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The Speed of Hereditary Properties of Graphs

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Given a property $\mathscr{P}$ of graphs, write $\mathscr{P}^n$ for the set of graphs with vertex set $[n]$ having property $\mathscr{P}$. The growth or speed of a property $\mathscr{P}$ can be discussed in terms of the values of $|\mathscr{P}^n|$. For properties with $|\mathscr{P}^n| < n^n$ hereditary properties are surprisingly well determined by their speeds. Sharpening results of E. R. Scheinerman and J. Zito (1994, *J. Combin. Theory Ser. B* **61**, 16–39), we prove numerous results about the possible functions $|\mathscr{P}^n|$ and describe in detail the properties exhibiting each type of growth. We also list minimal properties exhibiting each type of growth.

1. INTRODUCTION

A *graph property* is an infinite class of graphs closed under isomorphism. The property consisting of all finite graphs is said to be *trivial*; we shall always assume that our property is non-trivial. A property is *monotone* if it is closed under taking *subgraphs*, and it is *hereditary* if it is closed under taking *induced subgraphs*. Most of the "interesting" graph properties are hereditary. Thus being acyclic or planar are monotone properties (and so hereditary as well), while being perfect is a hereditary non-monotone property

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of graphs. In fact, rather trivially, every hereditary property can be defined in terms of *forbidden induced subgraphs*: given a (possibly infinite) family $\mathcal{F}$ of finite graphs, write Her($\mathcal{F}$) for the property of containing no member of $\mathcal{F}$ as an induced subgraph. Then Her($\mathcal{F}$) is a hereditary property, and every hereditary property is of this form for some family $\mathcal{F}$.

Given a property $\mathcal{P}$, write $\mathcal{P}^n$ for the set of graphs in $\mathcal{P}$ with vertex set $[n]=\{1, ..., n\}$. The sequence $(|\mathcal{P}^n|)_{n=1}^{\infty}$ is perhaps the most natural measure of the "size" of a property $\mathcal{P}$. We call the function $|\mathcal{P}^n|$ the *speed* of the property $\mathcal{P}$. An important but trivial fact to note is that if $G \in \mathcal{P}^n$, all graphs isomorphic to G are in $\mathcal{P}^n$, so $|\mathcal{P}^n|$ is at least as large as the number of non-isomorphic labelings of G.

Let us introduce some notation and terminology. If H is an induced subgraph of G, we shall write $H \leqslant G$. If $V_1, V_2 \subseteq V(G)$ are disjoint, then $G[V_1]$ will denote the induced subgraph of G on V_1 and $G[V_1, V_2]$ will denote the subgraph of G that contains exactly those edges which cross from V_1 to V_2. We take $G[V_i, V_i]$ to be the same as $G[V_i]$. We shall suppress G when it is clear from the context. In particular, we will write $E(V_i)$ for $E(G[V_i])$ and $E(V_i, V_j)$ for $E(G[V_i, V_j])$. Any graph or subgraph with no edges is called an *empty graph*, while any graph or subgraph with every possible edge is a *complete graph* or *clique*.

Needless to say, for a general property $\mathcal{P}$, nothing of interest can be said about the sequence $(|\mathcal{P}^n|)$, but Scheinerman and Zito [13] proved in 1994 that for a hereditary property $\mathcal{P}$ the sequence $(|\mathcal{P}^n|)$ is severely constrained. To be more precise, the following result is proved in [13].

THEOREM 1. *Let $\mathcal{P}$ be a hereditary property of graphs. Then one of the following holds.*

(i) *For all n sufficiently large $|\mathcal{P}^n|$ is identically zero, one or two.*
(ii) $|\mathcal{P}^n| = \Theta(1)\, n^k$ *for some positive integer k.*
(iii) *For some positive c_1, c_2, $c_1^n < |\mathcal{P}^n| < c_2^n$.*
(iv) *For some $c > 0$, $n^{cn} \leqslant |\mathcal{P}^n|$.*

Thus, putting it somewhat vaguely, the growth of $|\mathcal{P}^n|$ can be constant, polynomial, exponential, factorial or superfactorial.

Throughout this paper, we will abuse these terms slightly, describing the dominant factor of growth rather than the whole function. A *constant* function is one which, for suffincently large n, is a constant. A *polynomial* function is one which., for sufficiently large n, is a polynomial. Our notation for both polynomial and constant speeds is standard, but the following is not. An *exponential* function is one which for sufficiently large n, acts like the sum of exponential terms with polynomial coefficients. We similarly abuse the asymptotic notation Ω, O, and Θ. For example, for $k > 1$, the notation $\Omega(k^n)$ has its usual meaning ($f(n)=\Omega(k^n)$ means there is a $c>0$ such that $f(n)>ck^n$ for all n), but we will write $f(n)=O(k^n)$ if the fastest growing term in the expansion of $f(n)$ has order at most $n^t k^n$ for some t. We define $\Theta(k^n)$ similarly. A *factorial* function is one which is at least n^{cn} for some c.

Our aim in this paper is to show that there is a much finer hierarchy of speeds than that described in Theorem 1. In fact, we show that, for properties in which $|\mathscr{P}^n|<n^{(1+o(1))n}$ for all sufficiently large values of n, we can almost precisely determine the functions allowed to appear. In addition, this hierarchy is divided into discrete levels. Even more surprising, we are able describe the types of properties which occur at each level. Finally, we consider the question of which properties "force" a certain speed of growth for $|\mathscr{P}^n|$, to which end we define and characterize "minimal" properties.

We call a hereditary property *minimal* if it causes $\mathscr{P}$ to have a certain speed in the following sense. A hereditary property $\mathscr{P}$ is called a *minimal hereditary property* if, for every hereditary property $\mathscr{Q}$ it properly contains, the speed of $\mathscr{Q}$ is of lower order than the speed of $\mathscr{P}$. For example, if $|\mathscr{P}^n|=\Theta(5^n)$ for sufficiently large n, then $\mathscr{P}$ is minimal if all proper subproperties have speeds at most $O(4^n)$. We shall be more explicit when we describe each order of speed.

All of the results in this paper can be thought of as applying to equivalence classes of hereditary properties, rather than hereditary properties themselves, as the results always have the condition "for sufficiently large n." That is, two properties are considered equivalent if they differ on only a finite number of graphs, and hence agree on all graphs with at least n vertices for large n.

Scheinerman and Zito [13] have proved the best possible result when $|\mathscr{P}^n|$ is bounded. In this case, for n sufficiently large, $\mathscr{P}^n=\varnothing$ or $\{K_n\}$ or $\{\bar{K}_n\}$, or $\{K_n, \bar{K}_n\}$. So for large enough values of n, $|\mathscr{P}^n|\in\{0, 1, 2\}$ and the minimal properties for bounded growth are exactly $\{\varnothing\}$, $\{K_n\}$, and $\{\bar{K}_n\}$. Note that $\{K_n, \bar{K}_n\}$ is a property with bounded growth which is not minimal.

The bounded case looks quite clear, but it might seem that for faster speeds the result would be elusive. In fact, for all but the penultimate range of speeds (properties $\mathscr{P}$ satisfying $n^n \leqslant |\mathscr{P}^n| \leqslant 2^{o(n^2)}$) we are able to make similar characterizations. In this penultimate range, such a result seems impossible, particularly in view of some surprising results which answer some open questions (and contradict the conjectured answers) from [13]. This range and these questions are explored in [2].

We remark that at the other end of the speed spectrum, the fastest properties, those with $|\mathscr{P}^n|\geqslant 2^{\varepsilon n^2}$ for some $\varepsilon>0$, have been investigated in great detail. In fact, the study of hereditary properties really started with these rich properties (see Erdős, Frankel, and Rödl [8], Prömel and Steger

[9–11], Alekseev [1], Bollobás and Thomason [3]). In particular, in this range we once again can describe minimal properties.

In the paper, $\mathcal{P}$ denotes a general hereditary property, and, unless explicitly stated, any property mentioned is hereditary.

2. CANONICAL PROPERTIES

Let us begin by describing some general categories of hereditary properties and calculating their speeds.

Let G be an infinite graph. Then $\mathcal{P}(G) = \{H : H \leqslant G\}$. Clearly this is a hereditary property. As an example, if G is the infinite complete graph K_∞, then $\mathcal{P}(K_\infty) = \{K_n : n \in \mathbb{N}\}$ and $|\mathcal{P}^n(K_\infty)| = 1$ for each n. If G is the infinite star, then $\mathcal{P}(K_{1,\infty}) = \{H : H$ is a star or an empty graph$\}$ and $|\mathcal{P}^n(K_{1,\infty})| = n+1$ for each $n \geqslant 2$. If G consists of two infinite cliques, then $\mathcal{P}(G) = \{H : H$ is the union of at most two disjoint cliques$\}$ and $|\mathcal{P}^n(G)| = 2^{n-1}$ for each n.

Another class of properties, which we call *canonical properties*, is formed as follows. Let H be a labeled simple graph with loops allowed; in particular, $V(H) = \{1, 2, ..., k\}$. Let b be a function from $V(H)$ to $\mathbb{N} \cup \{\infty\}$ and c be a function from $\binom{V(H)}{2}$, the 2-element subsets of $V(H)$, to $\mathbb{N} \cup \infty$. Then the property $\mathcal{P}(H, b, c)$ contains all graphs G such that $V(G)$ can be partitioned into $\{V_1, V_2, ..., V_k\}$ (empty classes are allowed) and the following conditions hold:

(1) $|V_i| \leqslant b(i)$,

(2) $G[V_i]$ is complete if $ii \in E(H)$, otherwise $G[V_i]$ is an empty graph,

(3) if $ij \notin E(H)$, then the order of a component of $G[V_i, V_j]$ is at most $c(\{i, j\})$. If $ij \in E(H)$, the order of a component of the bipartite complement of $G[V_i, V_j]$ is at most $c(\{i, j\})$.

We shall say that the graph H is a *template* for the relationships of the classes of G. For example, if $b \equiv \infty$ and $c \equiv \infty$, then $\mathcal{P}(K_2, \infty, \infty)$ is the collection of bipartite graphs (since we have two independent sets of unbounded order and any combination of edges in between). For an example of what can happen when c changes, $\mathcal{P}(H, 1, \infty)$ consists of all graphs on at most $|V(H)|$ vertices (even replacing ∞ with 2 produces the same result), while $\mathcal{P}(H, 1, 1)$ is the hereditary property induced by H as a simple graph. If c is 2 on the edges of H and 1 on its nonedges, $\mathcal{P}(H, 1, c)$ consists of H and all of its simple (not only induced) subgraphs.

