Online Ramsey Games for Triangles in Random Graphs

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Abstract

In the online F-avoidance edge-coloring game with r colors, a graph on n vertices is generated by at each stage randomly adding a new edge. The player must color each new edge as it appears; his goal is to avoid a monochromatic copy of F. Let $N_0(F,r,n)$ be the threshold function for the number of edges that the player is asymptotically almost surely able to paint before he loses. Even when $F = K_3$, the order of magnitude of $N_0(F,r,n)$ is unknown for $r \geq 3$. In particular, the only known upper bound is the threshold function for the number of edges in the offline version of the problem, in which an entire random graph on n vertices with m edges is presented to the player to be r edge-colored. We improve the upper bound for the online triangle-avoidance game with r colors, providing the first result that separates the online threshold function from the offline bound for $r \geq 3$. This supports a conjecture of Marciniszyn, Spöhel, and Steger that the known lower bound is tight for cliques and cycles for all r.

Keywords: Ramsey number, online Ramsey-games, triangle-free, random graphs.

1 Introduction

Given a graph F and a natural number r, let $R_r(F)$ be the least positive integer m such that any r-coloring of the edges of the complete graph on m vertices contains a monochromatic copy of F. We call $R_r(F)$ the *Ramsey number*; Ramsey proved that $R_r(F)$ exists for any F and any r [8]. We consider an 'online' version of the problem. Friedgut, Kohayakawa, Rödl, Ruciński, and Tetali [3] studied the one-player game in which edges are presented one-at-a-time in an order chosen uniformly at random. The player, Painter, must color each edge as it is presented, trying to avoid a monochromatic triangle. More generally, in the

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F*-avoidance edge-coloring game with* r *colors* Painter tries to avoid a monochromatic copy of some fixed graph F.

Let r be a fixed positive integer and let n approach infinity. We say that $f(n) \ll g(n)$ if $\lim_{n\to\infty} f(n)/g(n) = 0$. We will call $N_0(F, r, n)$ a *threshold function* for the F-avoidance game if when $N \ll N_0(F, r, n)$ then there exists a strategy such that the player a.a.s. (asymptotically almost surely) wins the game played with N edges by following the strategy, and when $N \gg N_0(F, r, n)$ the player a.a.s. loses the game played with N edges. In [3] it was proved that if F is a triangle then $N_0(F, 2, n) = n^{4/3}$ and $N_0(F, 3, n) \geq n^{7/5}$, although the authors comment that their proof of the latter inequality can be improved to give the lower bound $n^{13/9}$.

We will follow standard notation; K_m denotes the complete graph, and C_m the cycle, on m vertices. For a graph G we let e_G denote the number of edges in G and v_G the number of vertices. If H is a subgraph of G we write $H \subseteq G$; if additionally $H \neq G$ then $H \subset G$. Let $m_2(G) = \max_{H\subseteq G} \frac{e_H - 1}{v_H - 2}$.

To find an upper bound for $N_0(F, r, n)$ we shall consider an offline version of the game. Let $G_{n,m}$ be the random graph on n vertices having m edges; Painter must color the edges of $G_{n,m}$ with r colors. The following theorem provides a threshold function for the offline game; because Painter has more information available to him when choosing colors in the offline game, this yields an upper bound for $N_0(F, r, n)$.

Theorem 1.1 ([5],[9],[10])**.** *Fix an integer* $r \geq 2$ *and a graph* F *that is not a star forest or, if* $r = 2$*, a forest of stars and paths with three edges. Let* $\mathcal{P}$ *be the graph property that any* r*-coloring of the edges of* G *has a monochromatic copy of* F*. Then there exist constants* $c = c(F, r)$ *and* $C = C(F, r)$ *such that*

$$\lim_{n\to\infty} \mathbb{P}[G_{n,m} \in \mathcal{P}] = \begin{cases} 1 & \text{if } m > Cn^{2-1/m_2(F)}, \\ 0 & \text{if } m < cn^{2-1/m_2(F)}. \end{cases}$$

Marciniszyn, Spöhel, and Steger [6] proved that for every graph F and integer $r \geq 1$, $N_0(F, r, n)$ exists and that

$$\lim_{r\to\infty} \lim_{n\to\infty} \frac{\log N_0(F, r, n)}{\log n} = 2 - \frac{1}{m_2(F)}.$$

In particular, for $F = K_3$ the following is known.

