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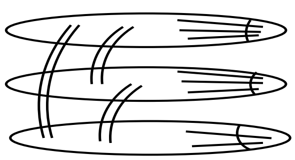
$X \subseteq V(G)$, $G[X]$ denotes the subgraph of G induced by X. For $X \subseteq V(G)$, we mostly shorten $e(G[X])$ to $e(X)$. We write G_n for a graph of order n; in fact, much of the time, the first suffix in our notation is the order of the graph, as in $K_p, T_{n,p}$ and H_k. The chromatic number of a graph L is denoted by $\chi(L)$, the order of L by $v(L)$; $\Gamma(x)$ is the set of neighbours of a vertex x, $d(x) = |\Gamma(x)|$ is its degree, and $d(x, A) = |\Gamma(x) \cap A|$ is the degree of x into a set $A \subseteq V(G)$. Also, $\Gamma^*(X)$ denotes the set of *common* neighbours of the vertices in X: $\Gamma^*(X) = \bigcap_{x\in X} \Gamma(x)$.

We write K_p for the complete graph on p vertices, and $T_{n,p}$ for the *p-class Turán graph*: Thus to obtain $T_{n,p}$ we partition n vertices into p classes so that their sizes are as equal as possible, and join two vertices if they belong to different classes. It is easy to see that

$$\left(1-\frac{1}{p}\right)\binom{n}{2} \le e(T_{n,p}) \le \left(1-\frac{1}{p}\right)\frac{n^2}{2} \quad \text{and} \quad e(T_{n,p}) = \left(1-\frac{1}{p}\right)\frac{n^2}{2} + O(n).$$

For a given graph G_n and p, a *p-partition* is a partition of $V(G_n)$ into p classes, a p-partition $(U_1, \dots, U_p)$ of $V(G_n)$ is *optimal* if $\sum e(U_i)$ is as small as possible. Sometimes, shortly we refer to such a partition as an *optimal p-partition.*



Given a partition $(U_1, \dots, U_p)$ of $V(G_n)$, we shall call the edges inside some partition-class U_i "horizontal edges".[1] Also, for a given partition $(U_1, \dots, U_p)$ we define the *horizontal degree* of $x \in U_i$ to be $|\Gamma(x) \cap U_i|$.

We say that a pair of vertex sets (A, B) is **completely joined** in a graph G_n if $A, B \subset V(G_n), A \cap B = \emptyset$, and each $x \in A$ is joined to each $y \in B$ in G_n. Having two vertex-disjoint graphs M and Q, $M \otimes Q$ denotes the graph obtained by joining each vertex of M to each vertex of Q.

In this paper the logarithms have always base 2. We shall often use the binary entropy function $H(x) = x \log_2 \frac{1}{x} + (1-x) \log_2 \frac{1}{1-x}$.

1.1. Turán type extremal problems

We say that the graph G *contains* L and write $L \subseteq G$ if L is a (not necessarily induced) subgraph of G. Given a family $\mathcal{L}$ of graphs, G is called **$\mathcal{L}$-free** if G contains no $L \in \mathcal{L}$, We call $\mathcal{L}$ the family of **forbidden graphs**. We assume that $e(L) > 0$ for each $L \in \mathcal{L}$. $\mathcal{P}(n, \mathcal{L})$ denotes the class of $\mathcal{L}$-free graphs with vertex set

[1]Mostly we call these as "horizontal degrees" that corresponds to specific figures of the optimal partition where these edges are almost horizontal.

$[n] := \{1, \ldots, n\}$; [2] $\mathbf{ex}(n, \mathcal{L})$ is the maximum number of edges an $\mathcal{L}$-free graph G_n can have, and an $\mathcal{L}$-free graph with $\mathbf{ex}(n, \mathcal{L})$ edges is **$\mathcal{L}$-extremal** or sometimes simply **extremal**. When $\mathcal{L}$ consists of a single graph L, we write $\mathbf{ex}(n, L)$ instead of $\mathbf{ex}(n, \{L\})$.

The basic Turán type extremal problem is as follows.

> For a given family $\mathcal{L}$, determine or estimate $\mathbf{ex}(n, \mathcal{L})$, and describe the (asymptotic) structure of extremal graphs, as $n \to \infty$.

We fix a forbidden family $\mathcal{L}$, and let

$$p := p(\mathcal{L}) = \min_{L \in \mathcal{L}} \chi(L) - 1. \tag{1}$$

For every $\mathcal{L}$ there is a constant $a > 0$ such that

$$\mathbf{ex}(n, \mathcal{L}) = e(T_{n,p}) + O(n^{2-a}) \tag{2}$$

and all the extremal graphs of order n can be transformed into $T_{n,p}$ by deleting and adding $O(n^{2-a})$ edges, as proved by Erdős [5] and Simonovits [8]. For a more detailed description of this field, see the book of Bollobás [3] or the surveys of Simonovits [9], [10], Füredi [7] and Bollobás [4].

The main idea of the results discussed here and in the preceding papers is that most of the $\mathcal{L}$-free graphs can be regarded as subgraphs of some extremal or almost extremal graphs for $\mathcal{L}$. Our starting point was the following theorem of Erdős, Frankl and Rödl [6].

Theorem 1. *For every* $\mathcal{L}$

$$2^{\mathbf{ex}(n,\mathcal{L})} \le |\mathcal{P}(n, \mathcal{L})| \le 2^{\mathbf{ex}(n,\mathcal{L})+o(n^2)}. \tag{3}$$

Note that the lower bound in (3) is trivial, as every subgraph of an $\mathcal{L}$-extremal graph is $\mathcal{L}$-free. In [2] we improved the upper bound in Theorem 1, see Theorem 3. Here we go one step further, and give a structural characterization of almost all graphs in $\mathcal{P}(n, \mathcal{L})$. To formulate our results, we need a definition.

Definition 2 (Decomposition Family)**.** Given a family $\mathcal{L}$ (and $p = p(\mathcal{L})$), let $\mathcal{M} := \mathcal{M}(\mathcal{L})$ be the family of minimal graphs M for which there exist an $L \in \mathcal{L}$ and a $t = t_L$ such that $L \subseteq M' \otimes K_{p-1}(t, \ldots, t)$, where M' is the graph obtained by adding t isolated vertices to M. We call $\mathcal{M}$ the *decomposition family* of $\mathcal{L}$.

[2] The vertices of our graphs are fixed, labelled and, for the sake of simplicity, we shall assume that $V(G_n) = \{1, \ldots, n\}$.

