Towards a weighted version of the Hajnal-Szemerédi Theorem

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Abstract

For a positive integer $r \geq 2$, a K_r-factor of a graph is a collection vertex-disjoint copies of K_r which covers all the vertices of the given graph. The celebrated theorem of Hajnal and Szemerédi asserts that every graph on n vertices with minimum degree at least $(1 - \frac{1}{r})n$ contains a K_r-factor. In this note, we propose investigating the relation between minimum degree and existence of perfect K_r-packing for edge-weighted graphs. The main question we study is the following. Suppose that a positive integer $r \geq 2$ and a real $t \in [0, 1]$ is given. What is the minimum weighted degree of K_n that guarantees the existence of a K_r-factor such that every factor has total edge weight at least $t\binom{r}{2}$? We provide some lower and upper bounds and make a conjecture on the asymptotics of the threshold as n goes to infinity. This is the long version of a "problem paper" in *Combinatorics, Probability and Computing*.

1 Introduction

Many results in graph theory study the relation between the minimum degree of a given graph and its spanning subgraphs. For example, Dirac's theorem asserts that a graph on n vertices with minimum degree at least $\lceil \frac{n}{2} \rceil$ contains a Hamilton cycle. Hajnal and Szemerédi [3] proved that every graph on $n \in r\mathbb{Z}$ vertices with minimum degree at least $(1 - \frac{1}{r})n$ contains a spanning subgraph consisting of $\frac{n}{r}$ vertex-disjoint copies of K_r (we call such a subgraph a K_r-factor).

In this note we propose investigating this relation in edge-weighted graphs. As a concrete problem, we study the particular case when the spanning subgraph is the graph formed by vertex-disjoint copies of K_r (in other words, we would like to extend the Hajnal-Szemerédi theorem to edge-weighted graphs). Suppose we equip the complete graph K_n with edge weights $w\colon E(K_n) \to [0, 1]$. For a given weighted graph and vertex v we let $\deg_w(v)$ denote the weighted degree of the vertex v. Let $\delta_w(G)$ be the minimum weighted degree of the graph G. The main question can be

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formulated as the following: How large must $\delta_w(K_n)$ be to guarantee that there exists a K_r-factor such that every factor has total edge weight at least $t\binom{r}{2}$ for some given $t \in [0,1]$?

More formally, for $n \in r\mathbb{Z}$ let $\mathcal{W}(r,t,n)$ be the collection of edge weightings on K_n such that every K_r-factor has a clique with weight strictly smaller than $t\binom{r}{2}$. We then define

$$\delta(r,t,n) = \sup_{w \in \mathcal{W}(r,t,n)} \delta_w(K_n) \quad \text{and} \quad \delta(r,t) = \limsup_{n \to \infty} \frac{\delta(r,t,n)}{n}.$$

The main open question that we raise is the following.

Question 1. Determine the value of $\delta(r,t)$ for all r and t.

Let $\mathcal{W}^*(r,t,n)$ be the collection of edge weightings of K_n such that every K_r-factor has a clique with weight at most $t\binom{r}{2}$ (instead of strictly smaller than $t\binom{r}{2}$), and define the functions $\delta^*(r,t,n)$ and $\delta^*(r,t)$ accordingly.

Proposition 1.1. *For all r, t, and n, $\delta(r,t,n) = \delta^*(r,t,n)$. Therefore, $\delta(r,t) = \delta^*(r,t)$.*

Proof. The inequality $\delta(r,t,n) \le \delta^*(r,t,n)$ easily follows from the definition. Noting that the complement of $\mathcal{W}^*(r,t,n)$ is open in the set of all real valued edge weightings, the set $\mathcal{W}^*(r,t,n)$ is compact. Thus there is a weight function $w \in \mathcal{W}^*(r,t,n)$ so that $\delta_w(K_n) = \delta^*(r,t,n)$. Let $\varepsilon < 1$ be an arbitrary positive real, and let w' be the weight function obtained from w by multiplying $1-\varepsilon$ to all the weights. One can easily see that $w' \in \mathcal{W}(r,t,n)$, and thus $\delta(r,t,n) \ge (1-\varepsilon)\delta^*(r,t,n)$. Thus as ε tends to 0, we see that $\delta(r,t,n) \ge \delta^*(r,t,n)$. This concludes the proof. $\square$