Theorem 1.2. *Let* $r \geq 1$*. Then the online* K_3*-avoidance edge-coloring game with* r *colors has a threshold function* $N_0(K_3, r, n)$ *that satisfies*

$$N_0(K_3, 2, n) = n^{\frac{4}{3}}, \tag{1}$$

$$n^{\frac{3}{2}\left(1-\frac{1}{3^r}\right)} \leq N_0(K_3, r, n) \leq n^{\frac{3}{2}}. \tag{2}$$

The lower bound in (1) is from [6], and the upper bound is from [3]. The upper bound of (2) is from Theorem 1.1. To prove the lower bound in (2), they consider a greedy strategy: as each edge is presented Painter colors it with color i if and only if for every $j > i$ using color j would close a monochromatic triangle, but using color i does not. In fact, they prove a more general lower bound for a general graph F, using a variant of a greedy strategy in which for each color Painter avoids a particular subgraph of F. This strategy is optimal for K_3 in two colors; see [3]. It is not optimal for every graph, however; for example, if F is the graph formed by two triangles sharing one vertex and $r = 2$ then this strategy provides the bound $n^{25/18} \leq N_0(F, 2, n)$, while a different strategy improves it to $n^{17/12} \leq N_0(F, 2, n)$. In [6] it was conjectured that their greedy strategy is optimal for K_3 and any number of colors. In fact, they conjecture that it is optimal for any clique and any cycle.

Grytczuk, Hałuszczak, and Kierstead [4] considered a two-player online Ramsey game, $(F, \mathcal{H})$, in which Builder presents edges one at a time and Painter colors them; Builder must ensure that the underlying graph remains at all times in the family $\mathcal{H}$, and Painter tries to avoid creating a monochromatic copy of F. Butterfield, Grauman, Kinnersley, Milans, Stocker, and West [2] considered the family of graphs with maximum degree at most k, denoted S_k. In particular, for triangles, they proved that there is a strategy for Builder to win the (K_3, S_4) game but that for $k < 4$ Painter can always win the (K_3, S_k) game. Belfrage, Mütze, and Spöhel [1] considered density conditions rather than degree ones, which connects the two-player online game with the one-player game. Let $\mathcal{H}_d$ denote the family of graphs that contain no subgraph H with $e_H/v_H > d$; we call $(F, \mathcal{H}_d)$ the deterministic F-avoidance game with density restriction d.

Theorem 1.3 ([1]). *Let F be a graph with at least one edge, and let $r \geq 2$. If $d > 0$ is a real number such that Builder has a winning strategy in the deterministic F-avoidance game with r colors and density restriction d, then the threshold for the F-avoidance edge-coloring game with r colors satisfies*

$$N_0(F, r, n) \leq n^{2-1/d}.$$

We use this result to improve the upper bound from Theorem 1.2 in the K_3-avoidance edge-coloring game for any number of colors. This is the first result separating $N_0(K_3, r, n)$ from the offline bound provided by Theorem 1.1 for $r \geq 3$. While there is still a gap between our upper bound and the lower bound in (2), this supports the conjecture that the lower bound is sharp.

Theorem 1.4. *For $r \geq 3$, there exists a constant $c_r > 0$ such that*

$$N_0(K_3, r, n) \leq n^{\frac{3}{2}-c_r}.$$

For $r = 3$ we provide Builder with a different strategy, yielding a better c_3 than Theorem 1.4 provides. We present it here because there is some chance that this is Builder's optimal strategy in the deterministic game, or at least gives an intuition for developing it.

Theorem 1.5.

$$n^{\frac{3}{2}-\frac{1}{18}} \leq N_0(K_3, 3, n) \leq n^{\frac{3}{2}-\frac{1}{42}}.$$

The lower bound in Theorem 1.5 follows from (2) with $r = 3$.

We prove Theorem 1.5 in Section 2 and Theorem 1.4 in Section 3.

2 Proof of Theorem 1.5

We prove Theorem 1.5 by providing a strategy for Builder in the two-player deterministic game with density restriction $d = 42/22$, which with Theorem 1.3 yields an upper bound of $n^{2-22/42} = n^{3/2-1/42}$.