In other words, a graph M belongs to $\mathcal{M}$ if whenever n is sufficiently large and we "place" M into a class U_i of $T_{n,p}$, then the obtained graph contains a forbidden $L \in \mathcal{L}$. ("Placing" means adding the edges of a copy of M into $T_{n,p}$, using only vertices of this U_i.) We emphasize that $\mathcal{M}$ always contains a bipartite graph, otherwise $\chi(L) \geq p+2$ for every $L \in \mathcal{L}$.

If $\mathcal{L}$ is finite, then $\mathcal{M}$ is also finite. The converse is not necessarily true. For example, if $\mathcal{L}$ is the family of all the odd cycles, then $\mathcal{M} = \{K_2\}$.

In [2] we gave the following improvement of Theorem 1.

Theorem 3. *For every $\mathcal{L}$ with $p = p(\mathcal{L}) \geq 2$, if $\mathcal{M}$, the decomposition family of $\mathcal{L}$, is finite, then*

$$|\mathcal{P}(n, \mathcal{L})| \leq n^{\mathbf{ex}(n,\mathcal{M})+c_{\mathcal{L}}\cdot n} \cdot 2^{\frac{1}{2}\left(1-\frac{1}{p}\right)n^2}, \tag{4}$$

for a sufficiently large constant $c_{\mathcal{L}} > 0$.

To see that this does strengthen Theorem 1, for a given $\mathcal{L}$, let $L \in \mathcal{L}$ have minimum chromatic number, (i.e. $\chi(L) = p+1$), and pick a t with $K_{p+1}(t, t, \dots, t) \supseteq L$. This implies that there is an $M \in \mathcal{M}$ with $M \subset K(t, t)$ and Theorem 3 implies that

$$|\mathcal{P}(n, \mathcal{L})| \leq |\mathcal{P}(n, L)| \leq n^{\mathbf{ex}(n,M)+c_L\cdot n} \cdot 2^{\frac{1}{2}\left(1-\frac{1}{p}\right)n^2} \leq 2^{\mathbf{ex}(n,\mathcal{L})+O(n^{2-c})},$$

where $0 < c < 1/t$. Here we used (2), $\mathbf{ex}(n, K(t, t)) = O(n^{2-1/t})$ and $n^{O(n^{2-1/t})} < 2^{O(n^{2-c})}$.

Remark 4. One knowing this field may ask: do we need/use the Szemerédi Regularity Lemma [11] in this paper or not? The answer is that in some crucial steps of our previous paper [2] we did use and here we use several results from that paper. However, here we do not need the explicit use of the Regularity Lemma.

1.2. Why do we need the finiteness of $\mathcal{M}$?

We construct a family $\mathcal{L}$ which shows that the condition of finiteness of $\mathcal{M}$ is needed in Theorem 3. Denote by I_ν the ν-vertex graph with no edges. Set $f(x) = 2^{x^2}$ and let $L_m := C_m \otimes I_{f(m)}$, i.e. join $f(m)$ independent vertices to an m-cycle C_m completely. Put

$$\mathcal{L} := \{C_m \otimes I_{f(m)} \ : \ m = 3, 4, 5, \dots\}.$$

We assert that for this $\mathcal{L}$ the conclusions of Theorem 3 (and Theorem 5 below) do not hold: for infinite $\mathcal{L}$ they are not necessarily true.

(a) Now, $\mathcal{M}$ is the family of all cycles, $p = p(\mathcal{L}) = 2$ and $\mathbf{ex}(n, \mathcal{M}) = n - 1$. Hence (4) would yield that

$$|\mathcal{P}(n, \mathcal{L})| \le n^{(c_{\mathcal{L}}+1)\cdot n} \cdot 2^{\frac{1}{2}\left(1-\frac{1}{p}\right)n^2} \le 2^{n^2/4+O(n\log n)}. \tag{5}$$

(b) On the other hand, let G_m be a graph on $m = \lceil n/2 \rceil$ vertices with average degree at least $\log^2 n$, and girth $g(G_m) \to \infty$. (Random graph methods or the Margulis-Lubotzky-Phillips-Sarnak type graphs imply that there are such graphs with girth $g(G_m) > c_1 \frac{\log m}{\log\log m}$.) Let $S_n := G_m \otimes I_{n-m}$. With our condition on $f(x)$ we have $f(g(G_m)) > 2m$. One can easily see that S_n is $\mathcal{L}$-free and

$$|\mathcal{P}(n, \mathcal{L})| \ge 2^{e(S_n)} > 2^{e(T_{n,2})} \cdot 2^{cn\log^2 n},$$

contradicting (5). So Theorem 3 does not hold for this infinite $\mathcal{L}$.

1.3. Results

The goal of this paper is to prove Conjecture 2.3 from [2]. We actually prove a stronger result, Theorem 9, which provides an estimate for the decay of the number of "bad graphs" compared to the number of $\mathcal{L}$-free graphs (and gives additional structural information on almost all $\mathcal{L}$-free graphs).

Theorem 5. *Let $\mathcal{L}$ be a finite family of graphs. Then there exists a constant $h_{\mathcal{L}}$ such that for almost all $\mathcal{L}$-free graphs G_n we can delete $h_{\mathcal{L}}$ vertices of G_n and partition the remaining vertices into p classes $(U_1, \dots, U_p)$ so that $G[U_i]$ is an $\mathcal{M}$-free graph for every $1 \le i \le p$.*

Observe that the family $\mathcal{L}$ constructed in Section 1.2 shows that in Theorem 5 at least the condition that $\mathcal{M}$ is finite is needed.

We shall define several classes of $\mathcal{L}$-free graphs; each one will be used to describe the similarity of typical $\mathcal{L}$-free graphs to random subgraphs of some $\mathcal{L}$-extremal graphs.

Definition 6 (Fixing the parameters I)**.** Let

$$t := \max_{L\in\mathcal{L}} v(L). \tag{6}$$

As in (1), let $p+1$ be the minimum chromatic number of a member of $\mathcal{L}$ and let $\delta < n/(p4^t)$ be a positive constant. We fix the constants

$$\beta_r := \frac{1}{2^{2r+1}}, \tag{7}$$

and an $\varepsilon > 0$ and a $\gamma > 0$ satisfying that

$$H(\varepsilon) < \frac{\beta_t}{100p^4t^2} \quad \text{and} \quad \gamma \le \frac{\beta_t}{100p^5t^5}. \tag{8}$$

Here, we made use of the fact that $\mathcal{L}$ is finite: for infinite $\mathcal{L}$ this definition gives $\varepsilon = 0$.