Therefore we have a great deal of flexibility in the types of properties we can construct. However, it is easy to see that $\mathcal{P}(H, b, c)$ is hereditary, since when G satisfies conditions (1)–(3) so does any induced subgraph of G.

As another example, the property $\mathscr{P}(K_{1,\infty})$ described above is also $\mathscr{P}(H, b, 1)$, where H is the graph consisting of an edge and two loops, and b is 1 at one vertex and ∞ at the other.

There are many advantages to properties that can be described in this way. For example, the structure of H and the values of b and c give us bounds on the speed of the property $\mathscr{P}(H, b, c)$, as we show in the following lemma.

LEMMA 2. *Let H be a graph on $k+1$ vertices and $x \in V(H)$. If $c \equiv 1$, $b(v) = 1$ for all $v \in V(H) - \{x\}$, $b(x) = \infty$, then $|\mathscr{P}^n(H, b, c)| = O(n^k)$.*

Proof. Let $G \in \mathscr{P}^n(H, b, c)$. Then G is completely defined by the labels that appear on vertices other than $v = b^{-1}(\infty)$. As there are at most $\binom{n}{k} k!$ ways to label those vertices, the result follows. Note that this number depends only on the structure of H. ∎

Although the canonical properties are easy to deal with, it may not be completely clear why these properties should be important. However, we shall show that the minimal (as described in Section 1) properties $\mathscr{P}$ of growth $|\mathscr{P}^n| < n^n$ can be expressed as canonical properties with appropriate restrictions on H, b, and c. We shall define another useful, but complex, category of hereditary properties in Section 5.

3. POLYNOMIAL GROWTH

We saw in the introduction that $|\mathscr{P}^n|$ is bounded in exactly four cases, and in each of these cases $|\mathscr{P}^n| \leqslant 2$. We shall show that in the next level of growth, the speed follows a polynomial.

If $|\mathscr{P}^n| \geqslant 3$, then there is a $G \in \mathscr{P}^n$ that is neither empty nor complete. Hence it contains a vertex $x \in V(G)$ with $d(x)$, $\bar{d}(x) \geqslant 1$. If x has label 1 then there are $\binom{n-1}{d(x)} \geqslant n-1$ ways to choose its neighborhood. Also, since $n > 2$, there is some vertex $y \in V(G)$ outside of the smaller of $\Gamma(x)$, $\bar{\Gamma}(x)$. That is, since $\mathscr{P}$ is hereditary, there is a graph $G' = G - y$ in $\mathscr{P}^{n-1}$ where $d_{G'}(x)$, $\bar{d}_{G'}(x) \geqslant 1$. Hence $|\mathscr{P}^{n-1}| \geqslant n-2$ and in fact, $|\mathscr{P}^{n'}| \geqslant n'-1$ for each $n' < n$. This proves the following two lemmas from [13].

LEMMA 3. *If $|\mathscr{P}^n| \geqslant 3$ then $|\mathscr{P}^n| \geqslant n-1$.*

LEMMA 4. *If for all N there is an $n > N$ such that $|\mathscr{P}^n| \geqslant 3$, then $|\mathscr{P}^n|$ has at least polynomial order growth.*

Based on the preceding and the fact that the case $|\mathscr{P}^n| \leqslant 2$ was settled in the introduction, we henceforth assume that the properties under consideration $|\mathscr{P}^n| \geqslant 3$ for all n. Thus they all have $|\mathscr{P}^n| = \Omega(n^j)$ for some j.

In this section we further assume that we have a hereditary property $\mathscr{P}$ such that $|\mathscr{P}^n| = O(n^k)$ for some positive integer k. Let us introduce some notation and definitions. Given a vertex x in a graph G, let $d(x) = d_G(x)$ be the degree of x and let $\bar{d}(x) = d_{\bar{G}}(x)$ be the codegree of x. Trivially $d(x) + \bar{d}(x) = |V(G)| - 1$. For $x \in V(G)$, $\Gamma(x)$ denotes the neighborhood of x in G. For $x, y \in V(G)$ we write $x \sim y$ to express the fact that $\Gamma(x) - \{y\} = \Gamma(y) - \{x\}$. It is easy to see that this relation is an equivalence relation. We call the equivalence classes of $\sim$ *homogeneous* sets. If $x \sim y$ in G we say that x is *G-equivalent* to y. Note that if x is G-equivalent to y, then they are in the same orbit of the automorphism group of G. It is easily checked that each homogeneous set is either independent or spans a complete subgraph.

Suppose $V(G) = A \cup B$ such that B is a homogeneous set. That means for each $x \in A$, x is adjacent to all or none of the vertices of B. If A has minimum order under that condition, then we call A a *head* and B a *body* of the graph G. A graph G can have an empty head, in which case G is either a complete graph or an independent set. In certain situations, the head is unique:

LEMMA 5. *If a graph G has a head with less than $V(G)/2$ vertices then G has a unique head.*

Proof. Suppose H_1, H_2 are two heads of G. Then, since $|H_i| < |V(G)|/2$ for $i = 1, 2$, we have $(V(G) - H_1) \cap (V(G) - H_2) = V(G) - (H_1 \cup H_2) \neq \emptyset$. Let $B = (V(G) - H_1) \cup (V(G) - H_2) = V(G) - (H_1 \cap H_2)$. By the definition of the head, for all $x_1 \in V(G) - H_1$, $x_2 \in V(G) - H_2$, and $x \in V(G) - (H_1 \cup H_2)$ we have $x_1 \sim x \sim x_2$. By the transitive property of $\sim$, $x_1 \sim x_2$, so B is a homogeneous set. Hence B could be a body of G and $H_1 \cap H_2$ would be a head of G, contradicting the minimality of H_1. (If $H_1 \cap H_2$ is empty, then G is a homogeneous set.) ∎

To get to our main result, we shall need a few technical lemmas and definitions. The assertion below constrains degrees in G.

LEMMA 6. *If $|\mathscr{P}^n|$ has $O(n^k)$ order growth, then there is an integer N such that, for $n > N$, if $G \in \mathscr{P}^n$ then each vertex of G has degree or codegree at most k.*

Proof. Let N' and c be constants such that $|\mathscr{P}^n| < cn^k$ for all $n > N'$. Suppose there is $G \in \mathscr{P}^n$ such that G contains a vertex x with $d(x), \bar{d}(x) > k$. Then we can label x as 1, and there are $\binom{n-1}{d(x)}$ ways to choose its neighborhood. So there are at least $\binom{n-1}{d(x)} \geqslant \binom{n-1}{k+1} > cn^k$ ways to label G, the latter inequality holding for sufficiently large n, say $n > N''$. Let $N = \max\{N', N''\}$. Then if $n > N$ and $G \in \mathscr{P}^n$, then for all $x \in V(G)$ we have that $d(x)$ or $\bar{d}(x)$ is at most k. ∎

The following lemma will be important throughout this paper since it will allow us to disregard some of the vertices of the graph.

LEMMA 7. *Given a graph G, either G or $\bar{G}$ has no more than $2k$ vertices of degree at most k.*

Proof. Suppose that the graph G is a counterexample. Clearly then $|V(G)| \geqslant 4k+2$. Let $A \subset \{x \in V(G) : d(x) \leqslant k\}$ and $B \subset \{x \in V(G) : \bar{d}(x) \leqslant k\}$ such that $|A| = |B| = 2k+1$.

By the definition of A, between A and B there are at most $|A|\, k$ edges. By the definition of B, between A and B there are at most $|B|\, k$ non-edges. Between A and B there are $|A| \cdot |B|$ places for edges, so $|A| \cdot |B| \leqslant (|A| + |B|) \cdot k$ and $2k+1 = |A| \leqslant 2k$. ∎

We shall use the preceding two lemmas to describe all $G \in \mathscr{P}^n$ for n sufficiently large ($n > N$). Without loss of generality, we can suppose G contains at most $2k$ vertices with codegree at most k. Let T denote that vertex set. Let G^* denote the graph that we get from G by deletion of T. By Lemma 6 we could choose N sufficiently large that each vertex in G^* has degree at most k. The idea is that we have not lost too much information by discarding T, as we show later.

Our next two lemmas constrain the edges of G^*. Let $M_t = tK_2$, the graph consisting of t disjoint edges.

LEMMA 8. *If G^* has at least $f(k, t) = (2(k-1)^2 + 1)(t-1) + 1$ edges, then it contains an induced M_t.*

Proof. Suppose G^* contains more than $f(k, t)$ edges. Pick an arbitrary edge $(u_1, v_1) \in E(G^*)$. Then choose another edge $(u_2, v_2) \in E(G^*)$ such that $u_2, v_2 \notin \Gamma(u_1) \cup \Gamma(v_1)$. Continue choosing edges; that is, in the i-th step choose a $(u_i, v_i) \in E(G^*)$, such that $u_i, v_i \notin \Gamma(u_1) \cup \Gamma(v_1) \cup \cdots \cup \Gamma(u_{i-1}) \cup \Gamma(v_{i-1})$.

Because for each vertex x we have $|\Gamma(x)| \leqslant k$, we remove at each step at most $|\Gamma(u_j) \cup \Gamma(v_j)| \cdot (k-1) + 1 \leqslant (2k-2)(k-1) + 1$ edges from the set of edges we can choose, and after $t-1$ steps there is still at least one edge left. So we have independent edges $(u_1, v_1) \cdots (u_t, v_t)$ which span an M_t. ∎

LEMMA 9. *There exists an m, independent from n and G, such that the number of edges in G^* is at most m.*

Proof. Assume that there is no such m. There is an N and a constant c such that for each $n > N$, we have $|\mathscr{P}^n| < cn^k < (n/2)!$. Since the number of edges in G^* is unbounded, there is a $G \in \mathscr{P}$ such that G^* has at least $f(k, N)$ edges. By the previous lemma G contains an induced M_N. Since $\mathscr{P}$

is hereditary, this gives $M_N \in \mathscr{P}^{2N}$, but there are $(2N)!/N!2^{-N} > N!$ ways to label a graph M_N, so $|\mathscr{P}^{2N}| > c(2N)^k$. Hence $m = f(k, N)$ will work. ∎

We are now ready to prove the following theorem.