Proof. First we present the strategy; we will check that it satisfies the density restriction afterwards.

Algorithm:

- Phase I:
  - Step 1: Builder places the edges of a star with center u and 25 leaves, and allows Painter to color the edges. There will be at least 9 edges in one color, say blue. Label the non-u endpoints of nine of them $v_1, \dots, v_9$.
  - Step 2: For each i, Builder gives 13 children to v_i, using new vertices, and lets Painter paint those edges. For each i, there is some 'majority color' such that v_i has at least 5 children whose edges to v_i are painted with the majority color. If there exist $v_{i_1}, v_{i_2}, v_{i_3}$ such that v_{i_j}'s majority color is red for $1 \leq j \leq 3$, or such that v_{i_j}'s majority color is green for $1 \leq j \leq 3$, move to Phase III.
  - Step 3: If this step is reached, at most four of $\{v_1, \dots, v_9\}$ do not have majority color blue, so there are five whose majority color is blue. Without loss of generality, assume they are v_1, v_2, v_3, v_4, v_5. Move to Phase II.
- Phase II: Set $j = 1$.
  - Step 1: For $1 \leq j \leq 5$, Builder gives 13 children, using new vertices, to each of the five children of v_j. If some child of v_j has majority color blue, call that child w_j (if there is more than one such child of v_j, choose one arbitrarily). If for any $1 \leq j \leq 5$ there is no such w_j then there are three children of v_j whose majority color is red, or three whose majority color is green. In that case, Builder moves to Phase III.
  - Step 2: If this step is reached, Builder adds the edges $\{uw_j\}_{j=1}^5$ and lets Painter paint them. If any of those edges is painted blue, then $\{u, v_j, w_j\}$ forms a blue triangle. Consequently, at least three of the uw_j edges must have the same color, red or green. Move to Phase III.
- Phase III: When this phase is reached, there is some rooted tree with root r that has three children c_1, c_2, c_3, each with five children $\{a_{i,j}\}_{j=1}^5$ for $1 \leq i \leq 3$. Additionally, edges of the form rc_i are all in one color (say blue) and edges of the form $c_i a_{i,j}$ for any i, j are all in another color (say red).
  - Step 1: Builder adds the edges $\{c_1 a_{3,j}\}_{j=1}^5$, $\{c_2 a_{1,j}\}_{j=1}^5$, and $\{c_3 a_{2,j}\}_{j=1}^5$. This connects c_1 to each child of c_3, c_2 to each child of c_1, and c_3 to each child of c_2.

If there is a red edge from c_1 to a child of c_3, as well as a red edge from c_3 to a child of c_2 and a red edge from c_2 to a child of c_1, then move to step 2. Otherwise move to Phase IV.

  - Step 2: If this step is reached, there is a red cycle of the form $c_1 a_{1,j_1} c_2 a_{2,j_2} c_3 a_{3,j_3} c_1$. In this case, Builder presents the edges c_1c_2, c_2c_3, c_1c_3. Painter now cannot avoid making a monochromatic triangle.

- Phase IV: If this phase is reached, then there is some $i \neq j$ and three vertices that are connected to c_i by red edges and to c_j by blue (or green) edges. In this case, Builder presents the edges of the triangle on those three vertices, and Painter now cannot avoid making a monochromatic triangle.

It remains to check that Builder played within the density restriction $d = 42/22$. Let G be the graph at the end of the game. Among all densest subgraphs of G, let H be chosen to be inclusion-minimal. Obviously H is connected, otherwise some component of H has density at least as high, contradicting the minimality of H. With some case analysis it can be checked that G contains no subgraph H with $e_H/v_H \geq 2$; the idea is that if we iteratively remove a vertex with degree at most 2 then we obtain a single edge, which is a graph with density less than 2. If H is a forest then its density is strictly less than $1 < d$. If H_1 is a connected graph that is not a tree and H_1' is obtained from H_1 by adding a pendant edge, then $e_{H_1'}/v_{H_1'} = (e_{H_1}+1)/(v_{H_1}+1) \leq e_{H_1}/v_{H_1}$. Consequently, H contains no pendant edge.