The proposition above shows that if an edge-weighting of K_n has minimum degree greater than $\delta(r,t,n)$, then there exists a K_r-factor such that every copy of K_r has weight greater than $t\binom{r}{2}$. Therefore, the Hajnal-Szemerédi theorem in fact is a special case of our problem when $t = (\binom{r}{2} - 1)/\binom{r}{2}$ where we only consider the integer weights $\{0,1\}$. Thus we believe that the following special case is an important and interesting instance of the problem corresponding to the Hajnal-Szemerédi theorem for $r = 3$ (which has been first proved by Corrádi and Hajnal [1]).

Question 2. What is the value of $\delta(3, \frac{2}{3})$?

In the rest of our note we describe our partial results toward answering Question 1.

2 Lower bound

It is not too difficult to deduce the bound $\delta(r,t) \ge (1 - 1/r)t$ from the graph showing the sharpness of the Hajnal-Szemerédi theorem. Our first proposition provides a better lower bound to this function.

Proposition 2.1. *The following holds for every integer $r \ge 2$ and real $t \in (0,1]$:*

$$\delta(r,t) \ge \frac{1}{r} + \left(1 - \frac{1}{r}\right)t.$$

Proof. Let $n \in r\mathbb{Z}$ with $n > r$ and let $k = \frac{n}{r}$. Let A be an arbitrary set of $k-1$ vertices and let B be the remaining $k(r-1)+1$ vertices. Consider the weight function w that assigns weight t to edges whose endpoints are both in B and weight 1 to all other edges. By the cardinality of A, we see that every K_r-factor must contain a clique that lies entirely within B. Since our weight function gives weight at most $t\binom{r}{2}$ to this clique, we see that $w \in \mathcal{W}^*(r,t,n)$. Further, $\delta_w(K_n) = \min\{n-1, k-1+t(n-k)\}$. But we have that

$$k - 1 + t(n-k) = \left(\frac{1}{r} - \frac{1}{n} + t(1 - \frac{1}{r})\right) n.$$

Therefore by Proposition 1.1, we have $\delta(r,t,n) \geq \left(\frac{1}{r} - \frac{1}{n} + t(1-\frac{1}{r})\right) n$ and $\delta(r,t) \geq \frac{1}{r} + \left(1 - \frac{1}{r}\right) t$. □

Proposition 2.1 illustrates the fundamental difference between the minimum degree threshold for containing a K_r-factor in graphs and edge-weighted graphs. For example, when $r = 3$, we see that $\delta(3, 2/3) \geq 7/9$, while the corresponding function for graphs has value $2/3$ by Hajnal-Szemerédi theorem. This difference suggests that we indeed need some new ideas and techniques to solve our problem.

3 Upper bound

Next, we establish an upper bound on $\delta(r,t)$. To do so, it is helpful to consider the graph induced by the edges of heavy weights in a given edge-weighted graph. Thus, given an edge-weighted graph G_w, we denote by $G_w(t)$ the subgraph of K_n consisting of edges of weight at least t. For $r = 2$, it is easy to establish the correct value of the function $\delta(2,t)$.

Observation 1. For every $t \in (0,1]$ we have $\delta(2,t) = \frac{1+t}{2}$.

Proof. The lower bound on $\delta(2,t)$ follows from Proposition 2.1, and thus it suffices to establish the upper bound. Let w be a weight function such that $\delta_w(K_n) \geq \frac{1+t}{2} n$. Now for any vertex $v \in G_w(t)$, we have $\deg_w(v) < (n - 1 - \deg(v)) \cdot t + \deg(v) \cdot 1$, were $\deg(v)$ is the degree of v in $G_w(t)$. But then the minimum weighted degree condition implies that $\deg(v) \geq \frac{n}{2}$, and so by the Hajnal-Szemerédi theorem there is a K_2-factor in $G_w(t)$. By the definition of $G_w(t)$, this establishes the bound $\delta(2,t) \leq \frac{1+t}{2}$. □

Even for $r \geq 3$, if t is small enough, then we can determine the correct value of the function $\delta(r,t)$.