THEOREM 10. *If $|\mathscr{P}^n|$ has $O(n^k)$ order growth, then there is an integer N such that, if $n > N$ and $G \in \mathscr{P}^n$, then G has a unique head A. Furthermore, $|V(A)| \leqslant k$.*

Proof. Let $G \in \mathscr{P}^n$ and define G^* and T as before. We complete our proof by partitioning the vertices of G into two sets.

Let $A = \{x \in V(G) : x \in T,\ x$ is not an isolated vertex in G^*, or $T - \Gamma(x) \neq \varnothing\}$. Let $B = V(G) - A$. Then B is a homogeneous set, because it is an independent set and each $x \in B$ is adjacent to every element of T and to no element of $G - T$. We can give an upper bound on the order of A which does not depend on n; $|A| \leqslant 2k + 2m + 2k^2$, where m is from Lemma 9. We can do much better, however. It is easy to check that B is a maximal homogeneous set, so if n is big enough, then, by Lemma 5, A is the unique head of G. Hence $|A| \leqslant k$ since there are at least $\binom{n}{|A|} = O(n^{|A|})$ ways to label G. If we initially assumed that G contains at most $2k$ vertices with degree (rather then codegree) k, then B would be a complete subgraph of G, but the rest of the argument remains the same. ∎

In order to settle the case $|\mathscr{P}^n| = O(n^k)$, all that remains is to prove that $|\mathscr{P}^n|$ is a polynomial in n (provided n is large enough), and discover which polynomials of degree k are allowed. We shall need the following lemma to establish a lower bound on the polynomials.

LEMMA 11. *If $|\mathscr{P}^n| = \Omega(n^k)$, then for each sufficiently large n, there is $G \in \mathscr{P}^n$ with a head of order k.*

Proof. If for every n there is an $N > n$ such that $\mathscr{P}^N$ has a graph with a head of order k, then since $\mathscr{P}$ is hereditary $\mathscr{P}^n$ contains such a graph as well. So assume not. Then for n large enough each $G \in |\mathscr{P}^n|$ has head of size at most $k - 1$.

To maximize $|\mathscr{P}^n|$, we allow every graph with a head of order at most $k - 1$. There are at most $\binom{n}{k-1} 2^{\binom{k-1}{2}}$ different graphs for a head of order $k - 1$, two choices for the homogeneous set (independent or clique), and each vertex of the head is either adjacent or not adjacent to every vertex of the homogeneous set, contributing a factor of 2^{k-1}. This last factor, together with allowing any graph in the head, accounts for heads of smaller order.

So $|\mathscr{P}^n| \leqslant \sum_{i=0}^{k-1} \binom{n}{i} 2^{i+1} 2^{\binom{i}{2}} = O(n^{k-1})$. ∎

From Lemma 11 we see that if $|\mathscr{P}^n|$ has a smaller speed than $\Omega(n^k)$, then it has speed $O(n^{k-1})$. Further, if $\limsup_n (|\mathscr{P}^n|/n^k) \geqslant 0$ then the speed of

$|\mathscr{P}^n|$ is at least $\Omega(n^k)$. Hence, the leading terms of the functions have integer powers.

We have seen in Theorem 10 that if $\mathscr{P}$ has polynomial order growth, then every $G \in \mathscr{P}$ can be split into a large homogeneous set B and a small remainder set A. Further, if $|\mathscr{P}^n| = \Theta(n^k)$, then there must be a graph in $\mathscr{P}$ with $|A| = k$. Since a homogeneous set is either a clique or an independent set, our main freedom in constructing $\mathscr{P}$ lies in the composition of A, and in partitioning $V(A)$ based on whether a vertex is adjacent to B. The growth of $\mathscr{P}$ will clearly be the slowest if we restrict A as much as possible; in particular by requiring that all vertices of A are G-equivalent. Conversely, $\mathscr{P}$ will grow the fastest if there is no restriction on the composition of A. This is the main idea of the following two results.

THEOREM 12. *Let* $\mathscr{L}_1 = \{G : G$ *is a star or an empty graph*$\}$. *Let* $\mathscr{U}_1 = \{G : G \in \mathscr{L}_1$ *or* $\bar{G} \in \mathscr{L}_1\}$. *If* $|\mathscr{P}^n| = \Theta(n)$, *then, for* n *sufficiently large,*

$$n + 1 = |\mathscr{L}_1^n| \leqslant |\mathscr{P}^n| \leqslant |\mathscr{U}_1^n| = 2n + 2.$$

In fact in this range, for sufficiently large n, $|\mathscr{P}^n| = p_{\mathscr{P}}(n)$, *where* $p_{\mathscr{P}}(n)$ *is either* $n+1$, $n+2$, *or* $2n+2$.

Proof. If $|\mathscr{P}^n| = \Theta(n)$ then, by Theorem 10, there is an N such that for all $n > N$ every graph in $\mathscr{P}^n$ consists of a homogeneous set and at most one other vertex. Note that $N \geqslant 3$. Further, for $n > N$ there is a graph $G \in \mathscr{P}^n$ which has a vertex outside of its homogeneous set. Without loss of generality, assume that for all sufficiently large n there is a graph in $\mathscr{P}^n$ with a vertex outside of its homogeneous set and whose homogeneous set has no edges. This implies that the vertex outside of the homogeneous set is connected to the homogeneous set, and thus since $\mathscr{P}$ is hereditary there is a $G \in \mathscr{P}$ such that G is a star. G can be labeled in n ways. Since $\mathscr{P}^n$ must also contain the empty graph, $|\mathscr{P}^n| \geqslant n + 1$. But the graphs just described constitute $\mathscr{L}_1$, and $\mathscr{L}_1$ is hereditary, so this is the smallest possible property. On the other hand, for n sufficiently large every graph in $\mathscr{P}^n$ has at most one vertex in its head, so the only other possibility for the structure of a given $G \in \mathscr{P}^n$ is for G to have a complete graph induced by the homogeneous set and a vertex independent of the homogeneous set. But then $\bar{G} \in \mathscr{L}_1$. Each of these graphs has n labellings, and the fact that $\mathscr{P}$ is hereditary guarantees $K_n \in \mathscr{P}^n$ as well, so we get $\mathscr{U}$ with speed $2n + 2$.

The argument above implies that the only other properties in this range is $\mathscr{P} = \mathscr{L}_1 \cup \{K_n\}_{n=1}^{\infty}$ or its complement ($\mathscr{Q} = \{G : \bar{G} \in \mathscr{P}\}$), each having speed $n + 2$. ∎

THEOREM 13. *For* $k > 1$, *let* $\mathscr{L}_k$ *and* $\mathscr{U}_k$ *be properties defined as follows:* $\mathscr{L}_k = \{G : G$ *contains a clique with order at most* k *and the remaining vertices* *are isolated*$\}$ *and* $\mathcal{U}_k = \{G :$ *in G all but at most k vertices are G-equivalent*$\}$. *If* $\mathcal{P}$ *is a property with* $|\mathcal{P}^n| = \Theta(n^k)$, *then, for sufficiently large values of n,*

$$|\mathcal{L}_k^n| \leqslant |\mathcal{P}^n| \leqslant |\mathcal{U}_k^n|,$$

where $|\mathcal{L}_k^n| = \binom{n}{k} + \cdots + \binom{n}{2} + 1$ *and* $|\mathcal{U}_k^n| \leqslant \frac{1}{k!}\,(2^{\binom{k+1}{2}+1})\, n^k$.

Proof. First we want to minimize $|\mathcal{P}^n|$ under the constraint that $|\mathcal{P}^n| = \Omega(n^k)$. So without loss of generality we can assume there exists a $H \in \mathcal{P}^n$ with head of size k and a large independent set. Let A be the head and $B = H - A$ the independent set. To minimize $|\mathcal{P}^n|$, we need to maximize the number of automorphisms of H.

Case 1. $E(A, B) = \emptyset$. Then the head must have an edge, and we maximize the number of automorphisms by making the head a clique. That is $\mathcal{P} = \{G : G$ has a clique of size at most k and there is no other edge of $G\}$ is the "smallest" property containing H. $\mathcal{P}^n$ then contains the independent set on n vertices and all graphs with heads of order 2 to k. The labellings of the graphs are determined completely by which labels appear in the head, so $|\mathcal{P}^n| = \sum_{i=2}^{k} \binom{n}{i} + 1$. Note that a head of order 1 is just the independent set, which is why there is no $\binom{n}{1}$ in the sum.

Case 2. $E(A, B) \neq \emptyset$. Then we maximize the number of automorphisms by saying $|E(A, B)| = |A| \cdot |B|$. If A has an edge, then as above $\mathcal{P} = \{G : G$ has a clique of order at most k completely joined to an independent set$\}$ which is the minimal property containing H. In this case $|\mathcal{P}^n| = \sum_{i=0}^{k} \binom{n}{i}$. On the other hand, if A has no edge, then as above $\mathcal{P} = \{G : G$ is a complete bipartite graph with one class of order at most $k\}$ is the minimal property containing H. In this case again $|\mathcal{P}^n| = \sum_{i=0}^{k} \binom{n}{i}$.

The smaller property is that described in Case 1.

To maximize $|\mathcal{P}^n|$, we should allow every graph with a head of order at most k, since by Theorem 10 the head has at most this order. The count in the proof of Lemma 11 gives us the upper bound $|\mathcal{P}^n| \leqslant \binom{n}{k}\, 2^{\binom{k+1}{2}+1}$. ∎

In fact, we can do even better than give bounds on properties with polynomial order growth; we can describe, using the canonical properties described in Section 2, exactly what properties with polynomial growth look like and hence which functions of polynomial order appear as speeds.

Let A be a simple graph. Any graph H which has one more vertex than A and has a vertex identified so that removing that vertex leaves a graph isomorphic to A will be said to be of the form $A * \{x\}$ and denoted $A * \{x\}$. We allow a loop at x but at no other vertex, and we use A to mean both the subgraph isomorphic to A and its vertex set $V(A)$, where the usage should be clear from context. When we write $A * \{x\}$, we mean one particular graph which can be formed as described.

Consider a graph G in a property $\mathscr{P}$ with growth $|\mathscr{P}^n| = \Theta(n^k)$. We are going to build a canonical property $\mathscr{P}(H, b, c)$ containing G.