On the other hand, if H_2 is a connected graph that is not a tree and H_2' is obtained from H_2 by adding a vertex connected to two distinct vertices in H_2, then $e_{H_2'}/v_{H_2'} = (e_{H_2} + 2)/(v_{H_2}+1)$. If H_2 satisfies $e_{H_2}/v_{H_2} < 2$ then $(e_{H_2}+2)/(v_{H_2}+1) > e_{H_2}/v_{H_2}$. Consequently, if H is a densest subgraph, then for every vertex $w \notin V(H)$ we know that w has at most one neighbour in H.

If Phase II terminates before step 2 then the tree in Phase III has root $r = v_j$ for some $1 \leq j \leq 5$. Now, step 1 of Phase III creates fifteen vertices of degree two whose neighbourhoods are connected. Consequently, H is densest if either Phase IV is never reached or Phase IV is reached at the end of step 1. In either case, H is a subgraph of one of the graphs in Figure 1. It has 19 vertices: the root and its three children, and 15 vertices of degree two. It has 36 edges: 3 from the root, 3 in the final triangle, and 30 from the vertices of degree two. With some work, using the fact that if $w \notin V(H)$ then w has at most one neighbour in H, one can check that both graphs in Figure 1 are strictly balanced. We omit the elementary but tedious details. Density is therefore at most $36/19 < 42/22$.

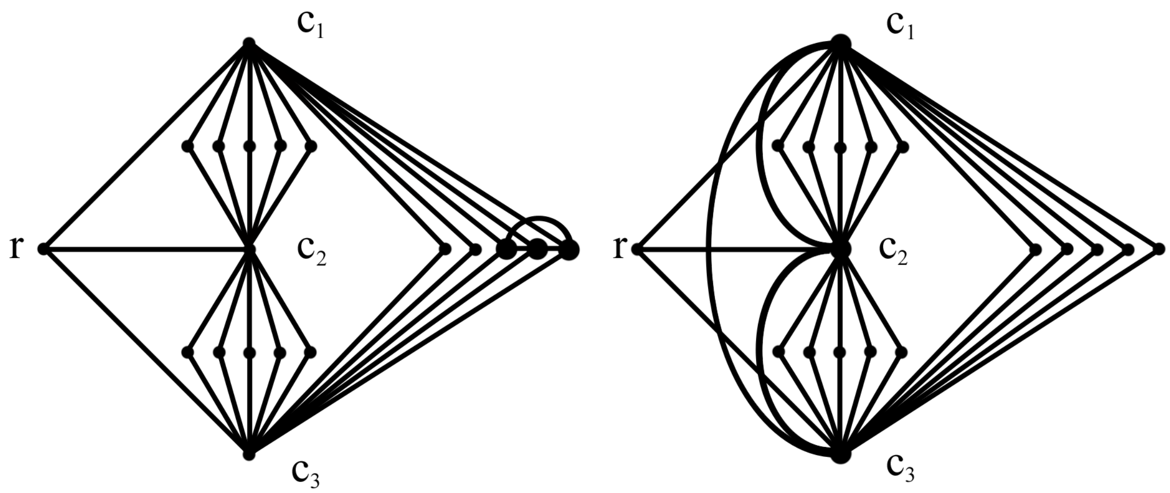

Figure 1: Larger vertices indicate position of final triangle.

If on the other hand Phase II reaches step 2 then the tree in Phase III has root $r = u$ and children from $\{w_j\}_{j=1}^{5}$. Again, a densest subgraph H will occur if either Phase IV is never reached or Phase IV is reached at the end of step 1. This time, however, for each w_j that is a child in the tree there is a vertex v_j connected to both u and w_j. Note that if $e_H/v_H \leq 3/2 < 42/22$ then the density restriction is satisfied. If on the other hand $e_H/v_H > 3/2$ then there is no triangle in G with exactly one vertex in H, as removing such a triangle would increase the density: $(e_H - 3)/(v_H - 2) > e_H/v_H$. Consequently, H is a subgraph of one of the graphs in Figure 2, both of which are strictly balanced. A densest subgraph in this case therefore has density at most 42/22.