Theorem 3.1. *For every $r \geq 3$, there exists a positive real t_r such that for every $t \in (0, t_r)$ we have*

$$\delta(r,t) = \frac{1}{r} + \left(1 - \frac{1}{r}\right) t.$$

Proof. We have $\delta(r,t) \geq \frac{1}{r} + \left(1 - \frac{1}{r}\right) t$ by Proposition 2.1. It remains for us to establish the upper bound. Let ε be an arbitrary positive real. For n sufficiently large, let w be a weight function such that $\delta_w(K_n) \geq \left(\frac{1}{r} + \left(1 - \frac{1}{r}\right) t + \varepsilon\right) n$. We will say a copy of a K_r is *heavy* if it has weight at least $\binom{r}{2}t$. A collection of vertex disjoint copies of K_r is *heavy* if each K_r in the collection is heavy. An edge is *overweight* if it has weight at least $\binom{r}{2}t$. Let t_r be a sufficiently small positive real depending on r to be determined later. We will find a heavy K_r-factor given that $t < t_r$ and n is a large enough integer divisible by r.

Take a maximum heavy collection of vertex-disjoint copies of K_r, that maximizes the number of overweight edges. Call this collection $\mathcal{R}$, and suppose that $|\mathcal{R}| = \rho$. Denote by V_R be the vertices covered by $\mathcal{R}$, thus $|V_R| = r\rho$. We may assume that $\rho < \frac{n}{r}$, as otherwise we have a heavy K_r. Then there exist r distinct vertices $v_1, v_2, \cdots, v_r \notin V_R$. Let $L = \{v_1, v_2, \cdots, v_r\}$. If there is an overweight edge whose both endpoints are in $V(K_n) \setminus V_R$, then we can find a larger collection than $\mathcal{R}$ by taking the union of this edge with $r-2$ vertices of L. Thus all the edges within $V(K_n) \setminus V_R$ have weight at most $\binom{r}{2}t$.

Fact 1. For every $R \in \mathcal{R}$, if there exists an overweight edge between $V(R)$ and L, then there exists a unique vertex in R which intersects every overweight edge between $V(R)$ and L.

Proof. Fix a copy of K_r in $\mathcal{R}$ and denote it by R. If there are two vertex-disjoint overweight edges between $V(R)$ and L, then we can find two heavy vertex-disjoint copies of K_r over $V(R) \cup L$. Therefore all the overweight edges between $V(R)$ and L share a common endpoint. In particular, there are at most r overweight edges between $V(R)$ and L.

Now suppose that there are at least two overweight edges between $V(R)$ and L, and that the common endpoint is in L. Without loss of generality, let $x \in V(R)$ and $v_1 \in L$ be vertices such that there are at least two overweight edges of the form $\{y, v_1\}$ for $y \in V(R) \setminus \{x\}$. Then by the assumption that we maximized the number of overweight edges, there are at least two overweight edges among the edges $\{y, x\}$ for $y \in V(R) \setminus \{x\}$ (otherwise we can replace R by $R \setminus \{x\} \cup \{v_1\}$). However, if this is the case, then we can find two independent overweight edges in $V(R) \cup L$, and this contradicts the maximality of $\mathcal{R}$. Thus if there are at least two overweight edges between $V(R)$ and L, then they share a common endpoint in $V(R)$. □

Let $\mathcal{R}'$ be the subset of copies of K_r of $\mathcal{R}$ which have at least $r-1$ overweight edges incident to it whose other endpoint is in L. Let $\rho' = |\mathcal{R}'|$.

Fact 2. For $R \in \mathcal{R}'$, there exists a unique vertex $x_R \in V(R)$ incident to all the overweight edges within $V(R) \cup L$. Moreover, all the edges incident to x_R within R are overweight.