By Theorem 10, G has a unique head of order at most k. We define the *type graph* of G as a graph $A * \{x\}$, where A is the graph of the head, and $xy \in E(A * \{x\})$ if and only if y is adjacent to the homogeneous set of G. Hence the type graph $A * \{x\}$ of G has a loop at x if and only if the homogeneous sit of G is a clique in G.

Clearly the type graph is well-defined and captures the structure of G. Further, with $b(a) = 1$ for all vertices a of the head A and $b(x) = \infty$, and with $c \equiv 1$, the canonical property $\mathscr{P}(A * \{x\}, b, c)$ contains G. Thus we call $(A * \{x\}, b, c)$ the *type* of G.

Since A is the head of the original graph, $\Gamma(x) \neq \Gamma(a)$ for all $a \in V(A)$. Every graph $A * \{x\}$ which appears as a type graph satisfies this condition (although there are graphs $A * \{x\}$ which do not satisfy this condition). Hence we can only talk of graphs of type $(A * \{x\}, b, c)$ if the condition holds. Also, if G has type $(A * \{x\}, b, c)$, then $G \in \mathscr{P}(A * \{x\}, b, c)$, but there are also other types $(A' * \{x\}, b, c)$ such that $G \in \mathscr{P}(A' * \{x\}, b, c)$. Only when A is minimal under the condition that $G \in \mathscr{P}(A * \{x\}, b, c)$ will $(A * \{x\}, b, c)$ be the type of G.

Recall from Lemma 2 that $|\mathscr{P}^n(A * \{x\}, b, c)|$ has polynomial order growth. These properties, in fact, form the basis for all properties with polynomial order growth. This surprising result has far reaching consequences, as the three corollaries following the proof demonstrate.

Note that each graph $A * \{x_i\}$ referred to in the statement of the theorem is the type graph of some graph in $\mathscr{P}$.

THEOREM 14. *If* $|\mathscr{P}^n| = O(n^k)$, *then there exist graphs* $A_1 * \{x_1\}, ..., A_r * \{x_r\}$ *such that, for n sufficiently large,* $\bigcup_{i=1}^r \mathscr{P}^n(A_i * \{x_i\}, b_i, 1) = \mathscr{P}^n$, *where* $b_i(A_i) \equiv 1$ *and* $b_i(x_i) = \infty$ *for all i.*

Proof. By Theorem 10, for sufficiently large n, every G in $\mathscr{P}^n$ has a unique head of order at most k. Since there are a finite number of graphs of order at most k, 2^k choices for how the body is connected to the head, and two choices for the loop at x, there are a finite number of types in $\mathscr{P}$. Each of these types are of the form $(A * \{x\}, b, 1)$, where $b(A) \equiv 1$ and $b(x) = \infty$.

If $2k < n < m$ and there is a graph $G \in \mathscr{P}^m$ of type $(A * \{x\}, b, 1)$, then there is a graph $G' \in \mathscr{P}^n$ of the same type, since $\mathscr{P}$ is hereditary. So for $n > 2k$ the number of types in $\mathscr{P}^n$ is a non-increasing function of n. Let the minimum number of types be r occuring in $\mathscr{P}^n$ for $n > 2k$. Let N be such that for all $n > N$ the number of types in $\mathscr{P}^n$ is r. Let $\{(A_i * \{x_i\}, b_i, 1)\}_{i=1}^r$ be the collection of types that occur in $\mathscr{P}^n$ for all $n > N$. Clearly

$\bigcup_{i=1}^{r} \mathscr{P}(A_i * \{x_i\}, b_i, 1) \subseteq \mathscr{P}$. Further, for $n > N$, if $G \in \mathscr{P}^n$ then there is an i for which G has type $(A_i * \{x_i\}, b_i, 1)$, so $\bigcup_{i=1}^{r} \mathscr{P}(A_i * \{x_i\}, b_i, 1) \supseteq \mathscr{P}^n$. ∎

The following corollary is immediate from the proof above. Recall that two properties $\mathscr{P}$ and $\mathscr{Q}$ are equivalent if their symmetric difference is finite. That implies that there exists an N such that $\mathscr{P}^n = \mathscr{Q}^n$ for all $n > N$.

COROLLARY 15. *For each $k \in \mathbb{N}$, there are only a finite number of non-equivalent hereditary properties with polynomial order growth $\Theta(n^k)$.*

Note that in the proof of the main theorem, if $\mathscr{P}(A * \{x\}, b, 1) \subseteq \mathscr{P}$, then for each $A' \leqslant A$ for which $(A' * \{x\}, b', 1)$ is a type, $(A' * \{x\}, b', 1)$ included in the list of types in $\mathscr{P}$, where b' is the restriction of b to $A' * \{x\}$. This gives us an easy way to count the number of graphs in $\mathscr{P}^n$.

COROLLARY 16. *If $|\mathscr{P}^n| = O(n^k)$ then, for n sufficiently large, $|\mathscr{P}^n|$ is a polynomial. In particular, for large enough n,*

$$|\mathscr{P}^n| = \sum_{i=0}^{k} a_i \binom{n}{i},$$

where $0 \leqslant a_j \leqslant 2^{\binom{i+1}{2}+1}$ is an integer for all j.

Proof. Let $r = \limsup_n |\{\text{types that occur in } \mathscr{P}^n\}|$. By the above argument, there is an N such that for all $n > N$ the number of types in $\mathscr{P}^n$ is r. Let $\{(A_i * \{x_i\}, b_i, 1)\}_{i=1}^{r}$ be the collection of types that occur in every $\mathscr{P}^n$ for $n > N$. We wish to count the elements of $\mathscr{P}^n$. For each $1 \leqslant j \leqslant r$ let l_j be the number of automorphisms of $A_j * \{x_j\}$, and $t_j = |A_j| + 1$, the size of the "template graph." Then the number of graphs of type $(A_j * \{x_j\}, b_j, 1)$ in $\mathscr{P}^n$ is $c_j \binom{N}{t_j}$, where $c_j = t_j!/l_j$. The sum of the values c_j over all heads of the same order i gives us a_i. Since $b_j < i!$, Theorem 13 gives us the bound on a_i. Hence $|\mathscr{P}^n| = \sum_{i=0}^{k} a_i \binom{n}{i}$. ∎

We have shown that the properties with polynomial growth do in fact have polynomial growth and described their structure. Hence we shall call properties $\mathscr{P}$ with polynomial order growth *polynomial properties*. All that is left is to describe the minimal properties with polynomial growth, which turn out to be exactly the canonical properties with proper order growth. For the polynomial order, since we have shown that all properties have polynomial growth, a *minimal* property shall be one in which all proper subproperties have a lower order polynomial growth.

COROLLARY 17. *The minimal properties for speed $\Theta(n^k)$ are those which consist of exactly one type, i.e., $\mathscr{P}(A * \{x\}, b, 1)$ where $(A * \{x\}, b, 1)$ is a polynomial order type.*

Proof. It is clear that a property which contains more than one type (of order n^k) can not be minimal. So we only need to show that given a type $(A * \{x\}, b, 1)$, the property $\mathcal{P} = \mathcal{P}(A * \{x\}, b, 1)$ is minimal. Suppose $|\mathcal{P}^n| = O(n^k)$ and let $\mathcal{P}' \subsetneq \mathcal{P}$. We need to prove $|\mathcal{P}'^n| = O(n^{k-1})$.

By Theorem 14, for large n, there are types $\{(A_i * \{x_i\}, b_i, 1)\}$ (with $b_i(A_i) \equiv 1$) such that $\mathcal{P}'^n = \bigcup \mathcal{P}(A_i * \{x_i\}, b_i, 1)$. Since $\mathcal{P}' \neq \mathcal{P}$, $A_i \neq A$ for all i. Hence $A_i \lneq A$ for each i. That is, $|A_i| \leqslant k-1$, so $|\mathcal{P}'^n| \leqslant O(n^{k-1})$. As $|\mathcal{P}^n|$ takes an integer value for each integer n, the values of a_i must be an integer for all i. ∎

Note that since there are only a finite number of polynomial properties of any order, the collection of minimal properties of a certain order is also finite.

4. EXPONENTIAL GROWTH

In the previous section we saw that $|\mathcal{P}^n|$ is polynomial or constant if and only if there is a bound on the size of the complement of a homogeneous set. Another way of describing this condition is to say that every $G \in \mathcal{P}$ has a bounded number of homogeneous sets (more precisely, at most $k+1$ if $|\mathcal{P}^n| = O(n^k)$) and only one of them has unbounded order. Note that this is exactly what is described by the polynomial canonical properties. Let us relax this condition and consider properties $\mathcal{P}$ in which every $G \in \mathcal{P}$ has a bounded number of homogeneous sets but at least two grow without bound. Let $l_{\mathcal{P}}$ be the maximal number of homogeneous sets of any graph $G \in \mathcal{P}$. It was shown in [13] that $l_{\mathcal{P}} < \infty$ if and only if $|\mathcal{P}^n| = O(k^n)$ for some k. In this section, we consider properties with $l_{\mathcal{P}} < \infty$ and find a stronger expression for the speed than $O(k^n)$, giving one direction of this result.

We do not use the other direction, but in this section assume that every property $\mathcal{P}$ has $|\mathcal{P}^n| = \Omega(n^j)$ for all j but $|\mathcal{P}^n| = O(k^n)$ for some k. Our approach shall be to look at the partition of each graph into collections of homogeneous sets. For each G, consider the partition of the vertex set into homogeneous sets, $H_1, ..., H_{l(G)}$, where the orders of the classes are in non-increasing order. Note that we have $l(G) \leqslant l_{\mathcal{P}}$ for all $G \in \mathcal{P}$. Let the order sequence of G be $a(G) = \langle a_1, ..., a_{l(G)} \rangle$, where $|H_i| = a_i$. Clearly $a_i \geqslant a_j > 0$ for each $i < j$ and $\sum_{i=1}^{l(G)} a_i = |V(G)|$.

Fix a property $\mathcal{P}$ with $l_{\mathcal{P}} < \infty$. For each graph $G \in \mathcal{P}$ redefine $a(G)$ to have length $l = l_{\mathcal{P}}$ by adding an appropriate number of zeros. Let $t_i(\mathcal{P}) = \limsup_{G \in \mathcal{P}} a_i(G)$. Since the number of the homogeneous classes is bounded, $t_1(\mathcal{P}) = \infty$. Let $k = k_{\mathcal{P}} = \max\{i : t_i = \infty\}$. Also let $t = t_{\mathcal{P}} = \max\{t_i : i > k\}$. That is, k is the number of unbounded coordinates of $a(G)$ over $\mathcal{P}$ and t is a bound on the bounded coordinates. We showed in the previous section that if $k_{\mathcal{P}} = 1$ then $|\mathcal{P}^n|$ is polynomial (of order at most $t(l-1)$).