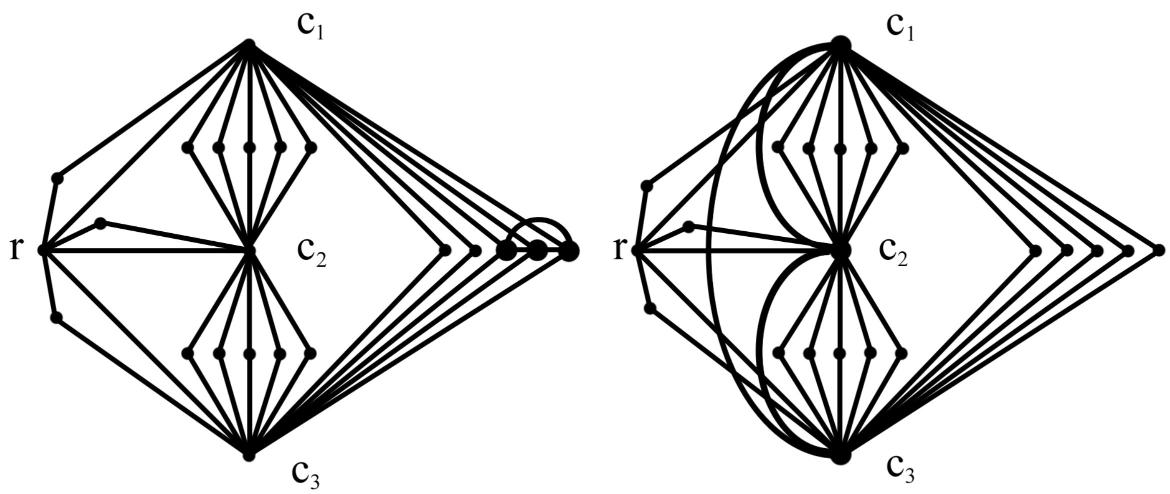

Figure 2: Larger vertices indicate position of final triangle.

□

The following lemma is due to Grytczuk, Hałuszczak, and Kierstead [4]; they proved it only for the two-color game, but their proof easily generalizes to the r-color game.

Lemma 2.1. *If $\mathcal{H}$ is the family of all forests and $F \in \mathcal{H}$, then there exists a strategy for Builder to win the $(F, \mathcal{H})$ game.*

An implication of this lemma is that Builder can win the two-player deterministic game with density restriction 1 when F is a forest, because every graph in $\mathcal{H}$ has density less than 1. This can be used to shorten our above proof: Builder can force a monochromatic tree with root u such that u has five children $v_1, \ldots, v_5$, each of which has one child which itself has five children. This is exactly the tree that our strategy requires for Phase II, step 2; Builder can therefore proceed with our strategy, starting with Phase II, step 2. We leave Phases I and II as they are for the sake of having a self-contained proof.

3 Proof of Theorem 1.4

We prove Theorem 1.4 by providing a strategy for Builder in the two-player deterministic game with r colors that maintains density strictly less than 2. The strategy is along the lines of a method from [4].

Proof. Let S_i be Builder's strategy to force a triangle in the two-player game with i colors, and let m_i be the number of vertices Builder needs for strategy S_i. The strategy S_1 is obvious, and $m_1 = 3$. We will define S_r recursively in terms of S_{r-1}. Builder begins with two large sets of vertices, X_1 and Y_1 (their size will be determined by what follows). In Phase 1, Builder will place edges only between X_1 and Y_1.

For each step in Phase 1, Builder chooses $r+1$ vertices in Y_1 and one vertex in X_1 and presents the $r+1$ edges between them. Painter must paint at least two of these r edges with the same color; Builder will discard and never reuse the $r-1$ vertices from Y_1 that are not painted with the majority color (if more than two edges get the same color, Builder still discards $r-1$ vertices). Discarded vertices will never be used again; vertices from X_1 will also never be reused.

Builder maintains an auxiliary edge-colored graph on Y_1 whose edges are the pairs of vertices kept at each stage, and whose color is the color of their edges to X_1. For example, if Builder's first move is to place the edges $xy_1, \ldots, xy_{r+1}$ and Painter paints xy_1 and xy_2 red then Builder will discard $\{y_3, \ldots y_{r+1}\}$ and will add the red edge y_1y_2 to his auxiliary graph. Notice that Builder does not present any edge from his auxiliary graph, and so these edges do not contribute to density. If yy' is a red edge in the auxiliary graph then there exists $x \in X_1$ such that xy and xy' are both red, so if Builder were to present the edge yy' then Painter would not be able to paint it red without creating a monochromatic triangle.