Definition 7 (Good r-tuples)**.** Given a graph G with a vertex partition $(U_1, \dots, U_p)$, call an r-tuple $X \subset V(G)$ GOOD if for every U_i disjoint from X the number of common neighbours of the vertices of X in U_i is

$$|\Gamma^*(X) \cap U_i| = \left| \bigcap_{x \in X} \Gamma(x) \cap U_i \right| > \frac{1}{4^{r+1}} |U_i|. \tag{9}$$

An r-tuple is BAD if it is not GOOD. We say that X is a BAD *r-tuple* for a class $U_{i(X)}$, if (9) is violated. Note that a set X may be BAD for several classes in a partition, and whether X is GOOD or BAD depends on $(U_1, \dots, U_p)$.

Definition 8 (GOOD graphs)**.** Denote by $\mathcal{P}^h_{\mathbf{GOOD}}(n, \mathcal{L})$ the family of $\mathcal{L}$-free graphs G_n having an optimal partition $(U_1, \dots, U_p)$ in which, if W is the set of vertices having horizontal degree at least εn, (where we use the ε fixed in Definition 6) then $|W| \le h$ and the vertex set $V(G) - W$ contains no BAD r-tuples for $1 \le r \le t$ in G_n.

Theorem 9. *There is an $h = h(\mathcal{L})$ such that almost all $\mathcal{L}$-free graphs are in $\mathcal{P}^h_{\mathbf{GOOD}}(n, \mathcal{L})$: there exist two positive constants, C and $\omega > 1$, such that*

$$\left| \mathcal{P}(n, \mathcal{L}) - \mathcal{P}^h_{\mathbf{GOOD}}(n, \mathcal{L}) \right| \le \frac{C}{\omega^n} \left| \mathcal{P}(n, \mathcal{L}) \right|. \tag{10}$$

The constants C and $\omega > 1$ in Theorem 9 can be computed from its proof (which is unlikely to provide the best possible values). In a forthcoming paper we plan to discuss some consequences of Theorems 5 and 9. In the proof of Theorem 9 we shall use some lemmas of [2].

1.4. Classes of $\mathcal{L}$-free graphs

We shall use several subclasses of $\mathcal{P}(n, \mathcal{L})$. Often we shall neglect indicating the dependence on all the parameters. We shall define ϑ later, in Definition 15, and then fix $\delta := 2\sqrt{H(\vartheta)}$.

1. Let $\mathcal{P}_\vartheta(n, \mathcal{L})$ be the family of $\mathcal{L}$-free graphs on $[n]$ having (optimal) partitions $(U_1, \dots, U_p)$ for which $\sum_i e(U_i) < \vartheta n^2$. These are the *$\vartheta$-Turán graphs.*

2. Let[3] $\mathcal{P}^\delta_{\mathbf{UNIF}}(n, \mathcal{L}) \subset \mathcal{P}_\vartheta(n, \mathcal{L})$ be the family of graphs for which *every optimal p-partition* is such that for every $1 \le i < j \le p$ and every pair of sets $A \subset U_i$, $B \subset U_j$ with $|A| = |B| \ge \lceil \delta n \rceil$ the inequality $e(A, B) > (1/4)|A|\cdot|B|$ holds. We shall call these graphs *δ-lower regular* (where "lower" refers to the fact that we have a lower bound on the density).

3. As in [2], we denote by $\mathcal{P}^\vartheta_{\mathbf{WP}}(n, \mathcal{L})$ the family of graphs $G_n \in \mathcal{P}_\vartheta(n, \mathcal{L})$ all optimal partitions $(U_1, \dots, U_p)$ of which satisfy

$$\left| |U_i| - \frac{n}{p} \right| < \left(\sqrt{\vartheta} \log \frac{1}{\vartheta} \right) n$$

for all i. (WP stands for "well partitioned".)

Let us fix a constant ϑ with $0 < \vartheta < (3p)^{-12}$. (Later we shall have some further restrictions on ϑ.) The "Main Lemma" of [2] asserts that almost all $\mathcal{L}$-free graphs are ϑ-Turán graphs. Note that *here* we quote the results that we actually *proved* in [2], not the weaker form as we stated them there.[4] Unfortunately, in [2] we often replaced $2^{-\rho n^2}$ by the rather weak bound 2^{-n}.

Lemma 10. *(Main Lemma in [2]) Let $0 < \vartheta < (3p)^{-12}$. Then, for a suitable positive constant $\rho = \rho(\vartheta) > 0$ and an integer $n_0(\vartheta)$, for $n > n_0(\vartheta)$ we have*

$$|\mathcal{P}(n, \mathcal{L}) - \mathcal{P}_\vartheta(n, \mathcal{L})| \le 2^{e(T_{n,p}) - \rho n^2}. \tag{11}$$

Lemma 11. *(Lemma 6.1 in [2]) Let $0 < \vartheta < (3p)^{-12}$. Then for $\delta \ge 2H(\vartheta)$ there is a positive constant $\rho = \rho(\vartheta, \delta)$ such that for n sufficiently large we have*

$$\left| \mathcal{P}(n, \mathcal{L}) - \mathcal{P}^\delta_{\mathbf{UNIF}}(n, \mathcal{L}) \right| < 2^{e(T_{n,p}) - \rho n^2}.$$

[3] As δ is a function of ϑ, here we neglect to show in the notation the dependency of the family on ϑ.

[4] The weaker bounds would be sufficient as well for our purposes, but now we think that stating the sharp results is better from point of view of understanding the proof better.

Lemma 12. *(Lemma 6.6 in [2]) Let $0 < \vartheta < (3p)^{-12}$. Then, for a suitable positive constant $\rho = \rho(\vartheta) > 0$ and for n sufficiently large we have*

$$\left|\mathcal{P}_\vartheta(n, \mathcal{L}) - \mathcal{P}^\vartheta_{\mathbf{WP}}(n, \mathcal{L})\right| < 2^{e(T_{n,p}) - \rho n^2}.$$

We shall say that a family of graphs is *negligible* if its cardinality is at most $2^{e(T_{n,p}) - \rho n^2}$ for some constant $\rho > 0$.

Remark 13. Lemmas 11 and 12 assert that the typical vertex-distribution and edge-distribution are very even in our optimal partitions.