Proof. For a fixed copy $R \in \mathcal{R}'$, let $x_R \in V(R)$ be the vertex guaranteed by Fact 1. If there is an overweight edge in R which does not intersect x_R, then we can find two heavy vertex-disjoint copies of K_r over the set of vertices $V(R) \cup L$, and this violates the maximality of $\mathcal{R}$. Therefore, all the overweight edges within R are incident to x_R. Moreover, if there are less than $r-1$ such edges, then we can find a copy of K_R over the vertex set $\{x_R\} \cup L$ which contains at least $r-1$

overweight edges. This contradicts the maximality of overweight edges of $\mathcal{R}$. Therefore, all the edges incident to x_R within R are overweight. □

Let X be the subset of vertices which are covered by copies of K_r in $\mathcal{R}'$ that are incident to an overweight edge (guaranteed by Fact 2), and let Y be the vertices which are covered by copies of K_r in $\mathcal{R}'$ that are not in X.

Fact 3. For every $y \in Y$ and $R \in \mathcal{R} \setminus \mathcal{R}'$, y is incident to R by at most one overweight edge.

Proof. Suppose that we are given vertices $x \in X$ and $y \in Y$ covered by $R \in \mathcal{R}$. Without loss of generality, suppose that x is adjacent to $v_1, \cdots, v_{r-1} \in L$ by overweight edges. By way of contradiction, suppose that there exists $R' \in \mathcal{R} \setminus \mathcal{R}'$ with $V(R') = \{z_1, z_2, \cdots, z_r\}$ such that $\{y, z_1\}$ and $\{y, z_2\}$ are both overweight. If R' contains an edge e other than $\{z_1, z_2\}$ that is overweight, then among the edges $\{x, v_1\}, \{y, z_1\}, \{y, z_2\}, e$ (which are all overweight), we can find at least three vertex-disjoint edges. Therefore we can find three vertex-disjoint copies of K_r over the vertex set $V(R) \cup V(R') \cup L$. However, this contradicts the maximality of $\mathcal{R}$. If there are no overweight edges within R' other than (possibly) $\{z_1, z_2\}$, then the two copies of K_r over the vertex sets $\{x, v_1, \cdots, v_{r-1}\}, \{y, z_1, z_2, \cdots, z_{r-1}\}$ contain at least $r+1$ overweight edges, while R and R' combined contain at most r overweight edges (see Fact 2). Therefore we conclude that there exists at most one overweight edge of the form $\{y, z_i\}$. □

Fact 4. There does not exist a heavy K_r over a vertex set of the form $\{v_i, y_1, y_2, \cdots, y_{r-1}\}$ for $v_i \in L$ and $y_1, \cdots, y_{r-1} \in Y$.

Proof. Suppose that for some $v_i \in L$ and $y_1, \ldots, y_{r-1} \in Y$ the vertices $\{v_1, y_1, \ldots, y_{r-1}\}$ induce a heave K_r. Suppose that $\{y_1, \cdots, y_{r-1}\}$ are contained in s disjoint copies $R_1, \cdots, R_s$ of K_r belonging to $\mathcal{R}$, and let $x_1, \cdots, x_s$ be the 'dominating' vertices of these K_r guaranteed by Fact 2 (note that $s \le r-1$). If $s \le r-2$, then since each x_i are incident to L by at least $r-1$ overweight edges, we can find $s+1$ vertex-disjoint copies of a heavy K_r over the vertices $L \cup V(R_1) \cup \cdots \cup V(R_s)$. On the other hand, if $s = r-1$, then there exists an index j such that there exists $z \in V(R_j) \setminus \{x_j, y_1, \cdots, y_{r-1}\}$. Then by using the overweight edge $\{x_j, z\}$ (see Fact 2) and the overweight edges between $\{x_1, \cdots, x_s\}$ and L, we can find at least $s+1 = r$ vertex-disjoint copies of a heavy K_r over the vertices $L \cup V(R_1) \cup \cdots \cup V(R_s)$. This contradicts the maximality of $\mathcal{R}$. □