Viewed another way, the above constants explain that we can divide any $G \in \mathscr{P}$ into at most k "large" homogeneous sets and have at most $t(l-k)$ remaining vertices. When we divide up the graph in this way, we get a *type* for exponential properties as follows:

For each $G \in \mathscr{P}$ consider $a(G)$, and let $i(G)$ be the number of classes with more than t vertices (i.e., $i(G) = \max\{i : a_i > t\}$). Let $T(G) = (H, b, 1)$ be the type of G, where H is the labeled graph (on at most l vertices) obtained by contracting the homogeneous classes of G (labeled in non-increasing order as in $a(G)$), $b(j) = \infty$ for $j \leqslant i$ and $b(j) = a_j$ for $j > i$.

For each G, the type of G is well-defined and, for a given l and t, there are a finite number of types. Note that saying that G is of type $(H, b, 1)$ gives the following information: G contains $|V(H)|$ homogeneous classes, $|b^{-1}(\infty)|$ of them have order $> t$ and the rest have order at most t. Also, if G is of type $(H, b, 1)$, then $G \in \mathscr{P}(H, b, 1)$. However, there is not a bijection between types and canonical properties, as if $H' \supset H$ and b' is a function on $V(H)$ such that $b'(h) \geqslant b(h)$ for all $h \in V(H)$, then $G \in \mathscr{P}(H', b', 1)$. If we always choose the values of b to be minimal, though, we can use the canonical properties to describe the composition of all properties with a bounded number of homogeneous classes.

THEOREM 18. *If $l = l_{\mathscr{P}} < \infty$, then $\mathscr{P} \supseteq \bigcup \mathscr{P}(H_i, b_i, 1)$ for some collection $\{H_i\}$ where each H_i has at most l vertices. Further, for n sufficiently large, $\mathscr{P}^n = \bigcup \mathscr{P}^n(H_i, b_i, 1)$.*

Proof. Recall that for each G, the type of G, $T(G) = (H, b, 1)$, is well-defined and that, for a given l and t, there are a finite number of types.

Let N be sufficiently large so that, for each $n > N$, if a type occurs in $\mathscr{P}^N$ it also occurs in $\mathscr{P}^n$. Let $\mathscr{C}$ be the collection of types occurring in $\mathscr{P}^N$. By the choice of N, if $n > N$ and $G \in \mathscr{P}^n$ then every graph of the same type as G also appears in $\mathscr{P}^n$.

Hence, for $n \geqslant N$, $\mathscr{P}^n = \bigcup_{\mathscr{C}} \mathscr{P}^n(H_i, b_i, 1)$, and for all n, $\mathscr{P}^n \supseteq \bigcup_{\mathscr{C}} \mathscr{P}^n(H_i, b_i, 1)$. ∎

We would now like to find the speed of such properties.

Consider a property of type $T = (H, b, 1)$. Graphs of this type are completely defined by the number of elements in the classes, so

$$|\{G : |V(G)| = n \text{ and } G \text{ has type } T\}| = \sum_{\substack{n_1, \ldots, n_i > t \\ cj \leqslant b(j)}} \binom{n}{n_1, \ldots, n_i, c_{i+1}, \ldots, c_l}.$$

This is what we calculate in the next lemma.

LEMMA 19. *Let $k, t, c_1, ..., c_l$ be integers. If $c_1, ..., c_l \leqslant t$ and $n \geqslant kt + c$, where $c = \sum_{i=1}^{l} c_i$, then*

$$\sum_{\substack{n_1, ..., n_k > t \\ \sum n_i = n - c}} \binom{n}{n_1, ..., n_k, c_1, ..., c_l} = \sum_{i=1}^{k} p_i(n)\, i^n,$$

where $p_1, ..., p_k$ are polynomials with rational coefficients.

Proof. We use induction on k. For $k=1$ the statement is obvious, because $n_1 = n - c$ and the sum has only one term, $\binom{n}{n_1, c_1, ..., c_l}$, which is a polynomial. So suppose we have the result for all integers less than k. We know

$$\sum_{\substack{n_1, ..., n_k > t \\ \sum n_i = n - c}} \binom{n}{n_1, ..., n_k, c_1, ..., c_l} = \frac{n(n-1)\cdots(n-c+1)}{c_1! \cdots c_l!} \sum_{n_1, ..., n_k > t} \binom{n-c}{n_1, ..., n_k}.$$

The first term is a polynomial in terms of n, so let us evaluate the second one:

$$\begin{aligned} \sum_{n_1, ..., n_k > t} \binom{n-c}{n_1, ..., n_k} &= \sum_{n_1, ..., n_k} \binom{n-c}{n_1, ..., n_k} - \sum_{A \subsetneq \{1, ..., k\}} \sum_{\substack{n_j > t \\ \text{for } j \in A}} \sum_{\substack{n_j \leqslant t \\ \text{for } j \notin A}} \binom{n-c}{n_1, ..., n_k} \\ &= k^{n-c} - \sum_{A \subsetneq \{1, ..., k\}} \sum_{\substack{n_j \leqslant t \\ \text{for } j \notin A}} \sum_{\substack{n_j > t \\ \text{for } j \in A}} \binom{n-c}{n_1, ..., n_k}. \end{aligned}$$

The first two sums on the last line give at most $t^k 2^k$ terms, so the number of the terms does not depend on n. Applying the induction statement to the innermost sum, we get the result. ∎

Using this lemma, we can count the number of graphs in our property.

THEOREM 20. *If $l = l_{\mathscr{P}} < \infty$ then there exist $k, t \in \mathbb{N}$ such that $|\mathscr{P}^n| = O(n^t k^n)$. In particular, there exist polynomials $\{p_i\}_{i=1}^{k}$ such that, for sufficiently large n, we have: $|\mathscr{P}^n| = \sum_{i=0}^{k} p_i(n)\, i^n$, where k is the maximal number of large homogeneous sets in $\mathscr{P}$.*

Proof. Let $\mathscr{P}$ be a property with $l = l_{\mathscr{P}} < \infty$.

As in Theorem 18, let N be sufficiently large so that, for each $n > N$, if a type occurs in $\mathscr{P}^N$ it also occurs in $\mathscr{P}^n$ and let $\mathscr{C}$ be the collection of types occuring in $\mathscr{P}^N$. Clearly if $G \in \mathscr{P}^N$ then every graph of the same type as G also appears in $\mathscr{P}^N$, by the definition of N and t. So we need to count how many graphs have the same type. Given the sizes of the classes, the graph is uniquely determined, as G, so the number of labellings is

$$\sum_{T \in \mathscr{C}} \frac{1}{i(T)!} \sum_{n_1, ..., n_{i(T)}, n_j > t} \binom{n}{n_1, ..., n_{i(T)}, a_{i+1}, ..., a_l},$$

with the $\frac{1}{i(T)!}$ appearing since a graph with a different permutation of the unbounded classes is counted once for each permutation. Since $i(T) \leqslant k$ for all $T \in \mathscr{C}$, applying the previous lemma gives the result. ▮

From the beginning of the section and the previous proof, we can see that if there is more than one large homogeneous set, $|\mathscr{P}^n| = \Omega(2^n)$. Also by Theorem 20, we know that if $|\mathscr{P}^n| = O(k^n)$, then there is a $k \in \mathbb{N}$ such that $|\mathscr{P}^n| = \Theta(k^n)$. Hence, for exponential properties, a minimal property is one which has growth $\Theta(k^n)$ for some k, but every proper subset has growth $O((k-1)^n)$.

We wish to describe these minimal exponential properties. The main idea is to choose t special vertices $v_1, ..., v_t$ and consider the partition of the vertex set of $G - \{v_1, ..., v_t\}$ defined by their adjacencies. That is, let $X = \langle x_1, ..., x_t \rangle$ be a $(0, 1)$ vector of length t. Then the set $U = \{u \in V(G) - \{v_1, ..., v_t\} : uv_i \in E(G) \text{ iff } x_i = 1\}$ is *distinguished* by $v_1, ..., v_t$, if it is not empty and we say that $\{v_1, ..., v_t\}$ *distinguishes* U. The following technical lemma shall be useful to describe our minimal properties here and instrumental later in discussing factorial growth.

LEMMA 21. *Let $k > 1$ and $m > 1$ be fixed. If there is a $G \in \mathscr{P}$ such that $v_1, ..., v_t \in V(G)$ distinguish m sets, each of order at least k, and t is as small as possible, then $t < m \leqslant 2^t$.*

Proof. We apply induction on m. If $m = 2$, then there is some vertex in $v_1, ..., v_t$ which is adjacent to all of the vertices in one set and nonadjacent to all the vertices in the second, or the two sets would not be distinguished. So assume the lemma is true for all numbers less than m. Let $v_1, ..., v_t$ be a minimal set which distinguishes m large sets. Consider the partition of the m sets defined by the adjacencies of v_1. Say v_1 is adjacent to the vertices in r (>0) of these sets and not adjacent to the vertices in $m - r$ (>0) of these sets. By induction, we can distinguish the subcollections of large sets by $r - 1$ and $m - r - 1$ vertices, respectively, and adding v_1 gives us a collection of $m - 1$ vertices which distinguish the m sets. Hence, $t < m$. The upper bound is trivial. ▮

To describe the minimal properties of speed $\Theta(k^n)$ let us briefly recap the structure of properties with growth $|\mathcal{P}^n| = \Theta(k^n)$. The graphs in $\mathcal{P}$ can be described as follows: they contain at most k large homogeneous classes and some boundedly many (say $\leqslant c$) other vertices. It is clear from our calculations that the speed $\Theta(k^n)$ is derived from the k large homogeneous sets. However, we may not simply look at the homogeneous sets, as they may be indistinguishable without other vertices. (For example, if G consists of a star and an independent set, we get growth 2^k, but ignoring the single vertex would give us only an independent set.) By Lemma 21, we need at most $k-1$ vertices to distinguish k large homogeneous classes. It is clear then that a minimal speed has as few of these distinguishing vertices as possible and k homogeneous classes.