Lemma 14. *(Lemma 7.1 in [2]) Given $\mathcal{L}$, let p be defined by (1). For any $\varepsilon > 0$ there is a $0 < \delta(\varepsilon) < 1/p$ such that if $\vartheta > 0$ satisfies that $\delta := 2\sqrt{H(\vartheta)} < \delta(\varepsilon)$, then the following holds: there exist two integers $h_0(\vartheta, \varepsilon, \mathcal{L})$ and $n_0(\vartheta, \varepsilon, \mathcal{L})$ for which, if $G_n \in \mathcal{P}^\delta_{\mathbf{UNIF}}(n, \mathcal{L})$ and $n > n_0$, and if $V(G_n) = (U_1, \dots, U_p)$ is an optimal partition of G_n, then for every $1 \le i \le p$*

$$\Big|\{x \in U_i : d(x, U_i) \ge \varepsilon n\}\Big| \le h_0(\vartheta, \varepsilon, \mathcal{L}).$$

Roughly speaking, Lemma 14 states that in an optimal partition "the number of vertices with 'high' horizontal degree is bounded".

Definition 15 (Fixing the parameters II)**.** In Definition 6 we already determined (for a given $\mathcal{L}$) the integers p and t and the constants β_r, γ and ε. For this ε, we choose a $\delta(\varepsilon)$ as in Lemma 14, and our $\delta > 0$ and ϑ satisfying $\delta = 2\sqrt{H(\vartheta)} < \delta(\varepsilon)$. Let $\rho > 0$ be defined to be the minimum of the $\rho' s$ provided by Lemmas 10, 11 and 12. Make sure that ϑ, δ and ρ are small enough (compared to γ) to satisfy

$$2H(\vartheta) + \rho < \frac{\beta_t \gamma}{4pt}. \tag{12}$$

All these constants should be (and can be) chosen small enough to satisfy

$$2pH(p\delta + 2p\gamma) + H(\varepsilon) + 4\sqrt{\vartheta}\log\frac{1}{\vartheta} + \gamma < \frac{\beta_t}{10pt}. \tag{13}$$

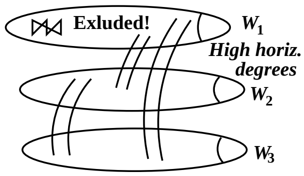
Finally, we fix $h := p \cdot h_0(\vartheta, \varepsilon, \mathcal{L})$, where h_0 is the constant whose existence is provided by Lemma 14, and let $\ell := \lceil \gamma n \rceil$.

Let

$$\mathcal{Q}^h_{\mathbf{GOOD}}(n, \mathcal{L}) := \mathcal{P}^h_{\mathbf{GOOD}}(n, \mathcal{L}) \cap \mathcal{P}^\vartheta_{\mathbf{WP}}(n, \mathcal{L}) \cap \mathcal{P}^\delta_{\mathbf{UNIF}}(n, \mathcal{L}). \tag{14}$$

The next lemma is essentially proved in [2].

Lemma 16 (M-Extension Lemma)**.** *Assume that* $n > n_0(\mathcal{L})$ *and* $G_n \in \mathcal{Q}^h_{\mathbf{GOOD}}(n, \mathcal{L})$. *Denote* $(U_1, \dots, U_p)$ *an optimal* p*-partition of* $V(G_n)$. *Let* W *be the set of vertices of* G_n *having horizontal degrees at least* εn *in this partition. Then for every* $M \in \mathcal{M}$ *and every* i *we have that* $M \not\subseteq G[U_i - W]$.



Proof. Let $W_i = W \cap U_i$. For a contradiction, assume that there is a graph $M \in \mathcal{M}$ with $M \subseteq G[U_1 - W]$. Then there is an $L \in \mathcal{L}$ such that $L \subset (M \cup I_y) \otimes K_{p-1}(t, \dots, t)$, i.e. L is a 'reason' that $M \in \mathcal{M}$. So there is a vertex partition of L into $L_1, \dots, L_p$ such that L_1 spans M and probably some additional isolated vertices, and each of $L_2, \dots, L_p$ is independent in L. By the assumption, we can embed M (spanned by L_1) into $U_1 - W_1$. We fix such an M. The set L_1 is a good ($\le t$)-tuple in $U_1 - W_1$, since $v(L) \le t$. Therefore, using that $G_n \in \mathcal{P}^h_{\mathbf{GOOD}}(n, \mathcal{L})$ we have $(\Gamma^*(L_1) \cap U_2) - W_2$ consists of at least $\beta_t U_2 - |W|$ vertices, and using that $G_n \in \mathcal{P}^{\vartheta}_{\mathbf{WP}}(n, \mathcal{L})$ it is at least $\beta_t n/(2p)$. So an L_2 could be chosen from it. Fixing L_2, the set $L_1 \cup L_2$ is a good ($\le t$)-tuple in $(U_1 \cup U_2) - W$, therefore $\Gamma^*(L_1 \cup L_2) \cap U_3 - W_3$ is 'large' and an $L_3 \subset \Gamma^*(L_1 \cup L_2) \cap U_3 - W_3$ could be chosen. This can be continued till we find L_p in U_p, therefore we have a copy of L in G_n as a subgraph, a contradiction. ∎

2. Some important lemmas

The following easy lemma says that if the size of a subclass of the $\mathcal{L}$-free graphs can be estimated by $2^{e(T_{n,p}) - \rho n^2}$ then this subclass is really negligible.

Lemma 17 (Many GOOD Graphs)**.** *For any fixed* t *and* $h > 0$,

$$\left|\mathcal{Q}^h_{\mathbf{GOOD}}(n, \mathcal{L})\right| > (1 - o(1)) 2^{e(T_{n,p})} \quad \text{as} \quad n \to \infty.$$

We shall need the following simple tail estimate (see for example [1]).

Lemma 18 (Tail Estimate)**.** *If* $\xi_1, \dots, \xi_m$ *are* m *independent random 0-1 variables for which* $Prob(\xi_i = 1) = u > 0$, *then*

$$Prob\left(\sum \xi_i < \frac{1}{2} u m\right) < e^{-\frac{1}{2} u^2 m}.$$

We shall use this lemma in the following setting.