For a set T vertices, let $w(T) = \sum_{v_1, v_2 \in T} w(v_1, v_2)$. By Fact 4, it suffices to show that

$$\sum_{i=1}^{r} \sum_{T \subset \binom{Y}{r-1}} w(\{v_i\} \cup T) \ge \binom{r}{2} tr \binom{|Y|}{r-1},$$

which contradicts the assumption that $\rho < \frac{n}{r}$. Note that

$$\sum_{i=1}^{r} \sum_{T\subset\binom{Y}{r-1}} w(\{v_i\}\cup T) = \binom{|Y|-1}{r-2} \sum_{i=1}^{r}\sum_{y\in Y} w(v_i,y) + \frac{1}{2}r\binom{|Y|-2}{r-3} \sum_{y_1\in Y}\sum_{y_2\in Y\setminus\{y_1\}} w(y_1,y_2)$$

$$= \binom{|Y|}{r-1}\left(\frac{r-1}{|Y|}\sum_{i=1}^{r}\deg_w(v_i,Y) + \frac{r(r-1)(r-2)}{2|Y|(|Y|-1)}\sum_{y\in Y}\deg_w(y,Y)\right)$$

$$\geq \binom{|Y|}{r-1}\left(\frac{r-1}{|Y|}\sum_{i=1}^{r}\deg_w(v_i,Y) + \frac{r(r-1)(r-2)}{2|Y|^2}\sum_{y\in Y}\deg_w(y,Y)\right) \tag{1}$$

where $\deg_w v, Y$ is the weighted degree of v to vertices in Y.

For the first term on the right hand side of (1), we have

$$\sum_{i=1}^{r}\deg_w(v_i,Y)$$
$$=\sum_{i=1}^{r}\Big(\deg_w(v_i) - \deg_w(v_i,X) - \deg_w(v_i,V_R\setminus(X\cup Y)) - \deg_w(v_i,V\setminus V_R)\Big)$$
$$\geq (1+(r-1)t+r\varepsilon)\,n - r\rho' - \left((r-2)+(r^2-r+2)\binom{r}{2}t\right)(\rho-\rho') - r\binom{r}{2}t(n-r\rho).$$

Since the coefficient of n is positive for small enough t, we can substitute $n > r\rho$ to get

$$\sum_{i=1}^{r}\deg_w(v_i,Y) > r(r-1)t\rho + \left(2-(r^2-r+2)\binom{r}{2}t\right)(\rho-\rho') + r\varepsilon n. \tag{2}$$

For the second term on the right hand side of (1), by Fact 3 and the fact $|Y| = (r-1)\rho'$, for a vertex $y\in Y$, we have

$$\deg_w(y,Y) \geq \left(\frac{1}{r}+\frac{r-1}{r}t+\varepsilon\right)n - \rho - \binom{r}{2}t(n-\rho-|Y|)$$
$$= \left(\frac{1}{r}+\frac{r-1}{r}t - \binom{r}{2}t+\varepsilon\right)n + \binom{r}{2}t(r-1)\rho' - \left(1-\binom{r}{2}t\right)\rho.$$

Since the coefficient of n is positive for small enough t, we can substitute $n > r\rho$ to get

$$\deg_w(y,Y) > (r-1)t\rho - (r-1)\binom{r}{2}t(\rho-\rho') + \varepsilon n. \tag{3}$$

Using (2) and (3) in (1),

$$\frac{1}{\binom{|Y|}{r-1}}\sum_{i=1}^{r}\sum_{T\subset\binom{Y}{r-1}} w(\{v_i\}\cup T)$$
$$\geq \frac{r-1}{|Y|}\left(r(r-1)t\rho + \left(2-(r^2-r+2)\binom{r}{2}t\right)(\rho-\rho') + r\varepsilon n\right)$$
$$+ \frac{r(r-1)(r-2)}{2|Y|}\left((r-1)t\rho - (r-1)\binom{r}{2}t(\rho-\rho') + \varepsilon n\right)$$