Claim 1: Builder can force an arbitrarily large star in the auxiliary graph, where he does not care how its edges are colored.

Proof. This follows from induction on the number of leaves in the star. Builder can force a star with one leaf because he forces an edge in each step. Suppose he can force a star with

$s-1$ leaves. He can therefore also force $r+1$ disjoint stars, each with $s-1$ leaves, by playing this star-forcing strategy repeatedly. Suppose the centers of these stars are $y_1, y_2, \ldots, y_{r+1}$; Builder can then choose a new vertex $x \in X_1$ and present the edges $\{xy_1, \ldots, xy_{r+1}\}$. At least two will be given the same color, say xy_1 and xy_2, and so Builder adds y_1y_2 to his auxiliary graph. Adding this edge to the star with center y_1 results in a star with s leaves. □

Claim 2: Builder can force an arbitrarily large clique in the auxiliary graph, where he does not care how its edges are colored.

Proof. This follows from Claim 1 by induction on the number of vertices in the clique. We already verified that Builder can force K_2, which is a star with one edge. Suppose Builder can force K_{m-1}, and suppose his strategy to do so involves s_{m-1} vertices from Y. By Claim 1, Builder can force a star with s_{m-1} leaves. He can then play his strategy to force K_{m-1} on the leaves of the star, resulting in a copy of K_m. □

Let m be the Ramsey number $R_r(m_{r-1})$; by Claim 2 Builder can force a clique on m vertices in the auxiliary graph. This clique will contain a monochromatic (say, red) copy of $K_{m_{r-1}}$. Notice that if Builder presents any edge among these vertices Painter may not color it red without creating a red triangle.

Builder may therefore play strategy S_{r-1} on these m_{r-1} vertices. Strategy S_{r-1} involves $r-1$ phases, which will be Phases 2 through r in strategy S_r. He begins by splitting these m_{r-1} vertices into two sets, Y_2 and X_2, each sufficiently large for what follows; this is the beginning of Phase 2. At the end of Phase 2, Builder will have obtained a set of m_{r-2} vertices in Y_2 such that Painter can make no edge between them, say, blue (and Painter can still not make them red). Builder may therefore play strategy S_{r-2} among these m_{r-2} vertices; this is Phase 3. By the end of Phase $r-1$, Builder will have obtained a set of $m_1 = 3$ vertices in Y_{r-1} such that Painter can only make edges between them, say, green. These three vertices are in Y_r, and X_r is an empty set. Builder therefore wins if he presents the three edges spanned by Y_r; this is Phase r. Phases 1 through r together form strategy S_r. Notice that $X_j, Y_j \subseteq Y_{j-1}$ for all $2 \le j \le r$ and that $X_j \cap Y_j = \emptyset$ for all $j \in [r]$.

It remains to show that throughout the course of the game Builder never creates a graph with density 2 or higher. Notice that the game, and therefore the graph, is finite. If $1 \le j \le r-1$ then in phase j of the game Builder places edges only between X_j and Y_j. For the purpose of analysis, we will orient these edges in the following way. Notice that if $x \in X_j$ is used in phase j then x has exactly $r+1$ neighbors in Y_j, because x is used only once. Of these, $r-1$ are discarded for having minority colors and never used again; orient those edges from X_j to Y_j and orient the remaining two edges from Y_j to X_j. If $y \in Y_j$ and y has non-zero in-degree then y is discarded during phase j. At the end of Phase j, therefore, vertices in X_j have in-degree 2 or 0 and vertices in Y_j have in-degree 1 or 0. Moreover, if $y \in Y_j$ has in-degree 1 then it is never used again.

Because X_j and Y_j are chosen from non-discarded vertices of Y_{j-1}, at the beginning of phase j each vertex in $X_j \cup Y_j$ has zero in-degree. By the end of Phase $r-1$, therefore, every vertex in the graph has in-degree at most 2. Phase r consists of placing the edges of a triangle in Y_r; orient these to be a cycle. Now vertices in Y_r have in-degree exactly one.

If there is a subgraph H in the final graph whose average in-degree is at least 2; then every vertex in H has in-degree exactly 2. This implies that H contains an oriented cycle. The only oriented cycle in the graph is the final triangle, however, and these vertices have in-degree 1, which is a contradiction. Consequently, $e_H < 2v_H$, as desired.

□

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