Lemma 19. *Let $G_{n,1/2}$ be a random graph where each edge is chosen independently, with probability $1/2$. Let $X := \{x_1, \dots, x_r\} \subseteq V(G_{n,1/2})$. Let $U \subseteq V(G_{n,1/2})$ be an m-element set disjoint from X. Then,*

$$Prob\left(|\Gamma^*(X) \cap U| < \frac{m}{2^{r+1}}\right) < e^{-\beta_r m}. \tag{15}$$

Recall that $\Gamma^*(X) := \cap_{x \in X} N(x)$ and $\beta_r = 2^{-(2r+1)}$. When we apply Lemma 19, we tend to take $|X|$ bounded, and $|U|$ linear in n.

Proof. Let

$$\xi_y = \begin{cases} 0 & \text{if } y \notin U \cap \bigcap_{i \le r} \Gamma(x_i), \\ 1 & \text{if } y \in U \cap \bigcap_{i \le r} \Gamma(x_i). \end{cases}$$

Clearly, $|\Gamma^*(X) \cap U| = \sum_y \xi_y$. Apply Lemma 18 with $u = 2^{-r}$:

$$Prob\left(\sum_y \xi_y < \frac{m}{2^{r+1}}\right) < e^{-0.5 \cdot 2^{-2r} m} = e^{-\beta_r m}.$$

∎

Proof of Lemma 17. The Turán graph $T_{n,p}$ has $2^{e(T_{n,p})}$ subgraphs. Take any of them at random: select each edge of $T_{n,p}$ independently, with probability $\frac{1}{2}$.

(*) We know that for all but $o(2^{e(T_{n,p})})$ subgraphs $G_n \subseteq T_{n,p}$, if $(U_1, \dots, U_p)$ is the original partition of $T_{n,p}$, then — in the random subgraph — each $x \in U_i$ is joined to each U_j $(j \neq i)$ by at least $\frac{n}{3p}$ edges.

(**) Similarly, if $A \subset U_i, B \subset U_j$, where $i \neq j$, and $|A|, |B| > n^{0.6}$, then in all but $o(2^{e(T_{n,p})})$ subgraphs $G_n \subseteq T_{n,p}$ has an edge between A and B.

Restricting ourselves to these subgraphs, an optimal partition of G_n coincides with the original partition $(U_1, \dots, U_p)$ of $T_{n,p}$. For this partition the number of horizontal edges is 0. If there is an other optimal partition $V_1, \dots, V_p$, then by property (**) there is a labelling of the classes, such that $\sum_{i=1}^p |U_i \Delta V_i| = o(n)$. But by property (*) if two partitions differ then their symmetric difference is at least $n/(3p)$, a contradiction, proving the unicity of the optimal partition. We need this because whether an r-tuple in G_n is BAD or GOOD depends on the partition as well.

A standard application of Lemma 18 implies that all but $o(2^{e(T_{n,p})})$ subgraphs $G_n \subseteq T_{n,p}$ belong to $\mathcal{P}^{\delta}_{\mathbf{UNIF}}(n, \mathcal{L})$, and trivially a typical G_n is well-partitioned, so $G_n \in \mathcal{P}^{\vartheta}_{\mathbf{WP}}(n, \mathcal{L})$. Recalling that $\mathcal{Q}^h_{\mathbf{GOOD}}(n, \mathcal{L}) = \mathcal{P}^h_{\mathbf{GOOD}}(n, \mathcal{L}) \cap \mathcal{P}^{\vartheta}_{\mathbf{WP}}(n, \mathcal{L}) \cap$

$\mathcal{P}^{\delta}_{\mathbf{UNIF}}(n,\mathcal{L})$, it remains to prove that $G_n \in \mathcal{P}^{h}_{\mathbf{GOOD}}(n,\mathcal{L})$ w.h.p.. We assert that the probability that G_n has a BAD r-tuple is $o(1)$ for every $r \le t$. By Lemma 19, only $o(2^{e(T_{n,p})})$ subgraphs have BAD r-tuples. Indeed, an r-tuple can be chosen in at most $\binom{n}{r}$ ways; fixing this r-tuple $X = \{x_1,\dots,x_r\}$, the expected size of $U_j \cap \Gamma^*(X)$ is around $(n/p)\cdot 2^{-r}$. So, for any fixed r-tuple $X \subset V(G_n)$, if $X \cap U_j = \emptyset$, then

$$Prob\Big(|\Gamma^*(X) \cap U_j| < \frac{n}{p2^{r+1}}\Big) < e^{-n/(p2^{2r+1})},$$

and

$$p \cdot \binom{n}{r} e^{-n/(p2^{2r+1})} = o(1).$$

∎

Definition 20 (ℓ-BAD graphs)**.** For given positive integer ℓ, let

$$\mathcal{R}^{\ell}_{\mathbf{BAD}}(n,\mathcal{L}) \subset \mathcal{P}_{\vartheta}(n,\mathcal{L}) \cap \mathcal{P}^{\vartheta}_{\mathbf{WP}}(n,\mathcal{L}) \cap \mathcal{P}^{\delta}_{\mathbf{UNIF}}(n,\mathcal{L})$$

be the family of graphs G_n having an optimal partition $(U_1,\dots,U_p)$ for which the following holds. For at least one $i \le p$, there are pairwise disjoint BAD $(\le t)$-tuples $X_1, X_2, \dots, X_s \subseteq V(G_n) - U_i - W$, with the (same) distinguished class U_i, such that

$$\Big|\bigcup_{j\le s} X_j\Big| \ge \ell. \tag{16}$$

The next lemma claims that, in most GOOD graphs, for a fixed optimal partition the BAD $(\le t)$-tuples can be represented by $o(n)$ vertices.

Lemma 21. *For the constants fixed in Definition 6, and $\ell := \lceil \gamma n \rceil$, there is a $\rho = \rho(\gamma) > 0$ such that*

$$\big|\mathcal{R}^{\ell}_{\mathbf{BAD}}(n,\mathcal{L})\big| \le 2^{e(T_{n,p})-\rho n^2} \quad \textit{for} \quad n > n_0.$$

Proof. Consider a graph $G_n \in \mathcal{R}^{\ell}_{\mathbf{BAD}}(n,\mathcal{L})$. By definition, G_n has an optimal partition $(U_1,\dots,U_p)$ and a class U_j such that there are pairwise disjoint U_j-BAD $(\le t)$-tuples $X_1,\dots,X_s$ with $|\bigcup X_i| \ge \ell$ and $s \le \ell$. We shall use an estimate of the form

$$\big|\mathcal{R}^{\ell}_{\mathbf{BAD}}(n,\mathcal{L})\big| \le p\cdot p^n \cdot n \cdot 2^{H(\vartheta)n^2+1}\cdot n^{tn}\cdot \mathbb{N}_1\cdot \mathbb{N}_2, \tag{17}$$

where on the right-hand side of (17), p stands for the number of ways of choosing a distinguished class U_j, p^n is a crude upper bound on the number of (optimal) p-partitions, n bounds the number of choices for s, and

$$\sum_{i\le\vartheta n^2}\binom{\binom{n}{2}}{i}\le 2\binom{\binom{n}{2}}{\vartheta n^2}\le 2\binom{n^2/2}{\vartheta n^2}\ <\ 2^{H(\vartheta)n^2+1} \tag{18}$$

bounds the number of ways of fixing the at most ϑn^2 horizontal edges (as $G_n \in \mathcal{P}_\vartheta(n,\mathcal{L})$). The explanation of the factor $n^{tn}\cdot\mathbb{N}_1\cdot\mathbb{N}_2$ is given below.