If t is small enough, then the coefficient of ρ in the right hand side is positive. Hence we can substitute $\rho \geq \rho'$ to get,

$$\begin{aligned}\frac{1}{\binom{|Y|}{r-1}}\sum_{i=1}^{r}\sum_{T\subset\binom{Y}{r-1}} w(\{v_i\}\cup T) \geq & \left(\frac{r(r-1)^2}{|Y|}+\frac{r(r-1)^2(r-2)}{2|Y|}\right)\left(t\rho'+\frac{\varepsilon n}{r-1}\right)\\ =&\frac{r^2(r-1)^2 t\rho'}{2|Y|}+\frac{r^2(r-1)}{2|Y|}\varepsilon n\\ \geq&\frac{r^2(r-1)^2 t\rho'}{2|Y|}\\ =&\frac{r(r-1)t}{|Y|}\binom{r}{2}.\end{aligned}$$

Since $|Y| = (r-1)\rho'$, we have

$$\frac{1}{\binom{|Y|}{r-1}}\sum_{i=1}^{r}\sum_{T\subset\binom{Y}{r-1}} w(\{v_i\}\cup T) > r\binom{r}{2}t.$$

Thus, by Fact 4, we contradict our assumption that $\rho < \frac{n}{r}$. Hence $\delta(r,t) \leq \frac{1}{r} + \frac{r-1}{r}t + \varepsilon$ for every positive ε, and our claimed upper bound follows. □

It is worth noting that this proof gives a value of t_r of $\frac{2}{\binom{r}{2}\left(3\binom{r}{3}+2\right)}$.

For general values of t, we suggest two approaches which establish some upper bound (that unfortunately does not match the lower bound given in Proposition 2.1).

3.1 First approach: hypergraphs

In our first approach, we reduce our problem into the problem of finding a perfect matching in hypergraphs as in Observation 1. The following lemma establishes the minimum number of heavy K_r's that each vertex must belong to in a given edge-weighted graph.

Lemma 3.2. *If w is a weight function with minimum weighted degree at least δn, then every vertex is in at least $\left(1-\frac{1-\delta}{1-t}\right)\binom{n-1}{r-1}$ cliques of size $r \geq 3$ with weight at least $t\binom{r}{2}$.*

Proof. Let w be an arbitrary weight function, v be an arbitrary vertex, and let α_v be the number of K_r's of weight at least $t\binom{r}{2}$ containing v. Now letting S_v be the sum of the weights of all $\binom{n-1}{r-1}$ K_r's containing v, we have that

$$S_v \leq \alpha_v\binom{r}{2} + \left(\binom{n-1}{r-1} - \alpha_v\right)t\binom{r}{2}.$$

Let W be the total weight of w. Since edges incident with v occur in $\binom{n-2}{r-2}$ K_r's containing v and the edges not incident to v occur in $\binom{n-3}{r-3}$ such K_r's, we have

$$S_v \geq \binom{n-3}{r-3}W + \left(\binom{n-2}{r-2} - \binom{n-3}{r-3}\right)\deg_w(v).$$

Combining these inequalities we have

$$\begin{aligned}\alpha_v &\geq \frac{1}{1-t}\binom{n-1}{r-1}\left(\frac{2(n-r)}{r(n-1)(n-2)}\deg_w(v) + \frac{2(r-2)}{r(n-1)(n-2)}W - t\right)\\ &= \frac{1}{1-t}\binom{n-1}{r-1}\left(\delta\frac{n}{n-1} - t\right)\\ &\geq \frac{\delta - t}{1-t}\binom{n-1}{r-1}.\end{aligned}$$

□

We now apply Daykin and Häggkvist's theorem [2] which asserts that an r-uniform hypergraph has a perfect matching if every vertex of it lies in at least $\left(1-\frac{1}{r}\right)\left(\binom{n-1}{r-1}-1\right)$ hyperedges. This gives the following bound:

Proposition 3.3. *For every $t \in (0,1]$ and $r \geq 3$ we have $\delta(r,t) \leq 1 - \frac{1-t}{r}$.*

Hàn, Person and Schacht [4] have conjectured that Daykin and Häggkvist's theorem can be improved, and an r-uniform hypergraph has a perfect matching if every vertex lies in at least $(1 - \left(\frac{r-1}{r}\right)^{r-1} - o(1))\binom{n}{r-1}$ hyperedges. If this conjecture were proved, then we would have $\delta(r,t) \leq 1 - (1-t)\left(\frac{r-1}{r}\right)^{r-1}$.

Since in [4] the conjecture was proved for $r = 3$, we have that $\delta(3,t) \leq \frac{5}{9} + \frac{4}{9}t$. It is worth noting that this technique cannot be applied to obtain an upper bound matching Proposition 2.1. Consider the case when $r = 3$ and $t = \frac{2}{3}$. The lower bound from Proposition 2.1 reads as $\delta(3, \frac{2}{3}) \geq \frac{7}{9}$. To obtain a matching upper bound using this method, we would need to improve the conclusion of Lemma 3.2 so that in every edge-weighted graph of minimum degree at least $\frac{7}{9}n$, every vertex is contained in at least $(\frac{5}{9} + o(1))\binom{n}{2}$ copies of K_3. However, the following graph has minimum degree $\frac{29}{36}n$, and there are vertices which are contained in at most $\frac{319}{648}n^2$ copies of K_3. Let $A \cup B$ be a vertex partition such that A has size $\frac{29}{36}n$ and B has size $\frac{7}{36}n$. First, assign weight 1 to all the edges connecting A and B and give weight 1 to an $\frac{11}{18}n$-regular graph on A. Give weight 0 to each of the remaining edges. The minimum weighted degree of this graph is $\frac{29}{36}n > \frac{7}{9}n > \frac{2}{3}n$, so this graph has a triangle factor by the Hajnal-Szemerédi theorem. However, every vertex in B is only in $\frac{29}{36}n \cdot \frac{11}{18}n \cdot \frac{1}{2} = \left(\frac{319}{648} + o(1)\right)\binom{n}{2} < \left(\frac{5}{9} + o(1)\right)\binom{n}{2}$ triangles. Similar constructions can be made for other values of r and t as well.

3.2 Second approach: induction

We improve the upper bound by using two reductive schemes to build a K_r-factor out of a $K_{r'}$-factor of the graph (or a large portion of the graph).

Scheme 1. Suppose $r = pq$ with $p, q > 1$, and let w be an arbitrary weight function with minimum weighted degree δn. Let $\mathcal{K}$ be an arbitrary K_p-factor of K_n with minimum average weight t_p and consider the weight function $w_{\mathcal{K}}$ on $K_{n/p}$ defined as follows. Associate to each vertex in $K_{n/p}$ a distinct clique in $\mathcal{K}$; the weight of an edge is the average weight in w of the edges between the corresponding cliques. Now the minimum weighted degree under w' is at least

$\frac{p\delta n - p(p-1)}{p^2} = \left(\delta - \frac{p-1}{n}\right)\frac{n}{p}$. Letting K' be an arbitrary K_q-factor of this graph with minimum average weight t_q, the factors K and K' induce a K_{pq}-factor in K_n with minimum weight at least $t_q\binom{q}{2}p^2 + t_p\binom{p}{2}q$. Thus $\delta(pq,t,n) \leq \max\left\{\delta(p,t,n), \delta(q,t,\frac{n}{p}) + \frac{p-1}{p}\right\}$. Consequently, we have $\delta(pq,t) \leq \max\{\delta(p,t), \delta(q,t)\}$.

