Combinatorics, Probability and Computing (2012) 00, 000–000. © 2012 Cambridge University Press
DOI: 10.1017/S0000000000000000 Printed in the United Kingdom

Towards a weighted version of the Hajnal-Szemerédi Theorem

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Many results in graph theory study the relation between the minimum degree of a given graph and its spanning subgraphs. For example, Dirac's theorem asserts that a graph on n vertices with minimum degree at least $\lceil \frac{n}{2} \rceil$ contains a Hamilton cycle. Hajnal and Szemerédi [4] proved that every graph on $n \in r\mathbb{Z}$ vertices with minimum degree at least $(1 - \frac{1}{r})n$ contains a spanning subgraph consisting of $\frac{n}{r}$ vertex-disjoint copies of K_r (we call such a subgraph a K_r-factor).

In this note we propose investigating this relation in edge-weighted graphs. As a concrete problem, we study the particular case when the spanning subgraph is the graph formed by vertex-disjoint copies of K_r (in other words, we would like to extend the Hajnal-Szemerédi theorem to edge-weighted graphs). Suppose we equip the complete graph K_n with edge weights $w\colon E(K_n) \to [0,1]$. For a given weighted graph and vertex v we let $\deg_w(v)$ denote the weighted degree of the vertex v. Let $\delta_w(G)$ be the minimum weighted degree of the graph G. The main question can be formulated as the following: How large must $\delta_w(K_n)$ be to guarantee that there exists a K_r-factor such that every factor has total edge weight at least $t\binom{r}{2}$ for some given $t \in [0,1]$?

More formally, for $n \in r\mathbb{Z}$ let $\mathcal{W}(r,t,n)$ be the collection of edge weightings on K_n such that every K_r-factor has a clique with weight strictly smaller than $t\binom{r}{2}$. We then

† Supported in part by NSF CAREER Grant DMS-0745185, UIUC Campus Research Board Grants 09072 and 11067, OTKA Grant K76099, and TAMOP-4.2.1/B-09/1/KONV-2010-0005 project. Work was partly done while at Department of Mathematics, University of California, San Diego La Jolla, CA, 92093.
‡ Supported in part by a Samsung Scholarship.

define

$$\delta(r,t,n) = \sup_{w \in \mathcal{W}(r,t,n)} \delta_w(K_n) \quad \text{and} \quad \delta(r,t) = \limsup_{n\to\infty} \frac{\delta(r,t,n)}{n}.$$

The main open question that we raise is the following.

Question 1. *Determine the value of $\delta(r,t)$ for all r and t.*

Let $\mathcal{W}^*(r,t,n)$ be the collection of edge weightings of K_n such that every K_r-factor has a clique with weight at most $t\binom{r}{2}$ (instead of strictly smaller than $t\binom{r}{2}$), and define the functions $\delta^*(r,t,n)$ and $\delta^*(r,t)$ accordingly. The compactness of the space $\mathcal{W}^*(r,t,n)$ gives the following result (whose proof we provide in the arXiv version of our paper [1]).

Proposition 0.1. *For all $r, t,$ and n, $\delta(r,t,n) = \delta^*(r,t,n)$. Therefore, $\delta(r,t) = \delta^*(r,t)$.*

The proposition above shows that if an edge-weighting of K_n has minimum degree greater than $\delta(r,t,n)$, then there exists a K_r-factor such that every copy of K_r has weight greater than $t\binom{r}{2}$. Therefore, the Hajnal-Szemerédi theorem in fact is a special case of our problem when $t = (\binom{r}{2} - 1)/\binom{r}{2}$ where we only consider the integer weights $\{0,1\}$. Thus we believe that the following special case is an important and interesting instance of the problem corresponding to the Hajnal-Szemerédi theorem for $r = 3$ (which has been first proved by Corrádi and Hajnal [2]).

Question 2. *What is the value of $\delta(3, \frac{2}{3})$?*

1. Lower bound

It is not too difficult to deduce the bound $\delta(r,t) \geq (1 - 1/r)t$ from the graph showing the sharpness of the Hajnal-Szemerédi theorem. Our first proposition provides a better lower bound to this function.

Proposition 1.1. *The following holds for every integer $r \geq 2$ and real $t \in (0,1]$:*

$$\delta(r,t) \geq \frac{1}{r} + \left(1 - \frac{1}{r}\right)t.$$

Proof. The following construction implies the bound. Let $n \in r\mathbb{Z}$ with $n > r$ and let $k = \frac{n}{r}$. Let A be an arbitrary set of $k-1$ vertices and let B be the remaining $k(r-1)+1$ vertices. Consider the weight function w that assigns weight t to edges whose endpoints are both in B and weight 1 to all other edges. □

Proposition 1.1 illustrates the fundamental difference between the minimum degree threshold for containing a K_r-factor in graphs and edge-weighted graphs. For example, when $r = 3$, we see that $\delta(3, 2/3) \geq 7/9$, while the corresponding function for graphs has value 2/3 by Hajnal-Szemerédi theorem.

2. Upper bound

Next, we provide some upper bounds on $\delta(r,t)$. For some of the bounds, in order to avoid distraction arising from technical issues, we omit the proofs and refer the reader to the arXiv version of our note [1] for details.