Each X_i can be chosen in at most $\sum_{r\le t}\binom{n}{r}\le t\binom{n}{t}\le n^t$ ways. So the system $\{X_i\}$ can be chosen in at most $n^{ts}\le n^{tn}$ ways.

Let $S := \bigcup_i X_i$. To count the graphs $G_n\in\mathcal{R}^\ell_{\mathbf{BAD}}(n,\mathcal{L})$, we fix an (optimal) partition $(U_1,\dots,U_p)$ in each such G_n and then the sets X_i described above. For each X_i put

$$\mathbb{E}_i := \bigcup_{i=1}^{s}\Big\{(x,u):\ x\in X_i,\ u\in U_j\Big\}\quad\text{and}\quad E_i := |\mathbb{E}_i|.$$

- $\mathbb{N}_1$ bounds the number of choices of the edges in $\mathbb{E} := \bigcup\mathbb{E}_i$.
- $\mathbb{N}_2$ bounds the number of choices for the edges in the remaining *vertical* pairs, i.e. between U_i and U_j for $i\ne j$. If $E := |\mathbb{E}| = \sum_i E_i$, then

$$\mathbb{N}_2\le 2^{e(T_{n,p})-E}. \tag{19}$$

The key step in our proof is our bound on $\mathbb{N}_1$. The crude bound would be $2^E = \prod 2^{E_i}$, but that is not sufficient for us. Therefore we shall sharpen this bound, checking, for each i, by how much we can decrease the bound 2^{E_i}.

Fixing $(U_1,\dots,U_p)$, the distinguished class U_j and the set-pairs (X_i,U_j), we count the number of ways the edges can be placed between X_i and U_j:

Assuming that the connection of X_i to U_j is random, the expected number of vertices $u\in U_j\cap\Gamma^*(X_i)$ (i.e. completely joined to X_i) is $|U_j|\cdot 2^{-|X_i|}$. However, as U_j is bad for X_i, the number of common neighbours is below half of the expected number, therefore the number of possibilities of these connections is at least $2^{-\beta_{|X_i|}\cdot|U_j|}$ times smaller, by Lemma 19. So, taking the total number (i.e. the product of the possibilities) we have an additional factor at most

$$\prod_{i\le s}(2^{-\beta_{|X_i|}\cdot|U_j|}) < 2^{-\sum_{i\le s}\beta_{|X_i|}|U_j|}\le 2^{-\beta_t s|U_j|}\le 2^{-\beta_t\ell n/(2pt)},$$

since, by (16), $s \geq \ell/t$ and as $G_n \in \mathcal{P}^{\vartheta}_{\mathbf{WP}}(n,\mathcal{L})$ we have $|U_j| \geq n/(2p)$. Hence, using $\ell := \lceil \gamma n \rceil$, we obtain

$$\mathbb{N}_1 \leq 2^{E-\beta_t \ell n/(2pt)} \leq 2^{E-\gamma\beta_t n^2/(2pt)}. \tag{20}$$

Combining inequalities (19), (20) and (12) with (17), we find that

$$\left|\mathcal{R}^{\ell}_{\mathbf{BAD}}(n,\mathcal{L})\right| \leq p^{n+1} n^{tn+1} \cdot 2^{H(\vartheta)n^2+1+e(T_{n,p})-E+E-\gamma\beta_t n^2/(2pt)} \leq 2^{e(T_{n,p})-\rho n^2},$$

if n is sufficiently large. ∎

3. Proof of Theorems 5 and 9

Proof of Theorem 5. By Lemmas 10, 11 and 12, almost all graphs from $\mathcal{P}(n,\mathcal{L})$ are in $\mathcal{P}^{\vartheta}_{\mathbf{WP}}(n,\mathcal{L}) \cap \mathcal{P}^{\delta}_{\mathbf{UNIF}}(n,\mathcal{L})$ (here we use that $|\mathcal{P}(n,\mathcal{L})| \geq 2^{e(T_{n,p})}$). By Theorem 9, almost all graphs from $\mathcal{P}(n,\mathcal{L})$ are in $\mathcal{P}^{h}_{\mathbf{GOOD}}(n,\mathcal{L})$, i.e. almost all of them are in $\mathcal{Q}^{h}_{\mathbf{GOOD}}(n,\mathcal{L})$. Now Lemma 16 implies Theorem 5. ∎

Proof of Theorem 9. The proof is based on a pseudo-symmetrization. Let

$$\mathcal{P}^{\ell}_{\mathbf{BAD}}(n,\mathcal{L}) := \mathcal{P}_{\vartheta}(n,\mathcal{L}) \cap \mathcal{P}^{\vartheta}_{\mathbf{WP}}(n,\mathcal{L}) \cap \mathcal{P}^{\delta}_{\mathbf{UNIF}}(n,\mathcal{L}) - \mathcal{P}^{h}_{\mathbf{GOOD}}(n,\mathcal{L}) - \mathcal{R}^{\ell}_{\mathbf{BAD}}(n,\mathcal{L}).$$

(Although we use $\ell = \lceil \gamma n \rceil$, we carry it in our notation.) We shall map each graph $G_n \in \mathcal{P}^{\ell}_{\mathbf{BAD}}(n,\mathcal{L})$ onto many $\mathcal{L}$-free graphs, changing at most γn^2 edges in G_n. The set of these graphs will be denoted by $\Phi(G_n)$. Roughly, the main idea is that we show that for most of the graphs H_n we have $|\Phi^{-1}(H_n)| = o(|\Phi(G_n)|)$. This will imply that $|\mathcal{P}^{\ell}_{\mathbf{BAD}}(n,\mathcal{L})| = o(|\mathcal{P}(n,\mathcal{L})|)$. We actually will show that $\mathcal{P}^{\ell}_{\mathbf{BAD}}(n,\mathcal{L})$ is an exponentially small part of $\mathcal{P}(n,\mathcal{L})$. We have to prepare the ground to carry out these ideas.