Scheme 2. Let $\delta' = \max\left\{\delta(r-1,t), \frac{1}{2} + \frac{t}{2}\right\}$. We prove that $\delta(r,t) \leq \delta'$. Let ε be an arbitrary fixed positive real, and assume that n_0 is large enough so that $\delta(r-1,t,n) \leq (\delta(r-1,t) + \frac{\varepsilon}{2})n$ for all $n \geq n_0$. Assume that we are given an edge-weighted graph G on $n \geq 2n_0$ vertices with minimum degree at least $(\delta' + \varepsilon)n$. We partition randomly the vertices of G into a set A of size $\frac{r-1}{r}n$ and a set B of size $\frac{1}{r}n = k$. By the Chernoff-Hoeffding inequalities, for large enough n, there is such a partition which additionally satisfies that for every vertex the weighted degree into A is at least $(\delta' + \frac{\varepsilon}{2})\frac{r-1}{r}n$ and into B is at least $\delta'\frac{1}{r}n$. By the assumption on δ', we can find a K_{r-1}-factor $\mathcal{K}_A$ on A with minimum average weight t.

Using $\mathcal{K}_A$ we construct a complete weighted bipartite graph H, where the vertices on one side are associated with cliques in $\mathcal{K}_A$ and the vertices on the other side are associated with vertices in B. For a clique $K \in \mathcal{K}_A$ and a vertex $v \in B$, we assign as weight of the edge (K,v), the average of the weights of the edges between v and the vertices in K. Notice that the minimum weighted degree of H is at least $\delta' k \geq (\frac{1}{2} + \frac{t}{2})k$. Recall that $H(t)$ is the unweighted subgraph of H consisting of edges with weight at least t. By a similar argument as in Observation 1, the minimum degree in $H(t)$ is at least $\frac{k}{2}$. Thus by Hall's theorem, there is a perfect matching $\mathcal{M}$ in $H(t)$.

Now notice that $\mathcal{K}_A$ and $\mathcal{M}$ lift to a K_r-factor of G with minimum weight $t\binom{r-1}{2} + t(r-1) = t\binom{r}{2}$. Consequently, $\delta(r,t,n) \leq (\delta' + \varepsilon)n$. Since ε can be arbitrarily small, we have $\delta(r,t) \leq \delta' = \max\left\{\delta(r-1,t), \frac{1}{2} + \frac{t}{2}\right\}$.

By Proposition 2.1, Observation 1, and Scheme 2, we obtain the following theorem.

Theorem 3.4. *For every $r \geq 3$ and $t \in (0,1]$,*

$$\frac{1}{r} + \left(1 - \frac{1}{r}\right)t \leq \delta(r,t) \leq \frac{1}{2} + \frac{t}{2}.$$

For the special case related to triangle factors that we discussed in the beginning, we have $\frac{7}{9} \leq \delta(3, \frac{2}{3}) \leq \frac{5}{6}$. We note that Theorem 3.4 has been proved without using Scheme 1, however, Scheme 1 implies that if there is an improvement on the upper bound for any r, then there is an improvement in the upper bound for an infinite class of r'. For example, for any fixed k, $\delta(r^k, t) \leq \delta(r,t)$. Because of the dependence on the bipartite matching result (which cannot be improved) a similar statement does not hold using just Scheme 2.

3.3 Open Question

In this article, we proposed the study of the function $\delta(r,t)$. Based on the evidence given by Proposition 2.1 and Theorem 3.1, we make the following conjecture.

Conjecture 1. For every $r \geq 2$ and $t \in (0, 1]$,

$$\delta(r,t) = \frac{1}{r} + \left(1 - \frac{1}{r}\right) t.$$

The function $\delta(r,t)$ shows different behavior from its non-weighted counterpart (which is related to the Hajnal-Szemerédi's theorem). As one can see from the discussion of Subsection 3.2, the approach of examining the function by fixing t and varying r, opens up new possibilities which has no counterpart in the Hajnal-Szemerédi theorem. We note that our results suggest, but does not quite establish, the fact that for fixed t, $\delta(r,t)$ is a *decreasing* function of r. Further note that the weighted case has an extra power coming from the ability to include any edge in a K_r-factor, even if that edge has weight 0. This suggests that there could be a relation to results of Kuhn and Osthus [5] on the existence of H-factors in graphs.

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