Given an edge-weighted graph G_w, we denote by $G_w(t)$ the subgraph of K_n consisting of edges of weight at least t. The following observation establishes the correct value of the function $\delta(2,t)$.

Observation 1. *For every $t \in (0,1]$ we have $\delta(2,t) = \frac{1+t}{2}$.*

Proof. The lower bound on $\delta(2,t)$ follows from Proposition 1.1, and thus it suffices to establish the upper bound. Let w be a weight function such that $\delta_w(K_n) \geq \frac{1+t}{2}n$. Now for any vertex $v \in G_w(t)$, we have $\deg_w(v) < (n-1-\deg(v))t + \deg(v)\cdot 1$, where $\deg(v)$ is the degree of v in G. Then the minimum weighted degree condition implies that $\deg(v) \geq \frac{n}{2}$, and so by the Hajnal-Szemerédi theorem there is a K_2-factor in $G_w(t)$. By the definition of $G_w(t)$, this establishes the bound $\delta(2,t) \leq \frac{1+t}{2}$. □

For general r, by using a similar argument to that of Observation 1, one can consider hypergraphs and try to use Daykin and Häggkvist's theorem [3] (or a stronger conjecture given in [5]) which relates the minimum degree of a hypergraph with the existence of a perfect matching. To a weighted graph we assign an r-uniform hypergraph: the r-edges of the hypergraph formed by the sufficiently heavy K_r's of the weighted graph. However, the bound obtained in this way turns out to be weaker than the following two bounds, see [1] for details.

The first bound below determines the correct value of the function $\delta(r,t)$ for small values of t.

Theorem 2.1. *For every $r \geq 3$, there exists a positive real t_r such that for every $t \in (0,t_r)$ we have $\delta(r,t) = \frac{1}{r} + \left(1-\frac{1}{r}\right)t$.*

The second bound is based on a simple reduction scheme.

Theorem 2.2. *For every $r \geq 3$ and $t \in (0,1]$, $\delta(r,t) \leq \frac{1}{2} + \frac{t}{2}$.*

Proof. Let $\delta' = \max\left\{\delta(r-1,t), \frac{1}{2}+\frac{t}{2}\right\}$. We prove that $\delta(r,t) \leq \delta'$. Let ε be an arbitrary fixed positive real, and assume that n_0 is large enough so that $\delta(r-1,t,n) \leq (\delta(r-1,t)+\frac{\varepsilon}{2})n$ for all $n \geq n_0$. Assume that we are given an edge-weighted graph G on $n \geq 2n_0$ vertices with minimum degree at least $(\delta'+\varepsilon)n$. We partition randomly the vertices of G into a set A of size $\frac{r-1}{r}n$ and a set B of size $\frac{1}{r}n = k$. By the Chernoff-Hoeffding inequalities, for large enough n, there is such a partition which additionally satisfies that for every vertex the weighted degree into A is at least $(\delta'+\frac{\varepsilon}{2})\frac{r-1}{r}n$ and into B is at least $\delta'\frac{1}{r}n$. By the assumption on δ' and n_0, we can find a K_{r-1}-factor $\mathcal{K}_A$ on A with minimum average weight t.

Using $\mathcal{K}_A$ we construct a complete weighted bipartite graph H, where the vertices on one side are associated with cliques in $\mathcal{K}_A$ and the vertices on the other side are associated with vertices in B. For a clique $K \in \mathcal{K}_A$ and a vertex $v \in B$, we assign as weight of the edge (K, v), the average of the weights of the edges between v and the vertices in K. Notice that the minimum weighted degree of H is at least $\delta' k \geq (\frac{1}{2} + \frac{t}{2})k$. Recall that $H(t)$ is the unweighted subgraph of H consisting of edges with weight at least t. A computation similar to that in Observation 1 shows that the minimum degree in $H(t)$ is at least $\frac{k}{2}$. Thus by Hall's theorem, there is a perfect matching $\mathcal{M}$ in $H(t)$.

Now notice that $\mathcal{K}_A$ and $\mathcal{M}$ lift to a K_r-factor of G with minimum weight $t\binom{r-1}{2} + t(r-1) = t\binom{r}{2}$. Consequently, $\delta(r, t, n) \leq (\delta' + \varepsilon)n$. Since ε can be arbitrarily small, we have $\delta(r, t) \leq \delta' = \max\left\{\delta(r-1, t), \frac{1}{2} + \frac{t}{2}\right\}$. Thus our conclusion follows from Observation 1, which asserts that $\delta(2, t) = \frac{1}{2} + \frac{t}{2}$. □

For the special case related to triangle factors that we discussed in the beginning, we have $\frac{7}{9} \leq \delta(3, \frac{2}{3}) \leq \frac{5}{6}$.

In this article, we proposed the study of the function $\delta(r, t)$. As seen in Section 1, this function shows different behavior from its non-weighted counterpart (which is related to the Hajnal-Szemerédi's theorem). Based on the evidence given by Proposition 1.1 and Theorem 2.1, we make the following conjecture.

Conjecture 1. *For every $r \geq 2$ and $t \in (0, 1]$,*

$$\delta(r, t) = \frac{1}{r} + \left(1 - \frac{1}{r}\right) t.$$

We refer the readers to the arXiv version of our paper [1] for further discussion.

Acknowledgement. We thank Wojciech Samotij for his valuable comments.

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