Let $G_n \in \mathcal{P}^{\ell}_{\mathbf{BAD}}(n,\mathcal{L})$. Since $V(G_n) = \{1,\dots,n\}$ is ordered, we may define $(U_1,\dots,U_p)$ as the "lexicographically first" optimal partition of G_n. (Of course, we do not care about the "lexicographical order": we just wish to fix one optimal partition.) As in Definition 8, let W denote the set of vertices of G_n of horizontal degree at least εn in $(U_1,\dots,U_p)$. Let $\{X_1,\dots,X_s\}$ be a **maximal** system of pairwise disjoint BAD ($\leq t$)-sets toward U_1. (Again, the first one in some well-defined ordering.) We define the mapping $\Phi : \mathcal{P}^{\ell}_{\mathbf{BAD}}(n,\mathcal{L}) \mapsto 2^{[\mathcal{P}(n,\mathcal{L})]}$ as follows.

For $G_n \in \mathcal{P}^{\ell}_{\mathbf{BAD}}(n,\mathcal{L})$, let $\Phi(G_n)$ be the family of graphs obtained by joining $X = X(G_n) := X_1 \cup \ldots \cup X_s$ to the vertices of $V(G_n) - U_1 - X - W$ in any way. More precisely, let $\Phi(G_n)$ denote the set of graphs obtained as follows:

First we remove all edges between X and $V(G_n)$. Then "put" the elements of X into U_1: join the vertices of X to the vertices of $V(G_n)-U_1-W-X$ arbitrarily.

To make our argument more transparent, we define ℓ directed graphs $\overrightarrow{\mathcal{D}}_i$ on the vertex set $\mathcal{P}(n,\mathcal{L})$: in the i^{th} graph $\overrightarrow{\mathcal{D}}_i$, there is an edge from G_n to H_n, if $G_n \in \mathcal{P}^{\ell}_{\mathbf{BAD}}(n,\mathcal{L})$, $H_n \in \Phi(G_n)$, and $|X(G_n)| = i$. Then our aim is to show that in each $\overrightarrow{\mathcal{D}}_i$ the outdegrees are large and the indegrees are small.

Perhaps the most important property of this map is that $\Phi(G_n) \subset \mathcal{P}(n,\mathcal{L})$: the graphs in $\Phi(G_n)$ are $\mathcal{L}$-free. Note that this is the part of the proof where we could not avoid using that $\mathcal{L}$ is finite.

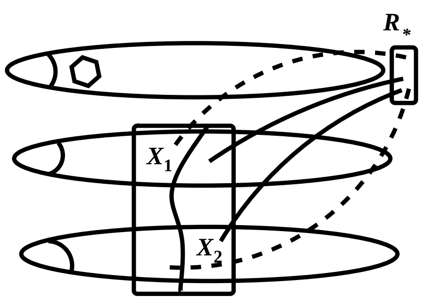
To show that any $H_n \in \Phi(G_n)$ is $\mathcal{L}$-free, observe that if we obtained some $L \in \mathcal{L}$ during our "symmetrization", i.e. if $L \subset H_n$, then the original G_n also contained an $L' \simeq L$. Indeed, $V(L)$ in H_n can be partitioned into four parts:

(i) $R_* = V(L) \cap X \neq \emptyset$,

(ii) $C_* = V(L) \cap U_1 - X$: the remaining part of L in U_1,

(iii) $W_* = V(L) \cap W - U_1 - X$,

(iv) $L_* = V(L) - U_1 - W - X$.



Observe that L_* was a GOOD ($\leq t$)-tuple in G_n, otherwise $\{X_1,\dots,X_s\}$ was not maximal. Hence $|\Gamma^*(L_*) \cap U_1 - X - W| > |V(L)|$. Therefore we can fix a set $Y \subset \Gamma^*(L_*) \cap U_1 - X - W$ with $|Y| = |R_*|$. In H_n there is no edge between X and W, and between X and U_1. So in G_n the graph spanned by $C_* \cup W_* \cup L_* \cup Y$ contains an L. This contradiction shows that $L \not\subseteq H_n$.

The next step is to give a lower bound on the outdegrees in $\overrightarrow{\mathcal{D}}_i$, i.e. to estimate $|\Phi(G_n)|$, given that $|X(G_n)| = i$. Creating the graphs in $\Phi(G_n)$, for any pair (a,b) with $a \in X$ and $b \in V(G_n) - U_1 - X - W$, we may include or exclude (a,b) as an edge. Hence, using that $|X| \leq \gamma n \leq \ell$ and that by $G_n \in \mathcal{P}^{\vartheta}_{\mathbf{WP}}(n,\mathcal{L})$ the classes are not big, i.e. $|U_1| \leq (1/p + \sqrt{\vartheta}\log(1/\vartheta))n$, we have

$$\begin{aligned} |\Phi(G_n)| &= 2^{|X|\cdot(n-|U_1|-|X|-|W|)} \geq 2^{in(1-1/p-\sqrt{\vartheta}\log(1/\vartheta)-i/n-o(1))} \\ &\geq 2^{in(1-1/p-\gamma-\sqrt{\vartheta}\log(1/\vartheta)-o(1))}. \end{aligned} \tag{21}$$

Our final aim is to bound the indegrees in $\overrightarrow{\mathcal{D}}_i$. In order to do this, first we bound the number of optimal partitions of graphs in $\mathcal{P}(n,\mathcal{L})$. Note that during the operation Φ the number of horizontal edges in the optimal partitions does not increase, hence $G_n \in \mathcal{P}_{\vartheta}(n,\mathcal{L})$ implies $\Phi(G_n) \subset \mathcal{P}_{\vartheta}(n,\mathcal{L})$.