As in the polynomial case, we can describe the type of a graph in terms of a special graph $A * \{x_i\}_{i=1}^k$, defined analogously to $A * \{x\}$. Then for any G in a property $\mathcal{P}$ with exponential growth, its type graph $A * \{x_i\}_{i=1}^k$ is obtained by contracting the (up to) k potentially large homogeneous classes into (up to) k distinct vertices $x_1, ..., x_k$, placing a loop at x_i if and only if the corresponding homogeneous set is a clique, and leaving the rest of the graph intact (and calling it A). The remaining edges are placed according to G (as in the polynomial case). Setting $b(A) \equiv 1$ and $b(x_i) = \infty$ for all i and $c \equiv 1$, yields the type $(A * \{x_i\}_{i=1}^k, b, c)$, where clearly $G \in \mathcal{P} * \{x_i\}_{i=1}^k, b, c)$.

As implied above, minimal properties correspond to those types $(A * \{x_i\}_{i=1}^k, b, c)$ *where* $|V(A)|$ *is as small as possible, so* $0 \leqslant |V(A)| \leqslant k-1$.

THEOREM 22. *Given* $k \in (\mathbb{N})$, *the minimal properties for exponential growth of order* $\Theta(k^n)$ *are* $\mathcal{P} = \mathcal{P}(A * \{x_i\}_{i=1}^k, b, c)$, *where* $c \equiv 1$, $b(x_i) \equiv \infty$, $b(A) \equiv 1$, *and* $|V(A)|$ *is minimal under the condition that it distinguishes* $\{x_i\}$. *For each k, there are only a finite number of minimal properties.*

Proof. As in the polynomial case, it is clear from Theorem 18 and the discussion above that if a property is minimal, it must be one of those described. What we need to prove is that these properties are in fact minimal. Let $\mathcal{P} = \mathcal{P}(A * \{x_{i=1}^k\}, b, c)$ as defined in the theorem and suppose $\mathcal{P}' \subsetneq \mathcal{P}$. By Theorem 18, $\mathcal{P}' = \bigcup_{j=1}^s \mathcal{P}(H_j, b_j, c)$ where for each j, $H_j \subsetneq (A * \{x_i\}_{i=1}^k)$ (with $c \equiv 1$ and $b_j \equiv b|_{H_j}$) since $\mathcal{P}' \neq \mathcal{P}$. If $H_j \not\supset A$, then H_j can not contain $\{x_i\}_i^k$ since A was minimal under the condition that it distinguishes the collection. If H_i contains A, then it cannot contain $\{x_i\}_1^k$ either. That means that each H_j has at most $k-1$ classes which can be arbitrarily large, so by Theorem 20, $|\mathcal{P}^n(H_j, b_j, c)| = \Theta(k-1)^n$ for each j. Since there are only a finite number of this type of subproperty, $|\mathcal{P}'^n|$ has the same speed, so $\mathcal{P}$ is a minimal property. ∎

5. FACTORIAL GROWTH

We have shown that if $l(G)$, the number of homogeneous classes in a graph G, is bounded for all $G \in \mathcal{P}^n$ then the growth of $|\mathcal{P}^n|$ is at most exponential. What happens if there is no upper bound on $l(G)$ for $G \in \mathcal{P}$? We begin with the following restatement of a result from [13].

LEMMA 23. *If there are graphs in $\mathcal{P}$ with an arbitrarily large number of homogeneous classes, then $|\mathcal{P}^n| \geqslant n^{(1/2 - o(1))\, n}$.*

The lemma implies that if $|\mathcal{P}^n|$ is not constant, polynomial or exponential, then it is at least factorial. In this section we shall describe the types of growth that occur if the growth of $|\mathcal{P}^n|$ is roughly factorial. Our main result provides a different proof of Lemma 23 as well. Before getting to our main result, however, we examine some simpler cases that are instrumental in proving our main result. Let $b(G) = (b_1, ..., b_{r(G)})$ be the sequence of the orders of the components of G in non-increasing order.

LEMMA 24. *If there is a c such that $b_1(G) < c$ for every $G \in \mathcal{P}$ and $k = \max\{s \leqslant c :$ for each t there is a $G \in \mathcal{P}$ with $|\{i : b_i(G) = s\}| > t\}$, then $|\mathcal{P}^n| = n^{(1 - 1/k + o(1))\, n}$.*

Proof. Note that since the number of vertices in a component is bounded, there is no bound on the number of components. Note also that k is the largest number such that there are graphs in $\mathcal{P}$ with arbitrarily many components of order k. In fact, since there are a finite number $(2^{\binom{k}{2}})$ of graphs on k vertices, there are graphs in $\mathcal{P}$ that have as components arbitrarily many copies of a particular graph, say L_k, on k vertices. Since $\mathcal{P}$ is hereditary, this means that $qL_k \in \mathcal{P}$ for every q. Let $G = (n/k)\, L_k$. We assume here, of course, that $k \mid n$ (and shall continue to do so), but the calculations are similar in other cases as well. We can label G in at least $\binom{n}{k, ..., k} \frac{1}{(n/k)!}$ ways, with $\frac{1}{(n/k)!}$ appearing because the components are isomorphic. Hence

$$|\mathcal{P}^n| \geqslant \binom{n}{k, ..., k} \frac{1}{(n/k)!} = \frac{n!}{(k!)^{n/k}\, (n/k)!} = n^{(1 - 1/k + o(1))\, n}.$$

Let us now examine how large $|\mathcal{P}^n|$ can be. By the definition of k, there is an w, independent of n, such that in each $G \in \mathcal{P}$ all but w vertices of G belongs to a component with size at most k. For each fixed distribution of the small components, the other w vertices create at most $\binom{n}{w} 2^{\binom{w}{2}}$ different graphs, so since the growth of $|\mathcal{P}^n|$ is at least factorial, we can ignore these vertices. Now consider the distribution of the remaining vertices into components. There are $l \leqslant \sum_{i=1}^{k} 2^{\binom{i}{2}}$ possible graphs that occur as components, so if a graph has components of orders $b_1, ..., b_n$ (some of them can be 0) and $a_1, ..., a_l$ is the multiplicity of appearance of a graph as a component, there are $\binom{n}{b_1, ..., b_n}(1/a_1! \cdots a_l!)$ ways to label it. However $a_1! \cdots a_l! \geqslant (((\sum a_i)/l)!)^l \geqslant ((n/kl)!)^l \geqslant ((n/kle)^{n/kl})^l = n^{n/k}(1/kle)^{n/k}$. So $\binom{n}{b_1, ..., b_n}(1/a_1! \cdots a_l!) \leqslant \binom{n}{b_1, ..., b_n}(1/n^{n/k}(1/kle)^{n/k}$. Putting all this together, and using the fact that the multinomial is maximized when $b_i = k$ for $i \leqslant n/k$ and 0 elsewhere, we obtain

$$|\mathscr{P}^n| \leqslant \binom{n}{w} 2^{\binom{w}{2}} \sum_{b_i \leqslant k} \binom{n}{b_1, ..., b_n} (2^{\binom{k}{2}})^n \frac{1}{(n/k)!} = n^{(1-1/k+o(1))\,n}$$

as claimed. ∎

Lemma 25. *If there exists k such that for every N, there is a connected $G \in \mathscr{P}^N$ with maximum degree at most k then $|\mathscr{P}^n| \geqslant n^{(1+o(1))\,n}$.*

Proof. Let s be a fixed integer. We show that for any t there are graphs in $\mathscr{P}$ with t components of size s. Let $N = (sk^2)^t$ and let $G \in \mathscr{P}^N$ be a connected graph with $\Delta(G) \leqslant k$. Let H_1 be a connected induced subgraph of order s. Since the degrees of the vertices of H_1 are at most k, there are at most $s(k-1)$ external neighbors of H_1. Removing these vertices creates at most sk^2 new components in G. By the pigeonhole principle, one of the components must have at least $(sk^2)^{t-1}$ vertices, and we can continue this method by induction. At each step we, remove a new component with order s, and a large component remains. Hence by the calculations in the proof of Lemma 24 we have $|\mathscr{P}^n| \geqslant n^{(1-1/s+o(1))\,n}$. Because s can be arbitrarily large, $|\mathscr{P}^n| \geqslant n^{(1+o(1))\,n}$. ∎

It is worth noting that the hypotheses of Lemmas 24 and 25 when $k > 1$ imply that the number of homogeneous sets is unbounded; the first by demanding many components, and the second by limiting the degrees within a connected graph (in both cases forcing the order of each homogeneous set to be small). However, they do not cover all properties in which there is an unbounded number of homogeneous sets. How can we approach the general case when $|\mathscr{P}^n|$ is factorial? Once again we need to divide our graphs into different classes. We shall show that much can be said about these classes and we can describe, in rough but acceptable terms, their structure.

Lemma 26. *Fix k. If for each $m > 0$ there is a $t \geqslant 0$ and a $G \in \mathscr{P}$ such that $v_1, ..., v_t \in V(G)$ distinguish m sets, each of order at least k, then $|\mathscr{P}^n| \geqslant n^{(1-1/k+o(1))\,n}$.*

Proof. The result follows from the same calculation as in Lemma 24. ∎

Now let $k = k_{\mathscr{P}}$ be the minimal number for which the condition of Lemma 26 fails, if it exists. Then we call a class distinguished by some set of vertices *large* if it has at least k vertices. We call any set V of vertices *dense* if $\Delta(\bar{G}[V]) \leqslant k$ and *sparse* if $\Delta(G[V]) \leqslant k$. We similarly call a pair of sets U, V of vertices *dense* (respectively *sparse*) if, for every $u \in U$, $v \in V$, $|\overline{\Gamma(u)} \cap V|$, $|\overline{\Gamma(v)} \cap U| \leqslant k$ (respectively $|\Gamma(u) \cap V|$, $|\Gamma(v) \cap U| \leqslant k$).

Any set $X = \{v_1, ..., v_t\} \subseteq V(G)$ distinguishes 2^t classes, but only some of them will be large. For a graph $G \in \mathscr{P}$, let $m(G) = \max\{m : \text{there is a number } t \text{ and } \{v_1, ..., v_t\} \subset V(G) \text{ which distinguish } m \text{ large classes}\}$ and $m = m_{\mathscr{P}} = \max\{m(G) : G \in \mathscr{P}\}$. If $k < \infty$, this must exist.