Lemma 22. *Given a graph $H_n \in \mathcal{P}_\vartheta(n, \mathcal{L})$, the number of optimal partitions of the graphs in $\Phi^{-1}(H_n)$ is at most $2^{2pH(\delta p+2\gamma p)n}$, where the same partition obtained from different graphs are counted only once.*

Proof. Let $H_n \in \mathcal{P}_\vartheta(n, \mathcal{L})$ and $G_1, G_2 \in \Phi^{-1}(H_n)$, where $G_1 = G_2$ is allowed. Recall that the domain of Φ was a subset of $\mathcal{P}^\delta_{\mathbf{UNIF}}(n, \mathcal{L}) \cap \mathcal{P}^\vartheta_{\mathbf{WP}}(n, \mathcal{L})$. Let $(U_1, \dots, U_p)$ be an optimal partition of G_1 and $(V_1, \dots, V_p)$ of G_2. For $j = 1, 2$, let X_j be the i-set of the vertices of G_j incident with the edges that were changed by Φ to obtain H_n. Note that as the optimal partitions of G_i are δ–lower regular and balanced, for every a there is at most one b such that $|V_a \cap U_b| > \delta n + 2i$ (for $1 \le a, b \le p$). Otherwise, if say $|V_a \cap U_{b_1}|, |V_a \cap U_{b_2}| > \delta n + 2i$, then $e_{G_2}(V_a \cap U_{b_1}, V_a \cap U_{b_2}) \ge \delta^2 n^2 \ge 4\vartheta n^2$, contradicting $G_2 \in \mathcal{P}_\vartheta(n, \mathcal{L})$.

This implies the existence of a labelling of the classes such that for every a, $1 \le a \le p$ we have $|V_a - U_a| \le (p-1)(\delta n + 2i)$.

As for a given optimal partition $(U_1, \dots, U_p)$ and $D_a := V_a - U_a$, the partition $(V_1, \dots, V_p)$ is determined, the number of optimal partitions $(V_1, \dots, V_p)$ is bounded by the number of ways the difference sets D_a can be chosen:

$$\left(\sum_{j=0}^{(p-1)(\delta n+2i)} \binom{n}{j} \right)^p < \binom{n}{\delta pn + 2ip}^p < 2^{2pH(\delta p+2p\gamma)n}.$$

∎

Our next step to bound the indegrees in $\overrightarrow{\mathcal{D}}_i$ is to give an upper bound on the number of ways of choosing the BAD ($\le t$)-tuples. The number of ways of choosing the index j in U_j which is the distinguished class is bounded by p. Then we can fix s, the number of sets in $\{X_1, \dots, X_s\}$ in less than n ways. The number of ways to choose $\{X_1, \dots, X_s\}$ is less than n^{ts}. Then for each $x \in \bigcup X_\ell$ we may choose the class U_m containing x in p ways: altogether in p^i ways. By Lemma 22, the number of ways to fix an optimal partition of $G_n \in \Phi^{-1}(H_n)$ is at most $2^{2pH(p\delta+2p\gamma)n}$.

Fixing an optimal partition of G_n and the sets $X_1, \dots, X_s$, we know all edges of G_n, except the ones adjacent to X. Note that by the definition of a BAD-tuple, each $x \in X$ had horizontal degree at most εn. Thus the number of ways of adding the horizontal edges with at least one end point in X is at most

$$\left(\sum_{j=0}^{\varepsilon n} \binom{n}{j} \right)^i \le 2^i \cdot \binom{n}{\varepsilon n}^i \le 2^{i+H(\varepsilon)in}.$$

We shall use that, as $G_n \in \mathcal{P}^{\vartheta}_{\mathbf{WP}}(n, \mathcal{L})$, we have

$$\frac{n}{p} - \sqrt{\vartheta}\log(1/\vartheta)n \le u_{min} := \min_{1\le j\le p}\{|U_j|\} \le u_{max} := \max_{1\le j\le p}\{|U_j|\} \le \frac{n}{p} + \sqrt{\vartheta}\log(1/\vartheta)n.$$

For a vertex $x \in X \cap U_j$ the number of possibilities of having the edge set in G_n between x and $V(G_n) - U_1 - U_j$ is at most $2^{n-2u_{min}}$. So the total number of ways of joining the elements of X to the rest of the graph excluding to its own class and U_1 is at most

$$2^{|X|(n-2u_{min})}.$$

For any $j \le s$ as $|\Gamma^*(X_j) \cap U_1|$ is smaller than half of its expected value in a random graph, by Chernoff's inequality (Lemma 19), the number of ways having the edges between X_j and U_1 is at most

$$2^{u_{max}(|X_j|-\beta_t)}.$$

We have to consider this for each X_j. Note that $\lceil i/t \rceil \le s \le i$. Putting these together, we have the following upper bound on the maximum indegree in $\overrightarrow{\mathcal{D}}_i$:

$$\begin{aligned} 2^{2pH(p\delta+2\gamma p)n} \quad &\cdot \quad p \cdot n \cdot n^{ts} \cdot p^i \cdot 2^{i+H(\varepsilon)in} \cdot 2^{|X|(n-2u_{min})} \cdot \Pi_{j=1}^{s} 2^{u_{max}(|X_j|-\beta_t)} \\ &\le \quad 2^{in[2pH(p\delta+2p\gamma)+o(1)+H(\varepsilon)+1-2/p+2\sqrt{\vartheta}\log(1/\vartheta)+1/p+\sqrt{\vartheta}\log(1/\vartheta)-s\beta_t/(ip)]} \\ &\le \quad 2^{in[1-1/p+o(1)+2pH(p(\delta+2\gamma))+H(\varepsilon)+3\sqrt{\vartheta}\log(1/\vartheta)-\beta_t/(pt)]}. \end{aligned}$$

With this bound our proof is essentially complete. Recall that the outdegree was bounded from below by

$$2^{in(1-1/p-\gamma-\sqrt{\vartheta}\log(1/\vartheta)-o(1))}. \tag{22}$$

Comparing the upper bound on the indegree and (22), the outdegree estimate in $\overrightarrow{\mathcal{D}}_i$, and using (13), we see that the ratio of them is at least $2^{\beta_t n/(2pt)}$, i.e. the number of BAD graphs with $|X| = i$ is at most $|\mathcal{P}_\vartheta(n, \mathcal{L})| \cdot 2^{-\beta_t n/(2pt)}$. Since $i \le n$, the number of BAD graphs is at most, $|\mathcal{P}_\vartheta(n, \mathcal{L})| \cdot 2^{-\beta_t n/(3pt)}$, say. Considering only the "good graphs" we neglected fewer than $4 \cdot 2^{e(T_{n,p})-\rho n^2}$ (other) graphs. This completes the proof. ∎

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