For $X = \{v_1, ..., v_t\}$ which distinguishes $m(G)$ large classes in $G \in \mathscr{P}$, let $a^X = (a_1^X, ..., a_m^X)$ be the sequence of the cardinalities of the distinguished classes in non-increasing order. Let $a(G) = (a_1(G), ..., a_m(G)) = \max\{a^X : X \text{ is as above}\}$ in lexicografic order and let $X(G)$ be the set that achieves this maximum and is minimal first in terms of order and then in lexicographic order considering the labels of G.

Finally, let $l = l_{\mathscr{P}} = \max\{i : \text{for every } r \text{ there is a } G \text{ such that } a_i(G) > r\}$. Note that $l \leqslant m$. Let $s = \max\{a_{l+1}(G) : G \in \mathscr{P}\}$ if $l < m$ and $s = k$ if $l = m$. That is in each G there are at most l classes that can be arbitrarily large and the rest of the classes have at most s vertices.

Our goal is to estimate the number of labelings of graphs in $\mathscr{P}$. Before we do this, we clean up each graph by removing vertices that are not easily characterised. As long as we do not remove too many vertices, the number of labelings of our new graph is approximately the number of labelings of our original graph. The method we use is described in the following lemma.

LEMMA 27. *If $k_{\mathscr{P}} < \infty$ then there is a $c = c_{\mathscr{P}} < \infty$ such that for each $G \in \mathscr{P}$, there is a graph $H = H(G) \subseteq G$ such that $|V(G) - V(H)| \leqslant c$ and $V(H)$ has a partition $\pi(H) = V_1, ..., V_{l(H)}$ such that $l(H) \leqslant l$, each V_i is either dense or sparse, and each pair V_i, V_j is either dense or sparse.*

Proof. From a given graph $G \in \mathscr{P}$, we succesively discard vertices to obtain a new graph. At each step the number of vertices we discard is bounded independently of n. Keep in mind that k, l, m, s are constants depending on $\mathscr{P}$. Furthermore, $t = t(G) = |X(G)| < m$ for all $G \in \mathscr{P}$, by Lemma 21.

Let $G \in \mathscr{P}$ and consider the large classes distinguished by $X(G)$. Discard all vertices not in these classes. There are no more than $k2^t$ such vertices, since $X(G)$ partitions $V(G)$ into 2^t parts (some perhaps empty). In fact, we discard all classes which have less than s vertices, losing at most sm more vertices. Consider the graph which remains, consisting of at most l classes.

If there is a vertex x in a class C with $\bar{d}_C(x)$, $d_C(x) \geqslant k$, then making x into v_{t+1} would create $m(G)+1$ classes each of size at least k, contradicting the maximality of $m(G)$. So for all x in each class C, either $d_C(x) \leqslant k$ or $\bar{d}_C(x) \leqslant k$. In each class we can discard either the vertices of high degree or low degree, as, by Lemma 7 there are at most $2k$ of these and the classes are arbitrarily large, that is, some graph has as many vertices in these classes as we like. This discards at most $2kl$ vertices and now each class has either maximum degree at most k or maximum codegree at most k.

Next, let A and B be two classes. If $a \in A$, either $|\Gamma(a) \cap B| \leqslant k$ or $|\bar{\Gamma}(a) \cap B| \leqslant k$, else we again contradict the maximality of $m(G)$. Let $A_1 = \{a \in A : |\Gamma(a) \cap B| \leqslant k\}$ and $A_2 = \{a \in A : |\bar{\Gamma}(a) \cap B| \leqslant k\}$ and B_1, B_2 defined similarly. We claim that for $i=1$ or $i=2$ we have $|A_i|, |B_i| \leqslant 2k$. Assume, for example, that $|A_1|$ and $|B_2|$ are both $2k+1$. There are at most $k\,|A_1|$ edges and $k|B_2|$ non-edges between the sets, but there are $|A_1|\,|B_2|$ spaces for the edges, hence $|A_1|\,|B_2| \leqslant k\,|A_1| + k\,|B_2|$ and $|A_1| \leqslant 2k$, which creates a contradiction. Since each class is arbitrarily large, we can discard the vertices in the smaller part, removing, again at most $4k\binom{l}{2}$ vertices.

The graph which remains consists of at most l large classes, $V_1, ..., V_l$, and the number of vertices we removed is bounded independently of n. For fixed i, j every $x \in V_i$ has $d_{V_i}(x) \leqslant k$ or $\bar{d}_{V_i}(x) \leqslant k$ and $|\Gamma(x) \cap V_j| \leqslant k$ or $|\bar{\Gamma}(x) \cap V_j| \leqslant k$. ∎

Note that if n is sufficiently large, then for all $G \in \mathscr{P}^n$, $H(G)$ is uniquely determined (by the choice of $X(G)$) and has a uniquely determined partition $\pi = \{V_1, ..., V_l\}$ as given in the proof.

We use this fact to define a type $T(G)$ for each graph as we have done in previous sections, only here, as we have a more complex structure, we can not base our types on the canonical properties. Instead, we develop a new class of properties which can be restricted to have factorial growth and which can be used to label graphs in properties with factorial growth.

We begin by defining an operator on graphs which can simplify their structure. Let G be a (possibly infinite) graph, K be a finite labeled simple graph with loops allowed, $V(K) = \{1, 2, ..., h\}$. Let $\pi = \pi(G, K)$ be a partition of $[|V(G)|] = \{1, 2, ..., |V(G)|\}$ into h nonempty parts $V_1, ..., V_h$. We define the *K, π-transform*, $S_{K,\pi}$ of a graph on n vertices (where $n = n(\pi)$ is the number of elements partitioned by π) as follows: $S_{K,\pi}(G)$ is a new graph G' such that $V(G') = V(G)$ and $E(G') = \{uv : u \in V_i, v \in V_j$, and one of the following holds: (i) $uv \in E(G[V_{...}, V_j])$ and $ij \notin E(K)$, or (ii) $uv \notin E(G[V_i, V_j])$ and $ij \in E(K)\}$. This definition might seem quite restrictive, but in fact many different graphs can be expressed in this way. For example, ij can be a loop, giving this construction great flexibility. Note that each of the canonical properties we have discussed thus far can be described as K, π-transforms. Also note that $S_{K,\pi}(S_{K,\pi}(G)) = G$, that is, S^2 is the identity. If G is the infinite empty graph, we refer to $S_{K,\pi}(G)$ simply as $S_{K,\pi}$. Finally, we call K *minimal* if there is no $K' \subsetneq K$ such that $S_{K',\pi}(G) = S_{K,\pi}(G)$.

We define a hereditary property $\mathscr{S}_{K,\pi}(G)=\{H\cong S_{K',\pi'}(G'): G'\leqslant G, \pi'$ is the restriction of π to G', and $K'\leqslant K$ such that $|V(K')|=$ the number of classes in $\pi'\}$. For example, if G is the infinite complete graph, K consists of two independent vertices, both having a loop, and $\pi=\{V_1, V_2\}$ such that $|V_1|=1$, then $\mathscr{S}_{K,\pi}(G)=\{H: H$ is a star or an empty graph$\}$. If G is the infinite empty graph, we write $\mathscr{S}_{K,\pi}$ rather than $\mathscr{S}_{K,\pi}(\overline{K_\infty})$.

We will use Lemma 27 to describe properties with a factorial rate of growth in terms of K, π-transforms. First we define a labeled graph $K(G)$ as follows. Let $V(K(G))=\{u_1, ..., u_l\}$. Let $u_iu_j\in E(K(G))$ if and only if $\{V_i, V_j\}$ is dense (so a loop occurs at u_i if and only if V_i is dense). Let $H'(G)=S_{K(G),\pi(H)}H(G)$. This is a graph which in some sense preserves the structure of $H(G)$ but has $\Delta(H'(G))\leqslant lk$. We call $T(G)=(K(G), H'(G), \pi)$ a *type* of G. A type $(K(G), H'(G), \pi)$ corresponds directly to the graph $H(G)=S_{K(G),\pi}H'(G)$. Note that if $G\in\mathscr{P}$, then $H(G)\in\mathscr{P}$ and $L\in\mathscr{P}$ for all $L\leqslant H(G)$. Hence $\mathscr{S}_{K(G),\pi}H'(G)\subseteq\mathscr{P}$. Based on this, we shall also say G has type $T=(K, H', \pi)$ for $K=K(G)$ and any $H'\geqslant H'(G)$ with partition corresponding to π on $H'(G)$. Further, given a type $(K, H'(G), \pi)$, we call $(K', H''(G), \pi')$ a subtype if $\mathscr{S}_{K',\pi}H''(G)\leqslant\mathscr{S}_{K,\pi}H'(G)$.

How many graphs have the same type? Since we discarded at most some constant number c of vertices, the number of graphs G for which $H(G)$ is the same is at most exponential in n. Also, the number of labellings of G is at most some polynomial function times the number of labellings of $H(G)$. In turn, a labelling of $H'(G)$ creates at most l^n non-isomorphic labellings of $H(G)$, the maximum occuring when $H'(G)$ is an independent set and $K(G)$ has no G-equivalent vertices. Allowing subgraphs only adds a constant 2^l factor. Hence

$$\frac{|\mathscr{P}^n|}{|\{(K(G), H'(G), \pi): G\in\mathscr{P}^n\}|}=n^{o(n)}.$$

Also, how many types have $H'(G)$ as the second entry? A given $H'(G)$ on n vertices can be paired with at most $2^{\binom{l}{2}+l}$ graphs K and at most l^n choices of partitions. Hence

$$\frac{|\{(K(G), H'(G), \pi): G\in\mathscr{P}^n\}|}{|\{H'(G): G\in\mathscr{P}^n\}|}=n^{o(n)}$$

as well.

We use these facts throughout the next two proofs. Note that here we do not use $k=k_{\mathscr{P}}$, but a new k depending on the transformed graphs.

THEOREM 28. *Assume $|\mathscr{P}^n|=\Omega(k^n)$ for all k. If there exist graphs $G\in\mathscr{P}$ such that $H'(G)$ has a component of arbitrarily large size, then $|\mathscr{P}^n|\geqslant n^{(1+o(1))\,n}$.*

Otherwise there exists a c such that $b_1(H'(G)) \leqslant c_{\mathscr{P}}$ *for all* $G \in \mathscr{P}$ *and* $|\mathscr{P}^n| = n^{(1-1/k+o(1))\,n}$ *for some k.*

Proof. Consider $\{H'(G) : G \in \mathscr{P}\}$. Note that this is not necessarily a hereditary property, as when vertices are removed from G to make some class of $H(G)$ no longer large, the resulting graph G' may have a completely different partition to make $H(G')$. However, it is clear that if $L \leqslant H'(G)$, then there is a $G' \leqslant G$ such that $H'(G') \subseteq L$ and $L - H'(G')$ has less than sl vertices. Hence if $\mathscr{H}$ is the hereditary closure of $\{H'(G) : G \in \mathscr{P}\}$, then $|\mathscr{H}|/|\{H'(G) : G \in \mathscr{P}\}| = n^{o(n)}$.

Now $\mathscr{H}$ is a hereditary property and every $G \in \mathscr{H}$ has maximum degree at most k. Examine two cases. If $H'(G)$ has a component of arbitrarily large order, then by Lemma 25, $|\mathscr{H}^n| \geqslant n^{(1+o(1))\,n}$. Hence by the discussion before this lemma, $|\mathscr{P}^n| \geqslant n^{(1+o(1))\,n}$.

So assume not, that is there is a $c < \infty$ so that $b_1(H'(G)) \leqslant c$. But then for all $H \in \mathscr{H}$, $b_1(H) \leqslant c$. Lemma 24 tells us that there is a k such that there are graphs in $\mathscr{H}$ with an infinite number of components of order k. Hence the calculations there tell us that $|\mathscr{H}^n| = n^{(1-1/k+o(1))\,n}$. Thus, in this case, $|\mathscr{P}^n| = n^{(1-1/k+o(1))\,n}$. ∎

Note that the second half of Theorem 28 applies to all cases where $|\mathscr{P}^n| < n^{(1+o(1))\,n}$. It says that for any property with $|\mathscr{P}^n| < n^{(1+o(1))\,n}$, there is a k such that $|\mathscr{P}^n| = n^{(1-1/k+o(1))\,n}$, where $k = 1$ for subfactorial cases. When $k = 1$ it is not very informative, but the previous sections give us sharp results.

As mentioned in the proof above, the k in the exponent of the speed is the maximum number for which there is a $G \in \mathscr{P}$ for which $H'(G)$ has an arbitrary number of components of order k. Consider a graph G_k on k vertices, a labeled graph with loops K on at most k vertices, and a labeled partition π of G_k into $|V(K)|$ parts. Then let H' be the graph consisting of an infinite number of copies of G_k and let π be extended to this graph so that it partitions each copy of G_k in the same way.

For $k \geqslant 2$, we construct the property $\mathscr{S}_{K,\pi}(H')$ and show that this is a minimal property for $n^{(1-1/k+o(1))\,n}$. By saying a property $\mathscr{P}$ is a *minimal property* for $n^{(1-1/k+o(1))\,n}$ (defined only when $k \geqslant 2$) we mean that for any proper subproperty $\mathscr{P}'$, its speed $|\mathscr{P}'^n| \leqslant n^{(1-1/(k-1)+o(1))\,n}$.

Notice that in constructing this type of property we can assume that K has exactly k vertices and π partitions G_k into exactly k nonempty sets. If not, we can split any vertex of K which is assigned more than one vertex of G_k by π, copy all original adjacencies to the new vertices and induce on the new vertices an independent set or a clique with loops, depending on whether the original vertex had a loop. This will give us K and π as described.

This fact provides the motivation for the following theorem.

THEOREM 29. *The collection* $\{\mathcal{M}=\mathcal{S}_{K,\pi}(H'): K$ *is a labeled graph with (possible) loops on k vertices, H' is the infinite graph where each component is a copy of some graph* G_k *on k vertices, and* π *is a partition of* $V(H')$ *which partitions each copy of* G_k *in the same manner*$\}$ *is the collection of minimal properties for speed* $n^{(1-1/k+o(1))\,n}$.

Proof. Let K, G_k, H' and π satisfy the conditions in the theorem and let $\mathcal{M}=\mathcal{S}_{K,\pi}(H')$. We first show that these properties have the proper speed. If $\mathcal{H}=\{H'(G): G\in\mathcal{M}\}$, then $\mathcal{H}=\{H: H\leqslant H'\}$ and by Lemma 24, $|\mathcal{H}^n|=n^{(1-1/k+o(1))\,n}$. Hence by the proof of Theorem 28, $|\mathcal{M}^n|=n^{(1-1/k+o(1))\,n}$.

Now we need to show that $\mathcal{M}$ is minimal. Let $\mathcal{Q}$ be a proper subproperty of $\mathcal{M}$ and let $M\in\mathcal{M}\backslash\mathcal{Q}$. Since $M\in\mathcal{M}$, we know $H'(M)\leqslant H'$. The graph M is finite, so $H'(M)$ has some finite number of components, each an induced subgraph of G_k. Say $H'(M)$ has $r=r(H'(M))$ components. Then since $M\notin\mathcal{Q}$, and $\mathcal{Q}$ is hereditary, the graph $rG_k\notin\{H'(G): G\in\mathcal{Q}\}$. That is, every graph $Q\in\mathcal{Q}$ has $H'(Q)$ with at most $r-1$ components which are copies of G_k and the rest of the components are proper induced subgraphs of G_k, that is, each of these remaining components has at most $k-1$ vertices. Hence by the proof of Theorem 28, $|\mathcal{Q}^n|\leqslant n^{(1-1/(k-1)+o(1))\,n}$.

Finally, we need to show that there are no other minimal properties at this speed. Let $\mathcal{P}$ be a property with $|\mathcal{P}^n|=n^{(1-1/k+o(1))\,n}$ and assume $\mathcal{P}$ is minimal. By the proof of Theorem 28, such a speed arises from the fact that there are graphs $G\in\mathcal{P}$ such that $H'(G)$ has arbitrarily many components of order k, but every $G\in\mathcal{P}$ has $H'(G)$ with a bounded number of components of larger order. Since there are a finite number of graphs on k vertices, there must be some particular graph on k vertices, say G_k, such that there are graphs $G\in\mathcal{P}$ such that $H'(G)$ has arbitrarily many components isomorphic to G_k. Similarly, there must be graphs $G\in\mathcal{P}$ such that an arbitrary number of the components of $H'(G)$ are congruent to G_k and some particular partition π of this G_k fits in the conversion of G to $H'(G)$. Finally, there are a finite number of ways to transform G_k with $\pi(G_k)$ to obtain a $G\in\mathcal{P}$, so one particular graph K must occur as a template graph in these graphs G an arbitrary number of times. Let $\mathcal{M}=\mathcal{S}_{K,\pi}(H')$, where H' is the infinite graph with all components congruent to G_k and π maps these copies of G_k to $V(K)$ in the same way. Note that $\mathcal{M}$ has the structure described in the theorem and the construction implies $|\mathcal{M}^n|\leqslant n^{(1-1/k+o(1))\,n}$. Since $\mathcal{M}\subseteq\mathcal{P}$, they both have the same speed, and $\mathcal{P}$ is minimal, $\mathcal{M}=\mathcal{P}$. ∎

In fact, as we saw in the polynomial and exponential cases, our minimal properties will in some sense characterize a basis for all properties of speed $n^{(1-1/k+o(1))\,n}$. In the polynomial case, there was an exact fit, as expressed in Corollary 14. In the exponential case, expanding the graphs $A*\{x_i\}$ to allow any structure in A, not just the minimal structure, provides an exact fit as well (see the proofs of Theorem 20 and Theorem 22 for details). In the small factorial case, as might be expected, we can not do as well. The large number of vertices we discard, while only a bounded number, complicate the structure greatly. However, except for these vertices, the structure is clear. The details are easily react out of the results of this section.

THEOREM 30. *Let $\mathcal{P}$ be a property with speed $n^{(1-1/k+o(1))\,n}$. There exists a finite collection $\mathcal{M}$ of minimal properties and a constant c such that for all $G \in \mathcal{P}$ there is a set $W \subseteq V(G)$ of at most c vertices so that $G-W$ is an element of some property in $\mathcal{M}$.*

6. HIGHER RATES OF GROWTH

In the previous sections we have seen that for properties $\mathcal{P}$ in which $|\mathcal{P}^n| < n^{(1+o(1))\,n}$, the structure of the property is well defined and the growth of the property can be completely described by a function in n. In particular, either $|\mathcal{P}^n| = \sum_{i=0}^{k} p_i(n)\, i^n$, where $\{p_i(n)\}_{i=1}^{k}$ are polynomials; or $|\mathcal{P}^n| = n^{(1-1/k+o(1))\,n}$.

As mentioned in the Introduction, the next level of growth, properties $\mathcal{P}$ where $n^n \leqslant |\mathcal{P}^n| \leqslant 2^{o(n^2)}$ are quite problematic. In [13], the question is raised as to whether the limits

$$\lim_{n\to\infty} \frac{\log |\mathcal{P}^n|}{n \log n} \qquad \text{and} \qquad \lim_{n\to\infty} \frac{\log \log |\mathcal{P}^n|}{\log n}$$

always have to exist. Clearly if $|\mathcal{P}^n| < n^n$ this question is answered in the affirmative. However, using random methods, we show in a forthcoming paper [2] that these limits do not necessarily exist and describe the limited statements one can make about properties in this range. Despite the quite predictable behaviour of properties of low speed, properties in this range can oscillate infinitely often, that is, there can be a big gap between $\liminf_n(|\mathcal{P}^n|/n \log n)$ and $\limsup_n(|\mathcal{P}^n|/n \log n)$.

Above this range, however, the properties do in fact settle down, as shown by Bollobás and Thomason [3]. In particular, in this range we once again have minimal properties.

THEOREM 31 [3]. *If $\limsup_{n\to\infty} \log |\mathcal{P}^n|/n^2 > 0$ and $\mathcal{P}$ is not the trivial property of all graphs, then there is an integer $k \geqslant 2$ such that $|\mathcal{P}^n| = 2^{(1-1/k+o(1))\,n^2/2}$. Furthermore, there is a hereditary property $\mathcal{Q} \subseteq \mathcal{P}$ such that $|\mathcal{Q}^n| = 2^{(1-1/k+o(1))\,n^2/2}$ but $|\mathcal{Q}_0^n| \leqslant 2^{(1-1/(k-1)+o(1))\,n^2/2}$ for every hereditary property $\mathcal{Q}_0$ properly contained in $\mathcal{Q}$.*

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